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A THERMAL ANALYSIS OF RAINWATER COLLECTION TANKS

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ABSTRACT

The transient and steady-state temperature distribution is given for a rainwater collection reservoir exposed to cold quiescent and moving air. The air is maintained at a constant temperature at the freezing temperature of water. The reservoir is approximated as a cylinder composed entirely of water, with the material bounding the reservoir being neglected. Convective heat transfer between the surface of the cylinder and the ambient air results in heat loss from the cylinder. The results are investigated at various spans of time, in which the temperature distribution is a function of both radial position and time. The time to reach the critical freezing temperature is investigated. Natural convection is found to result in very slow cooling of the water tank. Experimental results would likely see much more rapid cooling of the water due to combined methods of heat transfer. The time for the temperature to drop to the steady state value was as much as 200 hours for free convection. Forced convection would decrease this time to 30 hours.

NOMENCLATURE

$r - radial\ variable\ (meters)$
$t - time\ variable\ (seconds)$
$u(r,t)$
$- temperature\ (Kelvin), a\ function\ of\ both\ r\ and\ t$
$a - Thermal\ diffusivity\ of\ water\ (m^2/s)$
$h - heat\ transfer\ coefficient\ (W/m^2K)$
$k - thermal\ conductivity\ of\ water\ (W/mK)$
$u_0 - initial\ temperature\ distribution\ at\ t = 0$
$u_\infty - ambient\ air\ temperature, constant$
$\lambda_n - eigenvalues\ for\ this\ problem$
$R_n - eigenfunctions\ for\ this\ problem$

INTRODUCTION

Rainwater collection reservoirs are commonly implemented to store precipitation that has fallen over an area for later use. Common applications of these reservoirs include saving water for gardening or for household use in energy efficient homes. Many water reservoirs are cylindrical. The effects of cooling during a sudden change in temperature are investigated for these tanks, when the ambient air temperature results in cooling of the water in the tank. The time for the water to uniformly reach the temperature to freeze is investigated. In theory, freezing occurs when the entirety of the water in the tank reaches the freezing temperature. This study provides practical application to rainwater collection systems in which it is preferable to avoid freezing. The time for the tank to freeze is compared for ambient, still air as well as moving air, such as in a storm with high winds.

METHODS

Approximating the reservoir as a solid cylinder of water, the heat equation, given below, can be solved for both transient and steady-state temperature profiles in the tank.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

With the initial condition,

$$u(r, t = 0) = u_0$$

Indicating a uniform temperature distribution across the entire cylinder at time t=0. The equation is also subject to the type III (Robin) boundary condition,

$$-k\frac{\partial u(r_1)}{\partial r} + h(u_\infty - u(r_1)) = 0$$

By rearranging:

$$k\frac{\partial u(r_1)}{\partial r} + hu(r_1) = hu_\infty$$

Or,

$$\frac{\partial u(r_1)}{\partial r} + Hu(r_1) = Hu_\infty$$

Where,

$$H = \frac{h}{k}$$

The transient solution to the heat equation can be evaluated using the 0 order Hankel transform. For a Robin (III) boundary condition, the equation eigenvalues are found as solutions to the function,

$$-\lambda J_1(\lambda r_1) + HJ_0(\lambda r_1) = 0$$

For which there are n eigenvalues, denoted by λ_n. For each eigenvalue, an eigenfunction exists, denoted by R_n,

$$R_n(r) = J_0(\lambda_n r)$$

For which the norm,

$$\|R_n(r)\|^2 = \|J_0(\lambda_n r)\|^2 = \frac{r_1}{2}(\frac{H^2}{{\lambda_n}^2} + 1)J_0^2(\lambda_n r_1)$$

For this boundary condition, the operational property is given as,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = r_1 J_0(\lambda_n r_1)Hu_\infty - \lambda_n \overline{u_n}$$

So, the original heat equation becomes

$$r_1 J_0(\lambda_n r_1)Hu_\infty - \lambda_n \overline{u_n} = \frac{1}{\alpha}\frac{\partial \overline{u_n}}{\partial t}$$

$$r_1 J_0(\lambda_n r_1)Hu_\infty = \frac{1}{\alpha}\frac{\partial \overline{u_n}}{\partial t} + {\lambda_n}^2 \overline{u_n}$$

$$\alpha r_1 J_0(\lambda_n r_1)Hu_\infty = \frac{\partial \overline{u_n}}{\partial t} + \alpha {\lambda_n}^2 \overline{u_n}$$

Which is of the form

$$B = \frac{\partial \overline{u_n}}{\partial t} + A\overline{u_n}$$

A simple, first order ODE, whose solution is

$$\overline{u_n}(t) = \frac{B}{A} + Ce^{-At}$$

Where C is a constant, found by the initial condition

$$\overline{u_n}(t = 0) = \overline{u_{n,0}}$$

Which, using the 0-order finite Hankel transform, is found as

$$\overline{u_{n,0}} = \int_0^{r1} u_0 J_0(\lambda_n r) r dr$$

Solving for the constant, C, and substituting back in for A and B yields

$$\overline{u_n}(t) = \frac{r_1 J_0(\lambda_n r_1)Hu_\infty}{{\lambda_n}^2}\left(1 - e^{-\alpha {\lambda_n}^2 t}\right) + \overline{u_{n,0}}e^{-\alpha {\lambda_n}^2 t}$$

The inverse Hankel transform can then be applied to yield u(r,t)

$$u(r,t) = \sum_{n=1}^{\infty} \overline{u_n}(t)\frac{J_0(\lambda_n r)}{\|J_0(\lambda_n r)\|^2}$$

Through substitution,

$$u(r,t) = \sum_{n=1}^{\infty} \frac{r_1 J_0(\lambda_n r_1)Hu_\infty}{{\lambda_n}^2}\left(1 - e^{-\alpha {\lambda_n}^2 t}\right) + \overline{u_{n,0}}e^{-\alpha {\lambda_n}^2 t}\frac{J_0(\lambda_n r)}{\frac{r_1}{2}(\frac{H^2}{{\lambda_n}^2} + 1)J_0^2(\lambda_n r_1)}$$

For which

$$u(r, t = 0) = u_0$$

And

$$u(r, t \to \infty) = u_\infty$$

RESULTS AND DISCUSSION

Figures 1 and 2, given in the appendix (to preserve their size and for viewing ease) illustrate the initial and steady state temperature profiles of the tank, with curves shown at 1, 50, 100, and 200 hours for free convection as well as 1, 10, 20, and 30 hours for forced convection. The fluid properties of both water and air were carefully considered. For quiescent air, the heat transfer coefficient typically ranges between 2-25W/m^{2}K, with the minimum value of 2 W/m^{2}K considered here. The other limiting case is for moving air, in which h is maximized at 250 W/m^{2}K [1]. This range of heat transfer coefficient bounds the results for this problem. The minimum time to the critical temperature is found to be 30 hours, with a maximum time of 200 hours.

Initially, the temperature across the entire cylinder is uniform. As t (time) increases, the temperature profile near the cylinder edge becomes more pronounced, as the gradient increases. The Temperature across the cylinder changes slowly, especially if the value of r_1 were increased.

In practice, the temperature of the water in the cylinder would change much more rapidly due to the combined effects of convection, conduction, and possibly radiation heat transfer. This analysis neglects conduction at the cylinder base, which may be significant if not insulated. Additionally, the cylinder is assumed to be long (approaching infinite) in the axial direction, which assumption breaks down as real-world reservoirs are considered. Many have a length-to-radius ratio of about 2-3, which is not significant when approximating the height of the cylinder as infinite.

The significant times to cool the water tank is largely due to the boundary condition imposed on the domain. The water at the outer surface is cooled by convection. The driving force of convective heat transfer is the temperature difference between the surroundings and domain. As the initial temperature difference only measures 17K, the magnitude of convective heat transfer is small. Increasing the initial temperature of the fluid would result in different cooling behavior. However, as the fluid cools to near the temperature of the surroundings, the convective heat transfer diminishes regardless of the initial temperature. The heat flux from the tank to the surroundings approaches 0 as the tank temperature approaches that of the surroundings. This results in the behavior seen in figures 1 and 2, in which the temperature changes rapidly over the first hour, but then takes up to 200 hours in still air to approach the temperature of the surroundings.

CONCLUSIONS

A water tank suddenly exposed to colder temperatures will cool very slowly in natural and forced convection. The rate of cooling is dependent on the material properties, the temperature of the surroundings, the initial temperature of the fluid, as well as the radius of the domain. Fluid exposed to forced convection will cool much more rapidly than fluid exposed to natural (free) convection. In practice, the cooling of a tank of water would occur much more rapidly than predicted here. The model limitations and assumptions introduce error into the results; however, the model proves a useful prediction for ideal heat transfer behavior in a cylindrical tank.

ACKNOWLEDGEMENTS

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REFERENCES

[1] Bergman, T. L., &; Lavine, A. S. (2017). Fundamentals of heat and mass transfer. John Wiley & Sons.

APPENDIX

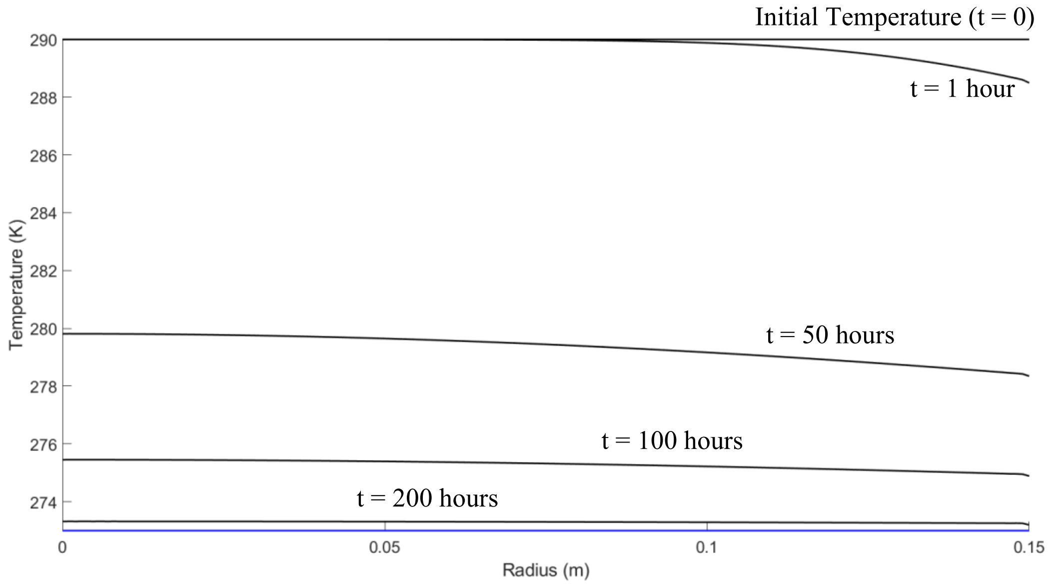

Figure 1. Plot of u(r, t) for t = 1, 50, 100, and 200 hours of cooling in still air. Blue line indicates the ambient air temperature.

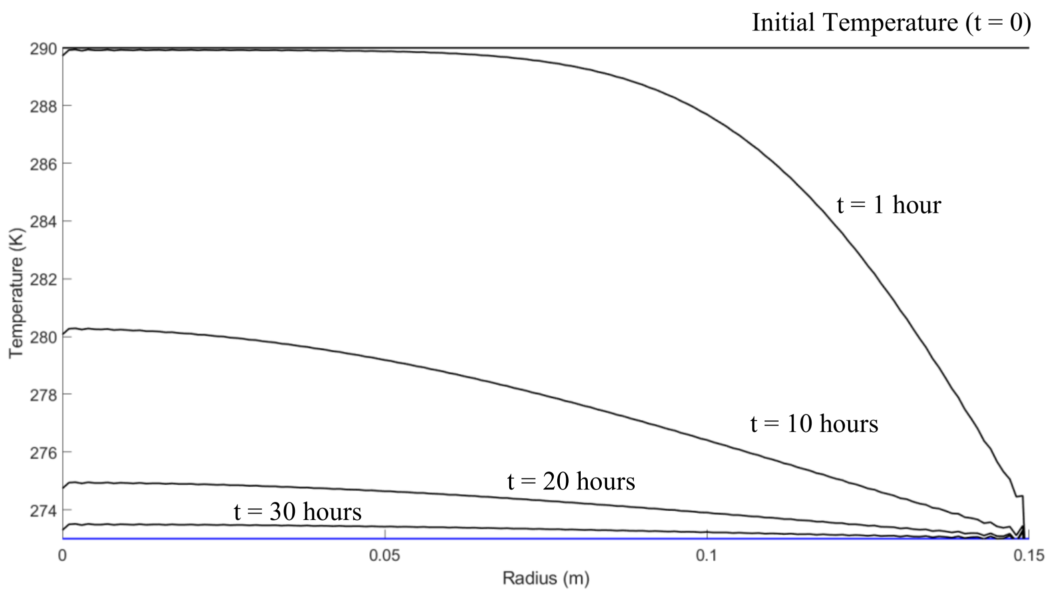

Figure 2. Plot of u(r, t) for t = 1, 10, 20, and 30 hours of cooling in moving air. Blue line indicates the ambient air temperature.

List of variables and values used in calculations

$a = 1.463 \cdot 10^{-7}\ m^2/s$

$h\ (free) = 2\ W/m^2K$

$h\ (forced) = 250\ W/m^2K$

$k = 0.60974\ W/mK$

$u_0 = 290K$

$u_\infty = 273K$