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# Doppler Effect with Gravitational waves in a 2D space

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## Abstract

This paper seeks to explore the possibility of the Doppler Effect in gravitational waves. The system is modeled using the wave equation and the two orbiting mass bodies are modeled as point sources. The model showed some interesting results. In addition to the expected Doppler Effect, there is an uneven change in amplitude of the waves. According to the model, the waves in direction of motion are significantly larger then the waves trailing behind.

## Nomenclature

| | |
|---|---|
| $\mathscr{F}$ | Fourier Transformation |
| $*$ | Convolution |
| $\omega$ | angular rate of orbiting objects |
| $\overline{u}$ | Applied Fourier Transformation |
| $a$ | propagation speed |
| $L$ | Boundary in the x-direction |
| $M$ | Boundary in the y-direction |
| $R$ | angular rate of orbiting objects |
| $S_0$ | Magnitude of the point source |
| $t$ | time |
| $x_0$ | initial x position for the center of the orbit |
| $X_n$ | Eigenfunction |
| $y_0$ | initial y position for the center of the orbit |
| $Y_m$ | Eigenfunction |

# Introduction

Gravitational waves have been theorized for over a hundred years, but their first direct observation was only made within the last ten years. This is because they are based on the still evolving field of quantum mechanics and require incredibly precise instruments. Gravitational wave are ripples in space time and are caused by large masses orbiting each other such as merging black holes or binary star systems[1]. Here a model of gravitational wave is made with the wave equation. The orbiting mass bodies are point sources and a velocity is added to the center of rotation to facilitate the Doppler Effect.

# Analysis

The Generalized Wave equation is given by

$$\nabla^2 u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

We will assume that the mass of the bodies is equal so that they will orbit with the same radius about their center of mass. We will approximate the movement of the bodies in a circular orbit.To represent a large mass object and the above assumptions into the wave equation a source term is added in the form

$$S_0\ \delta[x + R\cos(\omega t) - x_0]\ \delta[y - R\cos(\omega t) - y_0] \tag{2}$$

This equation will create a constant source that will rotate about the center $x_0$,$y_0$ with a radius of R. The other mass will then have a shift of $\pi$. The resulting wave equation then becomes

$$\nabla^2 u + S_0\ \delta[x + R\cos(\omega t) - x_0]\ \delta[y - R\cos(\omega t) - y_0] = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \tag{3}$$

In order to solve this equation boundary conditions and initial condition values are needed. We will assume that the objects are separated from other sources of gravitational influences or sources resulting in D-D boundary conditions. Our initial conditions we will assume are zero as well. Gravitational waves propagate indefinitely, so the boundaries would extend out to infinity[1]. But since we need to be able to plot our results, definite boundaries are needed. These will be represented by $L$ and $M$ as shown in Fig. 1.

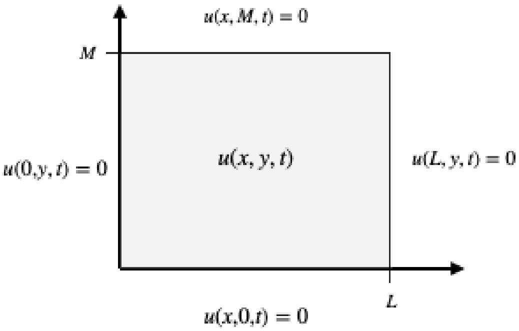


Figure 1: Boundary conditions Dirichlet, Dirichlet in both x and y boundaries.

The equation can now be solved by Finite Fourier Transformations and Laplace transformations[2], in which the solution should take on the form

$$u(x, y, t) = \sum_{n}^{\infty}\sum_{m}^{\infty} \overline{u}_{nm} \frac{X_n}{||X_n||^2}\frac{Y_m}{||Y_m||^2} \tag{4}$$

| | |
|---|---|
| Boundary Condition | $\mathrm{D}u(0) = f_0$<br>$\mathrm{D}u(L) = f_L$ |
| Eigenvalues | $\lambda_n = \frac{n\pi}{L}$ |
| Eigenfunctions | $\sin(\lambda_n x)$ |
| Norm | $\frac{L}{2}$ |
| Operational Property | $-\lambda_n^2 \overline{u}_n + f_0 X'(0)$<br>$-f_L X'(L)$ |

Solving the supplemental eigenvalue problem for either the x or y finite Fourier Transform will result in the properties listed in the table above[2].

Using Fourier transformation on the generalized wave equation, the second derivative with respect to x and y will result in

$$\mathscr{F}\{\nabla^2 u\} = \quad -\lambda_n^2 \overline{u}_{nm} \quad - \mu_m^2 \overline{u}_{nm} \tag{5}$$

Transformations of the source terms are

$$\begin{aligned}&\mathscr{F}\{\delta[x + R\cos(\omega t) - x_0]\}\\&= X_n\left(-R\cos(\omega t) + x_0\right)\end{aligned} \tag{6}$$

$$\begin{aligned}&\mathscr{F}\{\delta[y - R\sin(\omega t) + y_0]\}\\&= Y_m\left(R\sin(\omega t) - y_0\right)\end{aligned} \tag{7}$$

Using Fourier transformation in the x and y, results in a Ordinary Differential equation. This can be solved using the Laplace transformation. The solution of the ordinary differential equation for a single source becomes

$$\begin{aligned}&\overline{u}_{nm}\\&\quad = S_0\left[(\lambda_n^2 + \mu_m^2)a^2 \sin((\lambda_n^2 + \mu_m^2)t)\right]\\&\quad * \left[X_n\left(-R\cos(\omega t) + x_0\right)\right.\\&\quad \left.Y_m\left(R\sin(\omega t) - y_0\right)\right]\end{aligned} \tag{8}$$

# Conclusions

As a control, the linear velocity of the center of orbit was set to zero and the results can be seen in Fig. 2. As expected we have the large masses orbiting at the center with the waves spiraling outward uniformly.

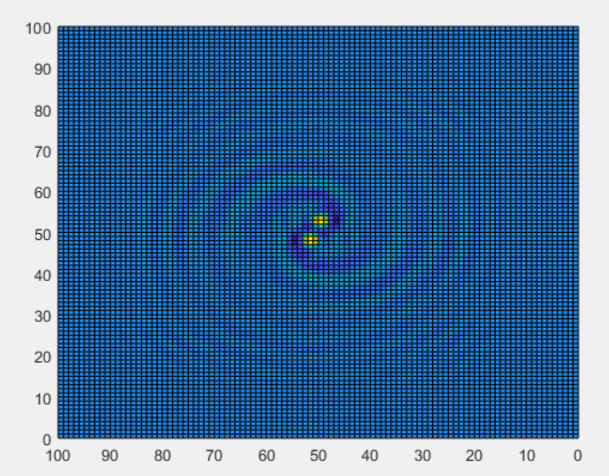


Figure 2: Two dimensional surface of the gravitational waves.

As the center of rotation begins to translate in the x-direction the waves begin to change shape. The largest change is in the amplitudes of the waves. As seen in Fig. 3 the waves in front of the orbiting masses are much larger while the trailing wave almost completely die out. This is likely due to the fact that the linear velocity of the masses due to angular velocity is combine with or subtract from the linear velocity of the entire system.

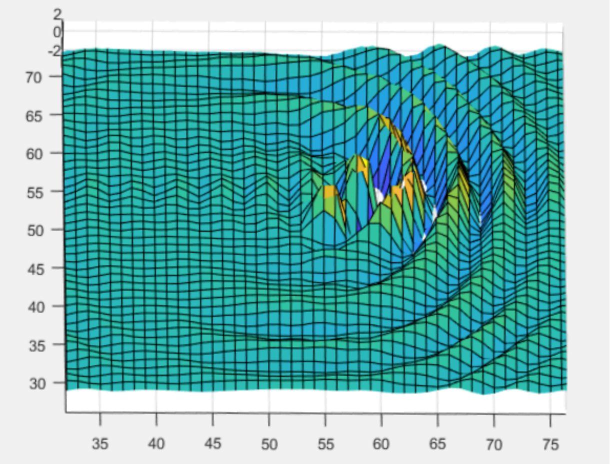


Figure 3: Two dimensional surface of the gravitational waves.

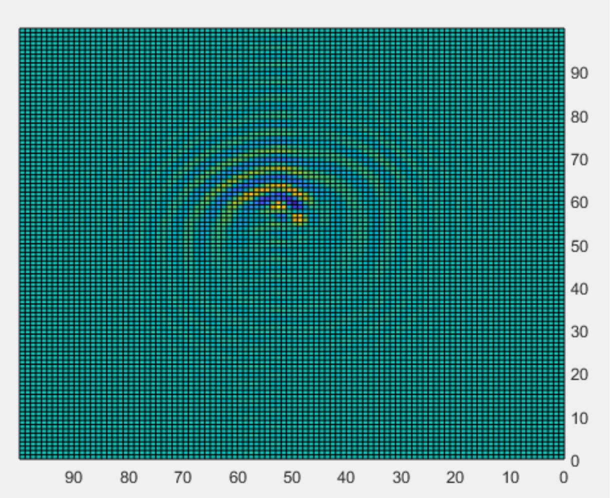


Figure 4: Top view of the two dimensional surface of the gravitational waves.

Comparing Fig. 2 and Fig. 4, the more traditional Doppler Effect is evident. the wave length of the stationary orbiting masses is longer than the translating orbiting masses.

# Future work

Other models to consider that are similar to this problem statement would be considering other orbiting paths of binaries systems. Many systems follow an elliptical path instead of a circular path. This will likely change the wave structure produced. Other things to include would be to analyze the non-dimensional plot. This would give a further analysis into the frequency of the system.

# Acknowledgements

A special thanks to Dr. Soloviev his help formulating the problem and for teaching our ME EN 505 Applied Engineering Mathematics class[2].

# References

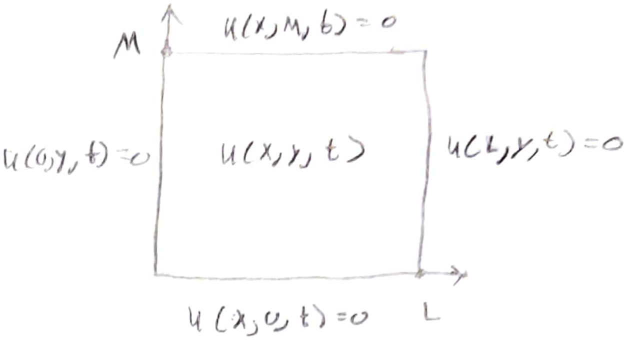
[1] C R Kitchin. *Understanding Gravitational Waves*. ISBN: 9783030742065.

[2] Vladimir Solovjov. *Integrated Engineering Mathematics*. 2021.

# Appendix

See attached pdf scans of hand worked math.

$$\frac{\partial^2 u}{dx^2} + \frac{d^2 u}{dy^2} + S_0\,\delta[X + R\cos(wt) - X_0]\,\delta[y - R\sin(wt) - y_0] + S_0\,\delta[x + R\cos(w(t - \frac{\pi}{w})) - x_0] \cdots = \frac{1}{a^2}\frac{d^2 u}{dt^2}$$



1) FFT in y

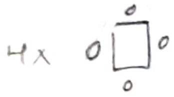
$$u(x,y) = \sum_{n=1}^{\infty} \bar{u}_n(x) K_n(y)$$

$$\bar{u}_n(x) = \int_0^M u(x,y) K_n(y)\,dy$$

I. C. $u(x, y, 0) = 0$

$$\frac{d}{dt} u(x, y, 0) = 0$$

(LE) $\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = 0$



$u_1 = \sum a_n X_n Y_n$

$$\bar{U}_n = \int_0^L u(x) X_n(x)\,dx$$

$$u(X) = \sum_n^{\infty} \bar{U}_n \frac{X_n(x)}{\|X_n(x)\|^2}$$

| Boundary Conditions | Eigenvalues | Eigenfunctions | Norm | Operational property |
|---|---|---|---|---|
| $u(0) = 0$<br>$u(L) = 0$ | $\lambda_0 = \frac{\pi}{L}$<br>$\lambda_n = \frac{n\pi}{L}$ $n = 1, 2, 3$ | $X_0 =$<br>$X_n = \sin(\lambda_n x)$ | $\frac{L}{2}$ | $-\lambda_n^2 \bar{u}_n + S_0 X_n'(0) - S_L X_n'(L)$<br>$X_n'(0) = \lambda_n$<br>$X_n'(L) = \lambda_n \cos(\lambda_n L)$ |
| $u(0) = 0$<br>$u(M) = 0$ | $\mu_0 = \frac{\pi}{M}$<br>$\mu_m = m\frac{\pi}{M}$ $n = 1, 2, 3$ | $Y_0 =$<br>$Y_m = \sin(\mu_m x)$ | $\frac{M}{2}$ | $-\mu_m^2 \bar{u}_m + S_0 X_m'(0) - S_m X_m'(M)$<br>$X_m'(0) = \mu_m$<br>$X_n'(M) = \mu_m \cos(\mu_m M)$ |

$$\mathcal{F}_x\{ \cdot\cdot\, \delta[x + R\cos(wt) - x_0] \} = X_n(-R\cos(wt) + x_0)$$

$$\mathcal{F}_y\{ \delta[y - (R\sin(wt) + y_0)] \} = Y_m(R\sin(wt) - y_0)$$

$$\mathcal{F}_x\{ \delta[x + R\cos(w(t - \frac{\pi}{w})) - x_0] = X_n(-R\cos(w(t - \frac{\pi}{w})) + x_0)$$

$$\mathcal{F}_y\{ \delta[y - R\sin(w(t - \frac{\pi}{w})) - y_0] = Y_m(R\sin(w(t - \frac{\pi}{w})) - y_0)$$

$$\mathcal{F}_x\left\{ \frac{d^2 u}{dx^2} \right\} = -\lambda_n^2 \bar{u}_n$$

$$\mathcal{F}_y\left\{ \frac{d^2 u}{dy^2} \right\} = -\mu_m^2 \bar{u}_m$$

Apply Fourier transformation in x

$$-\lambda_n^2 \bar{u}_n + \frac{d^2\bar{u}_n}{dy^2} + S_0 X_n(-R\cos(wt)+x_0)\,\delta[y-R\sin(wt)-y_0] + S_0 X_n(-R\cos(w(t-\tfrac{\pi}{w}))+x_0)\cdot\delta[y-R\sin(w(t-\tfrac{\pi}{w}))-y_0] = \frac{1}{a^2}\frac{d^2\bar{u}_n}{dt^2}$$

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Apply Fourier transformation in y

$$-\lambda_n^2 \bar{u}_{nm} - \mu_m^2 \bar{u}_{nm} + S_0 X_n(-R\cos(wt)+x_0)\,Y_m(R\sin(wt)-y_0) + S_0 X_n(-R\cos(w(t-\tfrac{\pi}{w}))+x_0)\cdot Y_m(R\sin(w(t-\tfrac{\pi}{w}))-y_0) = \frac{1}{a^2}\frac{d^2\bar{u}_{nm}}{dt^2}$$

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Apply the Laplace transformation

$$-\lambda_n^2 \bar{u}_{nm} - \mu_m^2 \bar{u}_{nm} + S_0 \mathcal{L}\{X_n(-R\cos(wt)+x_0)\,Y_m(R\sin(wt)-y_0)\} + S_0 \mathcal{L}\{X_n(-R\cos(w(t-\tfrac{\pi}{w}))+x_0)\cdot Y_m(R\sin(w(t-\tfrac{\pi}{w}))-y_0)\} = \frac{1}{a^2}\,S^2\hat{u}_{nm} - S\overset{0}{\cancel{f(0)}} - \overset{0}{f'(0)}$$

$$S_0 \mathcal{L}\{X_n(-R\cos(wt)+x_0)\,Y_m(R\sin(wt)-y_0)\} + S_0 \mathcal{L}\{X_n(-R\cos(w(t-\tfrac{\pi}{w}))+x_0)\,Y_m(R\sin(w(t-\tfrac{\pi}{w}))-y_0)\} \cdot \frac{a^2}{(S^2+\lambda_n^2+\mu_n^2)} = \hat{u}_{nm}$$

$$\bar{u}_{nm} = S_0\left[(\lambda_n^2+\mu_n^2)\cdot a^2\cdot \sin((\lambda_n^2+\mu_n^2)t)\right] * \left[X_n(-R\cos(wt)+x_0)\,Y_m(R\sin(wt)-y_0)\right] + S_0\left[(\lambda_n^2+\mu_n^2)\cdot a^2\cdot \sin((\lambda_n^2+\mu_n^2)t)\right] * \left[X_n(-R\cos(w(t-\tfrac{\pi}{w}))+x_0)\,Y_m(R\sin(w(t-\tfrac{\pi}{w}))-y_0)\right]$$

$$u(x,y,t) = \frac{2}{L}\frac{2}{M}\sum_{n=1}\sum_{n=1} \bar{u}_{nm}\, X_n\, Y_m$$