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Reverberations of Plucked Guitar String

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Abstract

This paper will explore the dynamics of a guitar string which is plucked by a player. A partial differential equation is derived to approximate the guitar string vibration through time. To derive this equation assumptions are made of completely rigid connections, hence no movement is occurring at the ends of the string. The approximation can be used for varying string diameters, tensions, and lengths. Information provided can be used to better understand the tones and notes played on a guitar string.

Nomenclature

d = string diameter
Δ = initial displacement
L = string Length
T = string tension
τ = string stiffness

Introduction

When a guitar is strummed, the strings vibrate at a frequency that can produce a harmonic melody. The string initially produces a sound which is a result of the plucking, after a certain period of time the string comes to a steady state vibration which dampens over time. A single point force, generated by the player, causes an initial displacement of the string. Tension causes wave propagation throughout time as the string attempts to return to steady state. This behavior can be altered by a number of different variables including string tension, location of displacement, string length, etc. Understanding the dynamics of the guitar string can help to understand the harmonic notes produced. For the purpose of this paper we will consider string lengths and diameters that are generally found in guitars see Figure 1. We will also assume no motion at the ends of the strings due to fully fixed connections. These limitations will give us boundary conditions at both ends of the string.

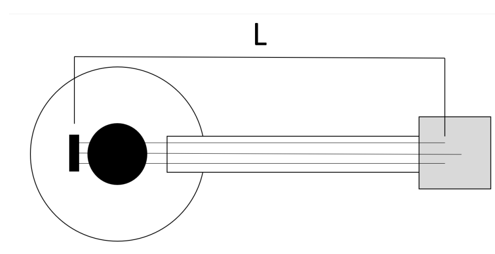


Figure 1: Guitar Diagram

Using the equations derived in this paper we were able to create a model and plot the guitar string's motion throughout time. Some of these plots are included in this paper to more easily visualize the motion. With this model it is possible to manipulate the parameters of the bow string to approximate different notes.

Method

To derive the equation of motion of a guitar string we start with the wave equation and add terms,b , for damping. α controls how the wave propagates through the string. This brought us to our governing equation.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

Boundary conditions:

$$u(0,t) = 0 \qquad t > 0$$
$$u(L,t) = 0 \qquad t > 0$$

Initial conditions:

$$u(x,0) = u_0(x) \qquad 0 \le x \le L$$
$$u'(x,0) = u_1(x) \qquad 0 \le x \le L$$

Eigenvalues and Functions:

$$\mu_n = \frac{n\pi}{L} \quad , \quad X_n(x) = sin(\mu_n x)$$

After taking the finite Fourier transform we arrive at.

$$-\alpha^2\mu_n^2\bar{u}_n - b\frac{\partial \bar{u}_n}{\partial t} = \frac{\partial^2 \bar{u}_n}{\partial t^2} \tag{2}$$

By selection $u_0(x) = 0$. Apply the Laplace transform and solve for $U_n(s)$

$$-\alpha^2\mu_n^2 U_n(s) - b\Big(sU_n(s) - \bar{u}_0\Big) = \Big(s^2 U_n(s) - s\bar{u}_0 - \bar{u}_1\Big)$$

$$U_n(s) = \frac{1}{s^2 + \alpha^2\mu_n^2 + bs}\bar{u}_1$$

$$U_n(s) = \frac{1}{s^2 + bs + C}\bar{u}_1 \tag{3}$$

Where $C = \alpha^2\mu^2$. Taking the inverse Laplace followed by the inverse Fourier arrive at.

$$\bar{u}_n = \bar{u}_1 \frac{2e^{-\frac{1}{2}bt} sinh(\frac{1}{2}t\sqrt{b^2 - 4C})}{\sqrt{b^2 - 4C}} \tag{4}$$

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \bar{u}_n sin(\mu_n x) \tag{5}$$

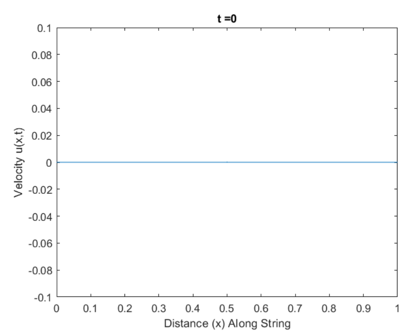
Because there is a singular point force in the middle of the string u_1 can be defined as $u_1(x) = \delta(x - \frac{L}{2})$ and transformed to be $\bar{u}_1 = sin(\mu_n \frac{L}{2})$. Dirac δ is commonly used to represent a single impulse function.

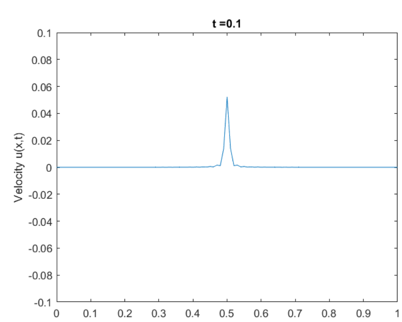
Results

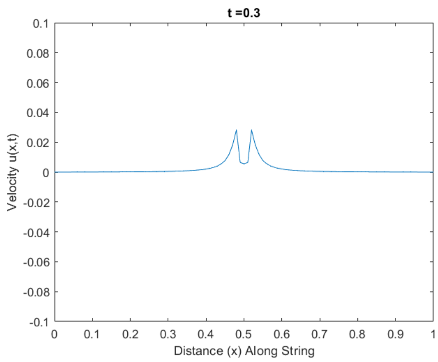
Plots of the PDE over time are included in this section. Values of variables used for these plots are found below.

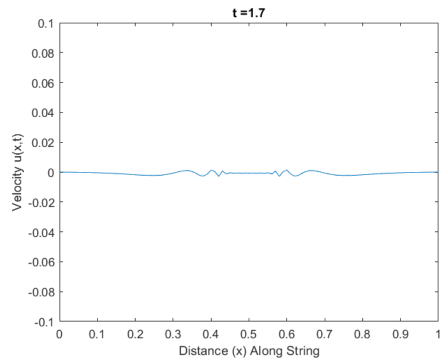
Table 1: Parameters used to create plots.

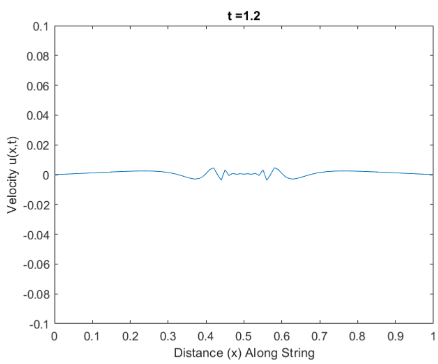
Parameter	Value
a	2
b	2
L	1
t	0 - 2.5

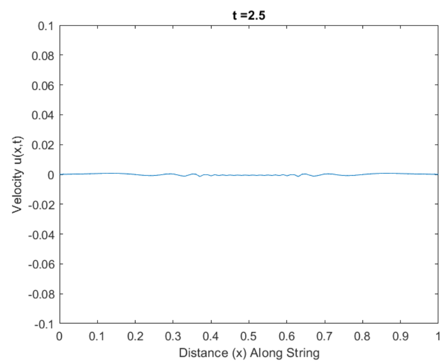












Conclusions

It can be seen that the string oscillates in a wave like pattern that is damped over time. The motion does not perfectly align with expectation. This PDE neglects the stiffness of the string so the plots behave more like a fluid. Tuning the coefficients it is possible to better approximate string motion. The difference made by string length as well as damping coefficients can be observed. We can also track through time the effects that each of these variables has on the string's motion. It is possible to use this model to approximate the motion of a guitar string after it has been plucked.

Acknowledgements

We would like to acknowledge the contributions of Dr. Soloviev for his help in setting up and solving the PDE as well as his book for providing equations and guidance.

References

[1] K. Lanc, S. Braut, R. Žigulić *MODELING AND TESTING OF THE MUSICAL INSTRUMENT STRING MOTION* AIAS 43rd National Convention (2014)

[2] Gràcia, Xavier and Tom'as SanzPerela *The wave equation for stiff strings and piano tuning* Classical Physics (2016)

Appendix

```
clc
clear
close all

L = 1;
a = 2;
b = 2;

t = 0:0.01:2.5;
x = 0:0.01:L;
for i = 1:length(t)
for j = 1:length(x)
sum = 0;
for n = 1:100
mu = n*pi/L;
A = b;
C = a*mu;
sum = sum + ((4/L)*.01*sin(mu*L/2)*exp(-
.5*A*t(i))*sinh(0.5*t(i)*sqrt(A^2 - 4 * C)) * sin(mu *
x(j))/sqrt(A^2 - 4 * C));
end
u(j, i) = sum;

    end
plot(x,u(:,i));
ylim([-0.1 .1])
xlim([0 L]);
time = strcat('t =', num2str(t(i)));
title(time)
xlabel('Distance (x) Along String')
ylabel('Velocity u(x,t)')
pause(0.001)
end
```

