ANALYSIS OF THEORY BEHIND LASER FLASH METHOD

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ABSTRACT

The laser flash method is a means of determining properties of a material, such as thermal conductivity. A disc of material is heated by a laser pulse, and then the temperature profile is measured on the surfaces. This method can be modeled by a partial differential equation and solved using integral transforms. Solutions can then be compared with experimental results to determine values of properties. This information would assist in the process of selecting materials for purposes of engineering, scientific studies, manufacturing, etc.

NOMENCLATURE

FLASH method
Thermal conductivity

INTRODUCTION

The laser flash method has been a useful means of determining the thermal conductivity or the thermal diffusivity of a material. It is done by heating one surface of a cylinder with the pulse of a laser then measuring the temperature profile on one of the faces on the cylinder. The temperature profile changes in time and can be compared to a mathematical model to determine the thermal property value. The purpose of this paper is to develop this mathematical model and show the viability of the laser flash method.

DEVELOPMENT OF THE MATHEMATICAL MODEL

It is assumed that the cylinder being examined has black absorptive surfaces and is homogenous in composition. It is also assumed that material is isotropic with respect to thermal conductivity and optically opaque.

The problem is based on the heat equation in cylindrical coordinates

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = \frac{\rho C_p}{k} \frac{\partial u}{\partial t}
\]

Where

\begin{itemize}
  \item r = radial position variable
  \item z = axial position variable
  \item t = time variable
  \item u = temperature relative to ambient temperature
  \item \rho = density of the material
  \item k = thermal conductivity
  \item C_p = specific heat
\end{itemize}

This equation is subject to the following boundary conditions:

At \( z = 0 \)

\[-k \frac{\partial u}{\partial z} + h_0 u = f_0(r, t)\]

At \( z = L \)

\[k \frac{\partial u}{\partial z} + h_L u = 0\]
At $r = r_1$
\[
k \frac{\partial u}{\partial r} + h_r u = 0
\]

At $r = 0$
\[
u(0, z, t) < \infty
\]

Where
- $L = \text{length of cylinder in meters}$
- $r_1 = \text{radius of cylinder in meters}$
- $h_0, h_L, h_r = \text{convective coefficient of the faces of the cylinder at } z = 0, z = L, r = r_1$ respectively
- $f_0(r, t) = \text{function describing surface boundary condition}$

With each of the boundary conditions being Robin type conditions, this model can be used to describe surfaces where convection occurring, insulated surfaces, or isothermal surfaces.

This equation can be solved using the Finite Hankel, Finite Fourier, and Laplace transforms together. Applying these transforms and solving for the transformed solution yields
\[
\bar{U}_{n,j} = \frac{\mu_n F_0}{\rho C_p \kappa} s + \lambda_j^2 + \mu_n^2
\]

Where
- $F_0$ indicates the Laplace transform of $f_0$ and the bar above $F_0$ indicates the Hankel transform has also been applied
- $\lambda_j$ are the eigenvalues used in the Finite Hankel transform
- $\mu_n$ are the eigenvalues used in the Finite Fourier transform

The solution is found by applying the inverse transforms, yielding
\[
u = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \int_0^T \mu_n f_0(\lambda_j, \tau) \exp\left(-\frac{k}{\rho C_p} (\lambda_j^2 + \mu_n^2)(t - \tau)\right) d\tau
\]
\[
\left(\mu_n \cos(\mu_n z) + \frac{h_0}{K} \sin(\mu_n z)\right) \left(f_0(\lambda_j r) \right)
\]
\[
* \left(\frac{(\mu_n^2 + \frac{h_0^2}{K^2})}{2}\left(L + \frac{h_L}{k \mu_n^2 + \frac{h_L^2}{K}}\right) + \frac{h_0}{2K}\right) \left(\frac{r_1^2}{2} \left(\frac{h_r^2}{k^2 \lambda_j^2} + 1\right) f_0^2(\lambda_j r_1)\right)
\]

If the boundary condition function at the irradiated surface is defined as
\[
f_0 = E \frac{\delta(r) \delta(t)}{2\pi r}
\]

Where
- $E$ is proportional to the energy emitted by the laser

Visualization and Determination of Thermal Conductivity

The solution can be plotted for $z = 0$ and/or $z = L$ to show the temperature profile at each circular face of the cylinder. If the density and specific heat of the material are known, these temperature profiles can be plotted for several values of thermal conductivity and several values of time, which can be compared with experimental data.

Following are several solution plots of the temperature profile of the unirradiated, circular face of a cylinder of steel. The thermal conductivity is varied between them, and the plot based on the actual conductivity value is shown below them all (all plots based on model). These profiles can be compared to determine the thermal conductivity of the material being examined.
Results

The value of the materials thermal conductivity is easily approximated using visual comparison with plots. Thus, comparison of actual data with mathematical model using high-speed thermal camera and computer software could provide fairly accurate determinations of a material’s thermal conductivity, assuming model accurately reflects temperature profile.

CONCLUSIONS

With appropriate equipment the laser flash method is a viable method for precisely measuring the thermal conductivity of a material. The method could potentially be a means of determining other properties (such as absorption coefficient for semi-transparent media) if appropriate adjustments to the mathematical model are made and other surface temperature profiles are measured.

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REFERENCES

APPENDIX

Maple 16 code for plotting and animating temperature profile for various values of constants.

```maple
> restart;
> with(plots):

> h0 := 3; hL := 3; hr := 3; k := 54; rho := 7833; C[p] := 0.465; L := 1/100; rL := 3/100; f0 := 1;

#Units in meters, kilograms, seconds, and kelvin. Values based on 0.5% C Steel (https://ncfs.ucf.edu/burn_db/Thermal_Properties/material_thermal.html) Actual value of k: k := 54

h0 := 3
hL := 3
hr := 3
k := 54
rho := 7833
C[p] := 0.465
L := 1/100
rL := 3/100
f0 := 1

> Zeq := x -> (h0*hL/k^2 - x^2)sin(x*L) + (h0 + hL)/k*x*cos(x*L);

#Characteristic equation for z variable Eigenvalues

Zeq := x -> (h0*hL/k^2 - x^2)sin(x*L) + (h0 + hL)*x*cos(x*L)/k

> mu := array(1..50);

mu := array(1..50, [ ])

> n := 1: for m from 0 to 500 do z := fsolve(Zeq(x) = 0, x = m*15..(m + 1)*15): if type(z,float) then mu[n] := z: n := n + 1 fi od:

#Solve characteristic equation for Eigenvalues and store in variable mu

> N := n - 1; #Number of Eigenvalues found in specified domain

N := 24

> n := 'n': m := 'm': z := 'z': N := 'N':

#Reset variables from previous calculations

> Z[n] := mu[n]*cos(mu[n]*z) + h0/k*sin(mu[n]*z);

#Define Eigenfunctions

Z_n := mu_n*cos(mu_n*z) + h0/k*sin(mu_n*z)
```

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> \[ Z_{\text{norm}}^2[n] := \frac{1}{2} \left( \frac{\mu_n^2}{k^2} + \frac{h \theta^2}{k^2} \right) \left( L + \frac{hL}{k \cdot \mu_n^2 + \frac{h \theta^2}{k}} \right) + \frac{h \theta}{2 \cdot k}; \] #Define the squared norm of the Eigenfunctions

\[ Z_{\text{norm}}^2 := \frac{1}{2} \left( \frac{\mu_n^2}{k^2} + \frac{1}{324} \right) \left( \frac{1}{100} + \frac{3}{54 \cdot \mu_n^2 + \frac{1}{6}} \right) + \frac{1}{36} \]

> \[ \text{Req} := x \rightarrow -x \cdot \text{BesselJ}(1, x \cdot r l) + \frac{h r}{k} \cdot \text{BesselJ}(0, x \cdot r l); \] #Characteristic equation for \( r \) variable Eigenvalues

\[ \text{Req} := x \rightarrow -x \cdot \text{BesselJ}(1, x \cdot r l) + \frac{h r \cdot \text{BesselJ}(0, x \cdot r l)}{k} \]

> \[ \text{lambda} := \text{array}(1 .. 50); \]

\[ \lambda := \text{array}(1 .. 50, [ ]); \]

> \[ n := 1 : \text{for} m \text{ from 0 to 50 do} \]

\[ z := \text{fsolve}((\text{Req}(x) = 0, x = m \cdot 50 .. (m + 1) \cdot 50) ; \text{if type}(z, \text{float}) \text{ then lambda}[n] := z \text{ : n} := n + 1 \text{ fi} \text{ od}; \]

#Solve characteristic equation for Eigenvalues and store in variable lambda

> \[ N := n - 1; \] #Number of Eigenvalues found in specified domain

\[ N := 25 \]

> \[ n := 'n'; m := 'm'; z := 'z'; N := 'N'; \]

#Reset variables used in previous calculations

> \[ R[j] := \text{BesselJ}(0, \text{lambda}[j] \cdot r); \] #Define Eigenfucntions

\[ R_j := \text{BesselJ}(0, \lambda_j \cdot r) \]

> \[ R_{\text{norm}}[j] := \frac{r^2}{2} \left( \frac{\theta^2}{k^2 \cdot \text{lambda}[j]^2} + 1 \right) \cdot (\text{BesselJ}(0, \text{lambda}[j] \cdot r)^2); \] #Define the squared norm of the Eigenfunctions

\[ R_{\text{norm}}^2 := \frac{9}{20000} \left( \frac{1}{324 \lambda_j^2} + 1 \right) \cdot \text{BesselJ}(0, \frac{3}{100} \lambda_j)^2 \]

> #Determining transformed boundary condition

> \[ f := f_0 \cdot \text{Dirac}(r) \cdot \text{Dirac}(x); \]

#Nonhomogeneous boundary function (\( t \) replaced with \( x \) to facilitate inverse laplace transform)

\[ f' := \frac{1}{2} \frac{\text{Dirac}(r) \cdot \text{Dirac}(x)}{\pi r} \]
> \( f_h := \int (J_0(r \cdot \text{lambda}[n]) \cdot r, r=-r\cdot r) \);  
>  
> #Finite Hankel transform, bounds of integration altered to account  
> for how maple integrates Dirac delta function. If other source were  
> desired to be used bounds would be \( r=0..r\);  
>  
> \( f_h := \frac{1}{2} \frac{\text{Dirac}(x)}{\pi} \)

> #Inverse Laplace transform for boundary source term  
>  
> \( u_f[n,j] := \frac{k}{\rho \cdot C[p]} \left( \text{mu}[n] \cdot \int \left( f_h \cdot \exp\left( -\frac{k}{\rho \cdot C[p]} \cdot (\text{lambda}[j])^2 \right. \right. 
> + \mu[n]^2 \cdot (t - x), x=0..t) \right) \);  
>  
> \( u_{f,n,j} := 0.002359569708 \mu_n (\text{Heaviside}(t) 
> - 0.5000000000) e^{-0.01482561372 \left( \lambda_j^2 + \mu_n^2 \right) t} \)

> #Inversion of Hankel transform through summation  
>  
> \( u_{total} := (r, z, t) \rightarrow \sum \left( \sum_{j=1}^{20} \left( \frac{R[j]}{\text{Rnorm2}[j]} \right) \right) \);  
>  
> #Upper limit of index of summation can be increased up to the total  
> number of Eigenvalues calculated above, (index \( n \) for \( \text{mu}[n] \), index  
> \( j \) for \( \text{lambda}[j] \))  
>  
> \( u_{total} := (r, z, t) \rightarrow \sum_{n=1}^{20} \sum_{j=1}^{20} \frac{u_{f,n,j} R_j}{\text{Rnorm2}_n Z_n} \)

> sampletime := 0.001;  
>  
> #Define the time at which the contour plots will be calculated  
>  
> sampledist := L;  
>  
> #Define the distance along the axis of the cylinder at which the  
> contour plots will be calculated
> t := sampletime : z := sampledist : contourplot([r, theta, u(x, y)], r = 0..1, theta = 0..2*Pi, coords = cylindrical, filledregions = true, coloring = [gray, red], contours = [500, 1000, 1500]); t := 't' ; z := 'z';
> z := sampledist : animate(contourplot, [[r, theta, utotal(r, z, t)], r = 0
   ..1, theta = 0 ..2*Pi, coords = cylindrical, filledregions = true,
   coloring = [gray, red], contours = [500, 1000, 1500]], t = 0.0001
   ..0.01); z := z;

\[ t = 0.010000 \]