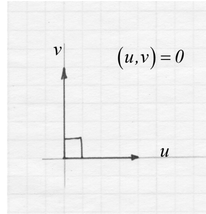
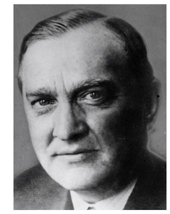
ME-505   FALL 2019   **FINAL EXAM**

1. (2) Banach and Hilbert Spaces.



David Hilbert
(1862 – 1943)



Stefan Banach
(1892 – 1945)

> Consider the Hilbert space $L_2(0,L)$ (Chapter VI, p. 432) with the inner product $(u,v)=\int_0^L u(x)v(x)\,dx$.
>
> If the function $f(x)\in L_2(0,L)$ is orthogonal to all functions $\sin\left(\frac{n\pi}{L}x\right)$ for all $n=1,2,\ldots$,
>
> then what is this function $f(x)$? Justify your answer.

i) **Given:** $f(x)$ is orthogonal to $\sin\left(\frac{n\pi}{L}x\right)$ on $(0,L)$ for all $n=1,2,\ldots$. That means that inner product

$$\left(f(x),\sin\left(\frac{n\pi}{L}x\right)\right)=\int_{x=0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = 0 \quad \text{for all } n=1,2,\ldots$$

ii) If $f(x)\in L_2(0,L)$, then $f(x)$ can be represented by the Generalized Fourier series

$$f(x)=\sum_{k=1} c_k\phi_k(x)\ \text{, where } c_k=(f,\phi_k)=\int_{x=0}^{L} f(x)\phi_k(x)\,dx$$

over any complete set of mutually orthogonal functions $\{\phi_n(x), n=1,2,\ldots\}$:

$$(\phi_m,\phi_n)=\int_{x=0}^{L}\phi_m(x)\phi_n(x)\,dx=0 \qquad \text{if} \quad m\neq n$$

iii) **Facts:** a) $\left\{\sin\left(\frac{n\pi}{L}x\right), n=1,2,\ldots\right\}\in L_2(0,L)$

b) The set $\left\{\sin\left(\frac{n\pi}{L}x\right), n=1,2,\ldots\right\}$ is a complete set of mutually orthogonal functions on $(0,L)$ as a solution of the Sturm-Liouville problem: $y''+\mu^2 y=0$, $y(0)=0$, $y(L)=0$.

iv) Then $f(x)\in L_2(0,L)$ can be represented as the sine Fourier series $f(x)=\sum_{n=1} a_n\sin\left(\frac{n\pi}{L}x\right)$,

where $a_n=\int_{x=0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx=\left(f(x),\sin\left(\frac{n\pi}{L}x\right)\right)=0$ according to i).

Therefore,

$$f(x)=0$$

> **General fact:** Let $\{\phi_n(x), n=1,2,\ldots\}$ be a complete set of mutually orthogonal functions:
>
> $(\phi_m,\phi_n)=\int_{x=0}^{L}\phi_m(x)\phi_n(x)\,dx=0$ if $m\neq n$.
>
> If $f(x)\notin\{\phi_n(x), n=1,2,\ldots\}$ and $(f,\phi_n)=0$ for all $n=1,2,\ldots$ then $f(x)=0$.

Philippe de Champaigne   Portrait of two men

Portrait of René Descartes

after Frans Hals

Portrait of Blaise Pascal

Philippe de Champaigne