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AN OSCILLATING PENDULUM

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ABSTRACT

Understanding the trajectories of a pendulum is a real-world problem that has shown to be more complex than often described by simple sine and cosine waves. In this study we examine the pendulum and take air-resistance into account and obtain a second-order differential equation. We solve this equation graphically by plotting vector fields and showing possible trajectories for the pendulum. Furthermore, we compare different trajectory maps with different input for air resistance and note that we can project how many times the pendulum will make turns on itself.

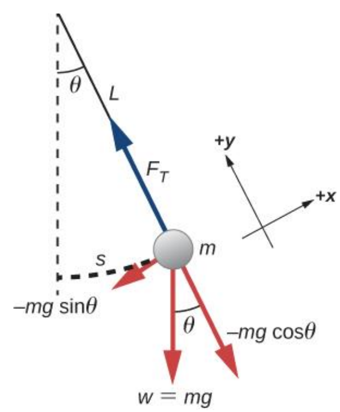
NOMENCLATURE

$$L = length\ of\ the\ string\ (m)$$
$$s = displacement\ of\ string\ (m)$$
$$g = gravity\ (\frac{m}{s^2})$$
$$mu = airdrag$$
$$m = mass\ (g)$$
$$\Theta = the\ angle\ of\ the\ pendulum$$
$$\Theta' = angular\ velocity$$
$$t = time\ (\text{sec})$$
$$a = acceleration\ (\frac{m}{s^2})$$

INTRODUCTION

A pendulum is some object that carries mass that is attached to a string with a certain length. The string is attached to a fixed point. The mass attached to the spring will oscillate back and forth as the force of gravity act on the object. This phenomenon has simplified solutions that are often solved in elementary mathematics courses using trigonometric waves. However, reality is not as simple as the sine and cosine waves continue oscillating back and forth across time. The pendulum does not oscillate forever because air resistance slows down the oscillating movements and which eventually results in the pendulum coming to a stop. This report aims to help visualize and understand how this pendulum will move across time with a given initial angle and initial angular velocity. Vector fields will be examined to visualize the trajectory of the pendulum as a function of the angle and angular velocity across time.

Figure 1.



A scheme of the pendulum with its associated angles and geometry.

Method

How does the angle Θ change as a function of time? This idea has been resembled and was originally thought of in terms of simple harmonic motion:

$$\theta(t) = \theta_0 \cos\,(g/\sqrt{\frac{g}{Lt}} \quad (1)$$

And with a period of:

$$P = 2\pi\sqrt{L/g} \quad (2)$$

These approximations do not resemble reality and if you made a measurement of a pendulum, you would notice that the period changes as you make it swing further out. Thus, this original formula works well when Θ is small.

To evaluate how the pendulum will move, we must solve a 2nd order differential equation. It can be derived from *Figure 1* where the acceleration can be written as:

$$a = s'' = -g\sin\Theta \quad (3)$$

$$-g\sin\theta = L\theta'' \quad (4)$$

$$\theta'' = -g/L\sin\theta \quad (5)$$

To reflect reality, another term $\boldsymbol{\mu}$ which represents the air resistance must be added. We also add the component $\boldsymbol{t}$ as the independent variable. This gives the second order differential equation:

$$\theta''(t) = -\mu\theta'(t) - \frac{g}{L\sin(\theta(t))} \quad (6)$$

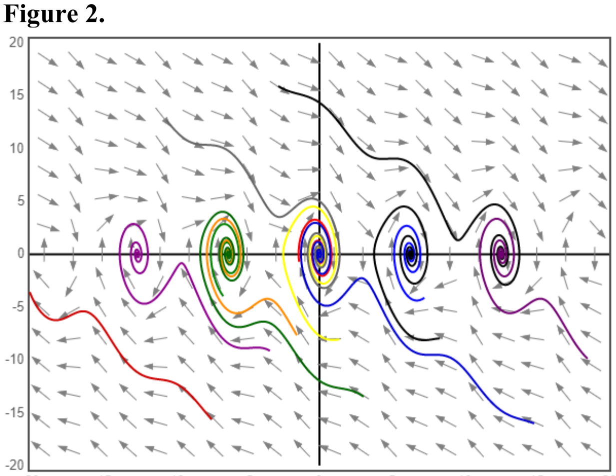
To understand how the pendulum will oscillate over time with an initial starting angle and initial angular velocity, we can graph a vector field that shows the trajectory of the pendulum.

To plot this vector field, we can rewrite this ordinary differential equation as a system of two equations.

$$\frac{d\theta}{dt} = w(t) \quad (7)$$

$$\frac{dw}{dt} = -\mu w - \frac{g}{L}\sin(\theta) \quad (8)$$

Figure 2.



Vector field showing different trajectories of different initial angles and initial angular velocity. The axes of this graph shows Θ (initial starting angle) as independent variable and Θ' (angular velocity) as the dependent variable. The pendulum stabilizes at any integer of 2nπ.

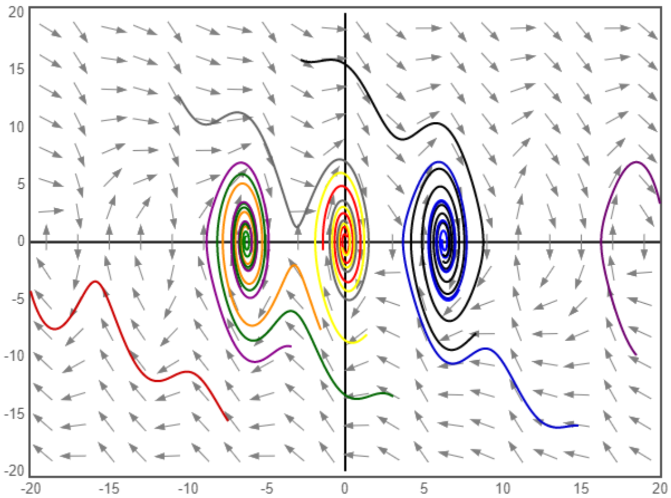
At any given starting point on the vector field, it is evident that the pendulum will go toward any multiple of 2π as *t* gets larger. That assumes that as time increases, the pendulum loses its energy over time.

Another observation that can be made by studying the vector field is that some trajectories tend to move across the Θ-axis and start decaying at a larger or smaller multiple of *2π* than its initial position. This happens when the pendulum is making one or more full turns before it starts decaying.

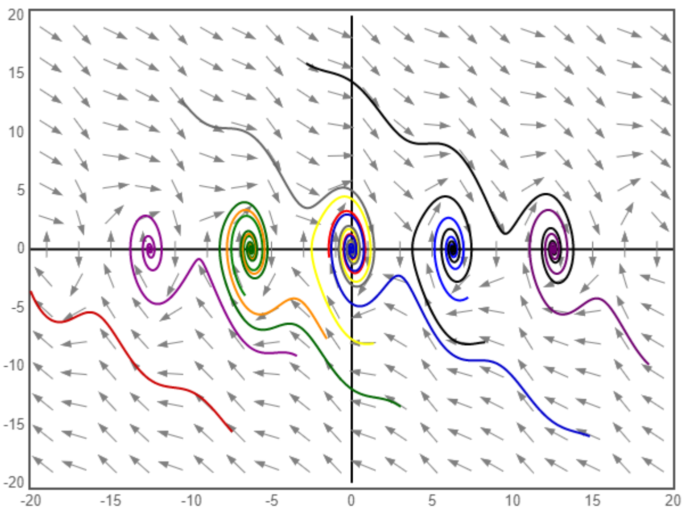
Furthermore, if we change the air resistance, we can make new observations:

Figure 3

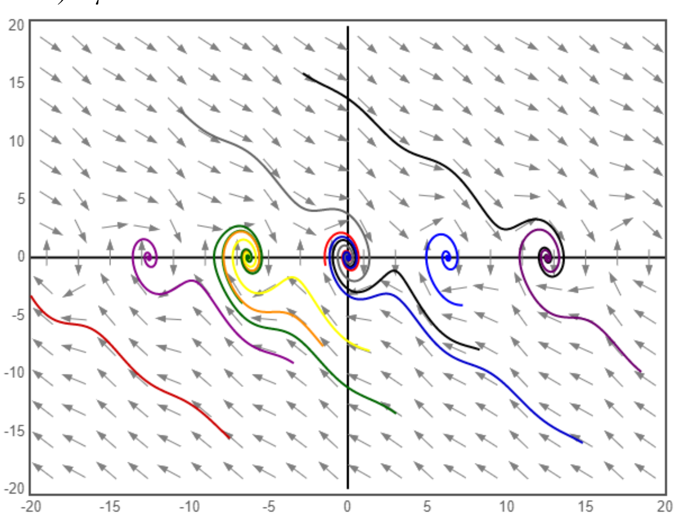
a) $\mu=0.05$



b) $\mu=0.1$



c) $\mu=0.2$



The result of air resistance a) $\mu=0.05$*, b)* $\mu=0.1$ *c)* $\mu=0.2$ *showing trajectories with ϴ (initial angle) as the x-variable and ϴ'(angular velocity) as our y-variable.*

By examining the three vector fields, air resistance will have an effect on how many turns each pendulum will make depending on its initial angle and initial angular velocity. For example, examine the yellow trajectory which decays around the origin when $\mu=0.05$ and $\mu=0.1$. However, when $\mu=0.2$ the yellow trajectory is decaying at -2π. Thus, by changing air resistance the pendulum will make turns around itself before starting to decay.

CONCLUSIONS

In conclusion it has been shown that a pendulum can be modeled by a second order ordinary differential equation and that it can be visualized using a vector field. This is useful for understanding the underlying physics behind oscillating objects and sets the stage for other physical observations that can be seen in the real world. This solution has taken air drag, angular and initial velocity into account. It becomes important for engineers as the apply these principles to oscillating objects. It can have important applications in the building of swings for children as it is useful to know how fast the swing will oscillate back and forth at different times and at what velocities the swing will turn on itself.

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