REVISITING STOKES’ FIRST PROBLEM

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ABSTRACT

Stokes’ first problem is a fundamental unsteady fluid problem from which an exact solution can be found. The main object of the study is to theoretically solve a variation of Stokes’ first problem. The variation of Stokes’ first problem being solved is a suddenly accelerated plate to a constant shear stress instead of a constant velocity. A solution of this problem was found using a similarity approach. Results have a variety of applications related to unsteady fluid-flow phenomena.

INTRODUCTION

In fluid dynamics, determining the flow of a suddenly accelerated, infinitely long, flat plate with an unbounded fluid above it is considered Stokes’ first problem or the Rayleigh problem. In this simple, unsteady problem, the plate is accelerated to a constant velocity. The solution to Stokes’ first problem is well known and is found through a similarity approach. Variations of solutions to Stokes’ problems have many applications to various academic researches [1, 2]. In this study, Stokes’ first problem is reconsidered but with a constant shear stress instead of constant velocity.

REVIEW OF STOKES’ FIRST PROBLEM

Stokes’ first problem consists of an infinitely long flat plate with an unbounded fluid above it. The plate is suddenly accelerated to a constant velocity, \( V \). The x-momentum equation and initial and boundary conditions are, respectively,
\[
\frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial y^2} \tag{1}
\]
\[
u(y,0) = 0 \tag{2.1}
\]
\[
u(0,t) = 0 \tag{2.2}
\]
\[
u(y \to \infty, t) = 0 \tag{2.3}
\]
where \( v \) is the kinematic viscosity. Following a similarity approach where,
\[
\eta = \frac{y}{2\sqrt{vt}} \tag{3}
\]
the boundary conditions of the problem become,
\[
u(\eta \to \infty) = 0 \tag{4.1}
\]
\[
u(\eta = 0) = V \tag{4.2}
\]
Solving the momentum equation for \( u \) yields,
\[
u = V(1 - \text{erf}(\eta)) \tag{5}
\]
This solution is illustrated in Figure 1 in the Appendix. There is no steady-state solution because the velocity profile will gradually develop [3].

VARIATION OF STOKES’ FIRST PROBLEM

The same similarity approach will be followed to solve Stokes’ first problem with constant shear stress instead of a constant velocity. The x-momentum of the flow is the same as equation (1), but with different boundary conditions,
\[
u(y,0) = 0 \tag{6.1}
\]
\[
u(0,t) = 0 \tag{6.2}
\]
\[
u(0,t) = K \tag{6.3}
\]
where \( K \) is a constant. The two similarity variables are needed to solve this equation are,
\[
\eta = \frac{y}{\sqrt{vt}} \tag{7.1}
\]
\[
\varphi = \frac{u}{\sqrt{t}} \tag{7.2}
\]
Using these similarity variables, the initial and boundary conditions and the x-momentum equation become, respectively,
\[
\varphi(\eta \to \infty) = 0 \tag{8.1}
\]
\[
\varphi'(\eta = 0) = K \sqrt{\nu} \tag{8.2}
\]
\[ \varphi'' + \eta \varphi' / 2 - \varphi / 2 = 0 \tag{9} \]

Equation (9) is a second-order, linear, homogeneous ODE and can be solved. First, the equation is converted to a linear, separable, first order differential equation through substitution:

\[ u' = -\frac{\eta^2 + 4}{2 \eta} \tag{10} \]

where \( u' = \frac{\varphi'}{\eta} \). Separating, solving, and back substituting yields,

\[ \varphi = \eta \int \frac{e^{-\frac{\eta^2 + c_1}{4}}}{\eta^2} \, dx \tag{11} \]

where \( c_1 \) is a constant. Integrating yields the final solution,

\[ \varphi = c_1 \left( -2e^{-\frac{\eta^2}{4}} - \sqrt{\pi} \eta \text{erf} \left( \frac{\eta}{2} \right) + c_2 \right) \tag{12} \]

where \( c_2 \) is a constant equal to \( K \sqrt{\nu} \), and \( c_1 \neq 0 \). Figure 2 shows a visualization of this solution with various values for \( c_1 \).

It is interesting to note the disagreement between the boundary condition in equation (8.1) and the solution given in equation (12). The boundary condition states that the solution is bounded, while the solution states otherwise. Therefore, the boundary condition could not be applied to the solution as it would yield an indeterminate solution. Further analysis would be needed to determine the correct value for constant \( c_1 \) to accurately describe the solution.

**CONCLUSIONS**

Variations of Stokes’ problems can be effectively used in a variety of research applications. A variation of Stokes’ first problem was considered. A solution for this equation was derived through similarity and solved through a substitution method. While the solution was determined, a disagreement between the boundary conditions and the solution was found, and, therefore, the correct value for the coefficient \( c_1 \) could not be determined. Further analysis is suggested to determine the value of \( c_1 \).

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**REFERENCES**


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**APPENDIX**

Figure 1. Visualization of Stokes’ first problem: \( u/V \) as a function of \( \eta \) for a suddenly accelerated plate to a constant velocity \( V \).

Figure 2. Visualization of a variation of Stokes’ first problem, \( \varphi \) as a function of \( \eta \) for a suddenly accelerated plate to a constant shear stress \( \frac{\partial u}{\partial y} = K \). Various values from 1-10 were used for the constant \( c_1 \).