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SIMPLIFIED HEAT TRANSFER MODEL TO TEST ASSUMPTIONS MADE FOR A MODIFIED TRANSIENT HOT WIRE METHOD

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ABSTRACT

A modified transient hot wire sensor known as the Needle Probe, developed to measure the thermal conductivity of molten salts, hinges its results based on the assumption that by limiting the amount of space a fluid has to move, convective heat transfer effects become negligible. This assumption was tested by modeling a simplified version of the Needle Probe system and varying the coefficient of convective heat transfer at small amounts. The results for a range of small heat transfer coefficients were compared. It has been deemed that the assumption that small convection coefficients can be ignored is valid.

NOMENCLATURE

$\boldsymbol{r_1} = \boldsymbol{radius\ from\ center\ of\ probe\ to\ heating\ wire}$
$\boldsymbol{r_2} = outter\ radius\ of\ needle\ probe$
$\boldsymbol{\alpha} = thermal\ diffusivity$
$\boldsymbol{\lambda} = eigenvalue$
$\boldsymbol{k} = thermal\ conductivity$
$\boldsymbol{q''} = heat\ flux$
$\boldsymbol{h} = convection\ coefficient$
$\boldsymbol{H} = h/k$

INTRODUCTION

In recent years, there has been growing interest in the use of molten salts for power generation, both in Concentrated Solar Power (CSP) and in Molten Salt Reactors (MSR), a fourth gen nuclear reactor. Both designs use molten salts as the working fluid, which means in order to make safe and effective designs, we need a thorough understanding of the thermophysical properties of molten salts. Due to the fact that molten salts must be at high temperatures (500-800°C), are electrically conductive, and are highly corrosive, some of these properties, such as thermal conductivity and thermal diffusivity, are notoriously hard to measure. Very few salts have any measurements, and those that do tend to be inconsistent between studies. In their literature review, Magnusson et al highlights this problem with the measurements of thermal conductivity found in the literature for the salt FLiNaK (a Lithium-Fluoride, Sodium-Fluoride, Potassium Fluoride mix), as seen in figure 1. It is commonly believed there is such a disparity due to a lack of radiative and convective heat transfer effects in the different researchers' models.

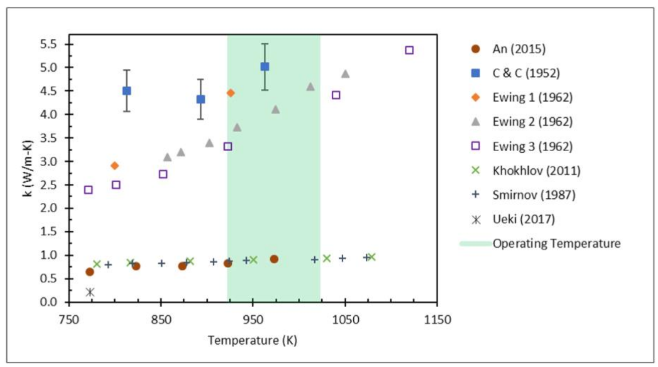

Figure 1: Graph of reported thermal conductivity values of FLiNaK, as reported by Magnusson et al.

Recently, Merritt et al designed a new sensor based on the transient hot wire method designed to measure thermal conductivity of molten salts, which they refer to as "the Needle Probe." This probe is submerged in molten salts in a steady state environment, and then begins to heat the salt while recording temperature response. By fitting the data to a solved heat diffusion equation for a 3-layer system, thermal conductivity of the salt can be found.

The model used is very complex, involving what is known as the thermal quadrupole technique (see Maillet et al for more details). The model also assumes that convective heat transfer between the probe and the salt is zero by use of what is known as the concentric cylinders technique, which greatly limits the amount that the salt can flow. While the flow is greatly limited, it is hard to imagine that there is no flow at all, but rather that it is low enough that all convective heat transfer effects are irrelevant. This study aims to verify that assumption.

A simplified model of the Needle Probe system was developed, looking only at conduction through the probe and the effects that convection at the boundary has on the temperature response.

METHODS

The basic concept of the Needle Probe is shown below in figure 2. The cylindrical probe is submersed in salts while heating wires down the length of the probe adds a transient response. A thermocouple measures the temperature response at the wires.

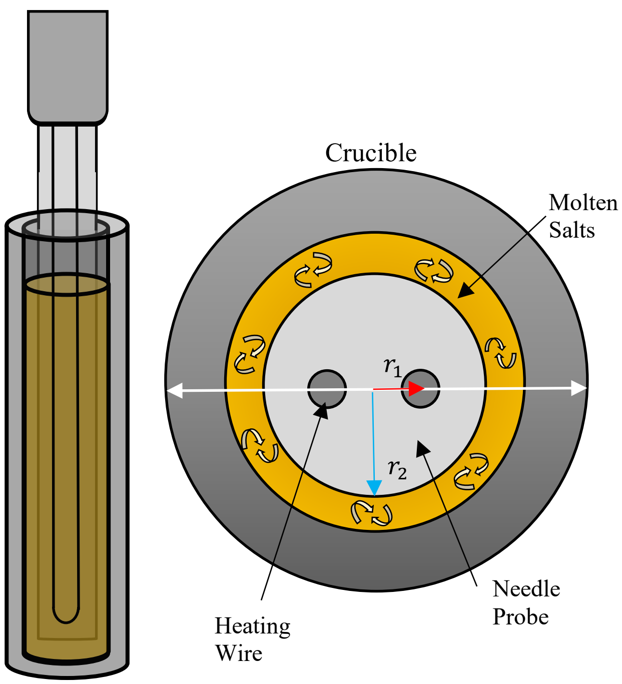

Figure 2. Model of Crucible, Needle Probe, and Molten Salts

To begin, the differential equation for a temperature distribution was considered. Because the crucible, salts, and probe are of annular nature (cylindrical coordinates most appropriate), this solution path was considered.

$$\frac{\partial^2 \boldsymbol{T}}{\partial \boldsymbol{r}^2} + \frac{\mathbf{1}}{\boldsymbol{r}}\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{r}} = \frac{\mathbf{1}}{\boldsymbol{\alpha}}\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{t}} \quad [1]$$

Initial values and boundary conditions were key to defining the solution path and were chosen as follows:

Initial value and Boundary conditions:

$$T(r,0) = 0$$

$$\frac{\partial T}{\partial r}(r,t) = -\frac{q''}{k}\ Neumann$$

$$k\frac{\partial T}{\partial r}(r_2) + T(r_2)h = 0\ Robin$$

Key Assumptions in this model were that:

1. The probe is an infinitely long cylinder
2. System is axisymmetric
3. Materials are homogeneous and isentropic (thermal conductivity does not vary within material)

With the ordinary differential heat conduction equation in mind, the needle probe and system were modeled with a steady state and transient solution. Those solutions are as derived below.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0$$

$$T_{ss} = c_1 + c_2 \ln(r) \quad [2]$$

Solving for the constants yielded for c2:

$$\boldsymbol{c_2} = -\frac{\boldsymbol{r_1 q''}}{\boldsymbol{k}}$$

With this first constant represented symbolically, the second constant could be solved by substituting in the solution such that,

$$k\frac{dT}{dr}(r_2) + T(r_2)h = 0$$

$$[\dots]$$

$$\boldsymbol{c_1} = \frac{\boldsymbol{q'' r_1}}{\boldsymbol{h r_2}} + \frac{\boldsymbol{r_1 q''}}{\boldsymbol{k}}\boldsymbol{ln}(\boldsymbol{r_2})$$

With constants derived, substituting c1 and c2 into the steady state temperature equation yields

$$T_{ss} = \frac{q'' r_1}{h r_2} + \frac{r_1 q''}{k} ln(r_2) - \frac{r_1 q''}{k}\ln(r_2) \quad [3]$$

Next the transient solution was to be derived. This would represent the solution path of the temperature that is changing over time.

Beginning with equation 1, the transient solution is derived with a combination of separation of variables and the Strum-Liouville Theorem.

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{\alpha}\frac{T'}{T} = \mu \quad [4]$$

To solve the constants in this SLP problem, a few key substitutions were used in combination with a linear algebra approach, using the determinate to simplify the solution path.

The solution path was greatly simplified and followed by referencing an annular-disk D-R approach, solved by Laura Hansen on page 526 of the course textbook. In general, this solution was:

$$R'(r_1) = -c_1\lambda J_1(\lambda r_1) - c_2\lambda Y_1(\lambda r_1) = 0$$

This would yield the following:

$$a_{11} = \lambda J_1(\lambda r_1)$$
$$a_{12} = \lambda Y_1(\lambda r_1)$$
$$a_{21} = -\lambda J_1(\lambda r_2) + HJ_0(\lambda r_2)$$
$$a_{22} = -\lambda Y_1(\lambda r_2) + HY_0(\lambda r_2)$$

Such that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 0$$

Positive roots at λ for this equation are the eigenvalues and the constants were solved to be

$$c_1 = \frac{1}{a_{21}} \quad and \quad c_2 = -\frac{1}{a_{22}}$$

With eigenfunction

$$R_N = \frac{J_0(\lambda_n r)}{a_{21n}} - \frac{Y_0(\lambda_n r)}{a_{22n}} \qquad [5]$$

Applying the SLP solution further, yields the Temperature function solution with respect to time.

$$T' - \alpha\mu T = 0 \;\; where \; \mu = -\lambda^2$$
$$T' + \alpha\lambda_n^2 T = 0$$
$$T_n(t) = e^{-\alpha\lambda_n^2 t}$$

Combining the Steady state and Transient Solutions then yielded

$$u(r,t) = U(r,t) + u_{ss}(r,t) \qquad [6]$$

$$\boldsymbol{u(r,t) = \sum a_n R_n T_n + \frac{q'' r_1}{h r_2} + \frac{r_1 q''}{k} ln(r_2)} \qquad [7]$$
$$\boldsymbol{-\frac{r_1 q''}{k}\ln(r_2)}$$

Where

$$a_n = \frac{\int_{r_1}^{r_2} -\frac{q''}{h}\left(\frac{J_0(\lambda_n r)}{a_{21n}} - \frac{Y_0(\lambda_n r)}{a_{22n}}\right) r dr}{\|R_n\|^2}$$

The complete derivation can be referenced in the appendix, but it is sufficient to say that this result was then used to solve eigenvalues and plot the solution of the temperature distribution with respect to radius, and with respect to time.

RESULTS

As the Needle Probe only takes measurements at the inner radius of the system for the first 15 to 30 seconds of the transient response, only the temperature response for the first 30 seconds at r_1 was analyzed. The response was ran for three different, relatively low convection values, namely h = 0.01, h = 0.1, and h = 1.0. The comparison of these results are referenced in figure 3.

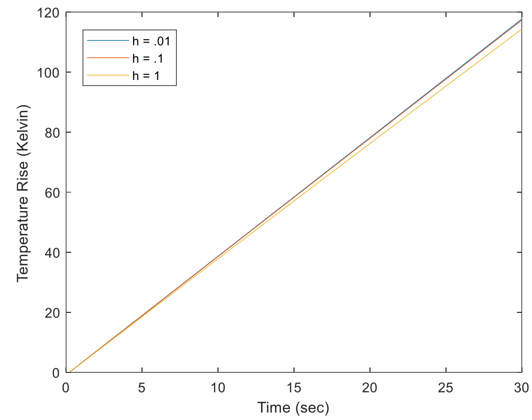

Figure 3: Transient response at location r_1 for t = 0 to 30 seconds for three different h values.

As can be seen in figure 3, for the first 30 seconds of the transient response, the effects of convective heat transfer for the three different values varied only a little. The effects of when h < 1 made very little difference to the overall response, while when h is equal to 1 only a relatively small difference was made.

CONCLUSIONS

This model is a highly simplified model, because of which this response is not accurate to what is actually occurring within the Needle Probe. For instance, it is not taking into effect the transient temperature response that the surrounding salts will have on the probe nor the radiative heat effects. This model does, however, provide a way to see to what extent low values of convective heat transfer has on the response of the probe.

Low values for convective heat flow between the salt and the probe does have little to no effect on the probe's temperature response. This validates the model used by Merritt et al which assumes due to the little motion present in the salt that the effects of convective heat transfer can be ignored.

FUTURE MODELING

Further mathematical iterations, namely extending and adapting the current model to study the conductive heat effects in the salt as a function of radius, can be explored. However, modeling this system as a whole gets increasingly complicated as the system is comprised of three independent layers, each with their own thermal conductivity and diffusivity, and the effects of radiative heat transfer between the probe and the crucible and the convective heat transfer between the crucible and the environment must all be accounted for. This leads to several individual yet related complex heat diffusion equations that must be solved. For more on this work, refer to the work done by Merritt et al.

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APPENDIX

Complete derivation of Steady State and Transient Temperature profile

$$\boldsymbol{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

With initial conditions:

$$T(r,0) = 0$$

$$\frac{\partial T}{\partial r}(r,t) = -\frac{q''}{k}\ Neumann$$

$$k\frac{\partial T}{\partial r_2} + T(r_2)h = 0\ Robin$$

Steady State Solution:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0$$

$$\boldsymbol{T_{ss} = c_1 + c_2 \ln{(r)}}$$

$$\frac{dT}{dr} = c_2(\frac{1}{r})$$

$$\frac{dT}{dr}(r_1) = -\frac{q''}{k} = c_2\left(\frac{1}{r}\right)\ where\ c_2 = -\frac{r_1 q''}{k}$$

$$k\frac{dT}{dr}(r_2) + T(r_2)h = 0$$

$$\boldsymbol{k\left(-\frac{r_1 q''}{k}\left(\frac{1}{r_2}\right)\right) + (c_1 + c_2\ln(r_2))h = 0}$$

$$-\boldsymbol{\frac{r_1 q''}{r_2}} + \boldsymbol{c_1} h - \frac{r_1 q''}{k}\ln(r_2)\,h$$

$$c_1 h - \frac{r_1 q''}{k} h ln(r_2) = \boldsymbol{\frac{r_1 q''}{r_2}}$$

$$c_1 h = \frac{q'' r_1}{r_2} + \frac{r_1 q''}{k} ln(r_2) h$$

$$c_1 = \frac{q'' r_1}{h r_2} + \frac{r_1 q''}{k} ln(r_2)$$

$$T_{ss} = \frac{q'' r_1}{h r_2} + \frac{r_1 q''}{k} ln(r_2) - \frac{r_1 q''}{k}\ln(r_2)$$

$$such\ that\ T_{ss} = \frac{q'' r_1}{h r_2} + \frac{r_1 q''}{k} ln(r_2) - \frac{r_1 q''}{k}\ln(r_2)$$

Transient Solution:

$$\boldsymbol{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{\alpha}\frac{T'}{T} = \mu$$

$$R'(r_1) = 0$$

$$kR'(r_2) + hR(r_2) = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \mu$$

$$r^2 R'' + rR' - r^2 R\mu = 0$$

Note that $\lambda^2 = -\mu$

$$r^2 R'' + rR' + [r^2\lambda^2 - 0]R = 0$$

$$where\ v = 0$$

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_o(\lambda r)$$

$$R'(r) = -c_1\lambda J_1(\lambda r) - c_2\lambda Y_1(\lambda r_1)$$

$$R'(r_1) = -c_1\lambda J_1(\lambda r_1) - c_2\lambda Y_1(\lambda r_1) = 0$$

$$c_1\lambda J_1(\lambda r) + c_2\lambda Y_1(\lambda r_1) = 0$$

$$-c_1\lambda k J_1(\lambda r_2) - c_2 k\lambda Y_1(\lambda r_2) + c_1 h J_0(\lambda r_2) + c_2 h Y_o(\lambda r_2) = 0$$

$$c_1[\lambda J_1(\lambda r_2) + H J_o(\lambda r_2)] + c_2[\lambda Y_1(\lambda r_2) + H Y_0(\lambda r_2)] = 0$$

$$c_1\lambda J_1(\lambda r_1) + c_2\lambda Y_1(\lambda r_1) = 0$$

$$a_{11} = \lambda J_1(\lambda r_1)$$

$$a_{12} = \lambda Y_1(\lambda r_1)$$

$$a_{21} = -\lambda J_1(\lambda r_2) + H J_0(\lambda r_2)$$

$$a_{22} = -\lambda Y_1(\lambda r_2) + H Y_0(\lambda r_2)$$

$$H = \frac{h}{k}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 0$$

$$a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = 0$$

$$[\lambda J_1(\lambda r_1) \cdot -\lambda Y_1(\lambda r_2) + HY_0(\lambda r_2)] - [\lambda Y_1(\lambda r_1) \cdot -\lambda J_1(\lambda r_2) + HJ_0(\lambda r_2)] = 0$$

Positive roots at λ for this equation are the eigenvalues

$$a_{21}c_1 + a_{22}c_2 = 0$$

$$c_1 = \frac{1}{a_{21}} \quad such\ that\ c_2 = -\frac{1}{a_{22}}$$

With eigenfunction

$$R_N = \frac{J_0(\lambda_n r)}{a_{21_n}} - \frac{Y_0(\lambda_n r)}{a_{22_n}}$$

$$Norm = \ \|R_n\|^2 = \int_{r_1}^{r_2} rR_n^2 dr$$

$$T' - \alpha\mu T = 0 \ \ where\ \mu = \ -\lambda^2$$

$$T' + \alpha\lambda_n^2 T = 0$$

$$T_n(t) = e^{-\alpha\lambda_n^2 t}$$

Combining the Steady state and Transient Solutions

$$U(r,t) = R_n T_n = \sum a_n R_n T_n$$

$$U(r,0) = u_o - u_s = 0 - \frac{q''}{h} = -\frac{q''}{h}$$

$$a_n = \frac{\int_{r_1}^{r_2} -\frac{q''}{h} R_n r dr}{\|R_n\|^2} = \frac{\int_{r_1}^{r_2} -\frac{q''}{h} R_n r dr}{\int_{r_1}^{r_2} rR_n^2 dr}$$

$$u(r,t) = U(r,t) + u_{ss}(r,t)$$

$$u(r,t) = \sum a_n R_n T_n + \frac{q'' r_1}{hr_2} + \frac{r_1 q''}{k} ln(r_2) - \frac{r_1 q''}{k} \ln(r_2)$$

$$a_n = \frac{\int_{r_1}^{r_2} -\frac{q''}{h}\left(\frac{J_0(\lambda_n r)}{a_{21_n}} - \frac{Y_0(\lambda_n r)}{a_{22_n}}\right) rdr}{\|R_n\|^2} = \frac{\int_{r_1}^{r_2} -\frac{q''}{h}\left(\frac{J_0(\lambda_n r)}{a_{21_n}} - \frac{Y_0(\lambda_n r)}{a_{22_n}}\right) rdr}{\int_{r_1}^{r_2} r\left(\frac{J_0(\lambda_n r)}{a_{21_n}} - \frac{Y_0(\lambda_n r)}{a_{22_n}}\right)^2 dr}$$

