2. (5) The Rock – Sturm-Liouville problem and integral transform method (see Outline in Section IX.5, p.862)

a) Consider the following initial boundary value problem for function $u(x,t)$:

$$x\frac{\partial^2 u}{\partial x^2}+w\frac{\partial u}{\partial x}+S(x,t)=\frac{1}{a^2}\frac{\partial u}{\partial t}, \qquad 0\le x<L,\quad t>0$$

$$u(0,t)<\infty \qquad t>0$$

$$u(L,t)=f(t) \qquad t>0$$

$$u(x,0)=u_0(x) \qquad 0\le x\le L$$

b) Analyze the differential operator with respect to variable x. **Use value** $w=2$.

Rewrite it in self-adjoint form and find the weight function.

c) Set the supplemental Sturm-Liouville problem for construction of the integral transform.

Find the general solution of the set Sturm-Liouville problem (hint: Generalized Bessel Equation can be used).

Apply boundary conditions to generate eigenvalues and corresponding eigenfunctions.

Define the weighted inner product.

Define the norm of eigenfunctions.

d) Use the found eigenfunctions to represent the function $h(x)=1-H(x-L/2)$ by the Generalized Fourier series on the interval $0\le x\le L$. Sketch the graph of $h(x)$ and truncated Fourier series (with 20 terms).

e) Define the Finite Integral Transform pair based on found eigenfunctions.

f) Derive the operational property of the defined integral transform: apply transform to operator $Lu\equiv x\frac{\partial^2 u}{\partial x^2}+w\frac{\partial u}{\partial x}$ subject to boundary conditions in part a).

g) Use the defined Finite Integral Transform and the Laplace transform to solve the given i.b.v.p.

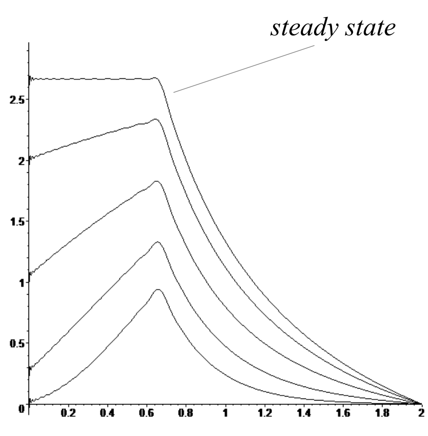
Find the steady state solution.

h) Visualize solution for $L=2,\ w=2,\ S(x)=S_0\delta(x-x_0),\ S_0=4,\ x_0=L/3,\ a=0.5,\ f(t)=0,\ u_0(x)=0$

Sketch the solution curves for different moments of time: $t=0.5,\ t=1,\ t=2,\ t=4$, and the steady state solution.

Solution:

$$u(x,t)=\frac{S_0}{2}\sqrt{\frac{x_0}{x}}\sum_{n=1}\frac{J_1\left(2\mu_n\sqrt{x_0}\right)J_1\left(2\mu_n\sqrt{x}\right)}{\mu_n^2\,J_0^2\left(2\mu_n\sqrt{L}\right)}\left(1-e^{-a^2\mu_n^2 t}\right)$$



b) Operator. Treat $x=0$ as a boundary point.

$$Lu \equiv x\frac{\partial^2 u}{\partial x^2} + w\frac{\partial u}{\partial x}$$

$a_0(x) = x > 0\,,\quad a_1(x) = w\,,\quad a_2(x) = 0$

$$p(x) = \frac{1}{a_0}e^{\int\frac{a_1}{a_0}dx} = \frac{1}{x}e^{\int\frac{w}{x}dx} = \frac{1}{x}e^{\int\frac{w}{x}dx} = \frac{1}{x}x^w = x^{w-1}$$

$q(x) = 0$

$r(x) = a_0(x)p(x) = x\cdot x^{w-1} = x^w$

$$Lu \equiv \frac{1}{x^{w-1}}\frac{\partial}{\partial x}\left(x^w\frac{\partial u}{\partial x}\right)$$ **self-adjoint form**

for $w=2$

$p(x) = x$

$q(x) = 0$

$r(x) = x^2$

$$Lu \equiv \frac{1}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)$$

c) Supplemental Eigenvalue Problem

$Ly = \lambda y\,,\qquad y(0) < \infty,\ y(L) = 0$

$$\frac{1}{x^{w-1}}\left(x^w y'\right)' = \lambda y$$

$\left(x^w y'\right)' + \left(0 - \lambda x^{w-1}\right)y = 0$ ***Sturm-Liouville Problem*** has non-trivial solution only when $\lambda = -\mu^2$

$\left(x^w y'\right)' + \mu^2 x^{w-1} y = 0 \qquad y(0) < \infty,\ y(L) = 0$

$w x^{w-1}y' + x^w y'' + \mu^2 x^{w-1} y = 0$ reduce this equation to the Generalized Bessel Equation:

$$y'' + \frac{w}{x}y' + \frac{\mu^2}{x}y = 0$$ ***Generalized Bessel Equation*** with $\alpha = 0,\ m = \dfrac{1-w}{2}$

$2p - 2 = -1 \quad\Rightarrow\quad p = \frac{1}{2}$

$m^2 - p^2\nu^2 = 0 \quad\Rightarrow\quad \nu^2 = (1-w)^2 \quad\Rightarrow\quad \nu = |w-1| = w-1 \ge 0$

$p^2 a^2 = \mu^2 \quad\Rightarrow\quad a = 2\mu$

General solution:

$$y(x) = x^{\frac{1-w}{2}}\left[c_1 J_{w-1}\left(2\mu x^{1/2}\right) + c_2 Y_{w-1}\left(2\mu x^{1/2}\right)\right]$$ use w=2

$$y(x) = x^{\frac{-1}{2}}\left[c_1 J_1\left(2\mu x^{1/2}\right) + \not{c_2}Y_1\left(2\mu x^{1/2}\right)\right]$$

$$y(x) = \frac{1}{\sqrt{x}}J_1\left(2\mu\sqrt{x}\right)$$ bounded at x=0

B.c. at $x = L$

$$y(L) = \frac{1}{\sqrt{L}}J_1\left(2\mu\sqrt{L}\right) = 0$$

Eigenvalues $\quad J_1\left(2\mu\sqrt{L}\right) = 0 \quad\Rightarrow\quad 0 < \mu_1 < \mu_2 < \ldots$

Eigenfunctions $\quad y_n(x) = \frac{1}{\sqrt{x}}J_1\left(2\mu_n\sqrt{x}\right)$

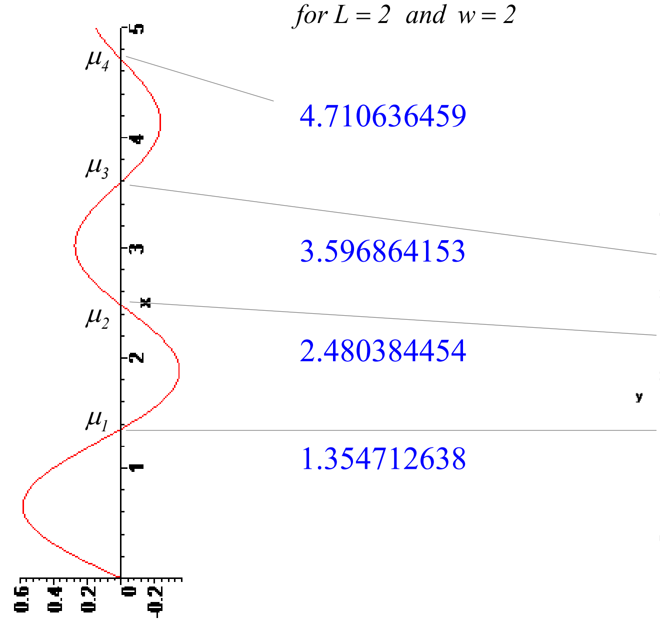
Inner Product $\quad (f,g)_p = \int_0^L f(x)g(x)\overbrace{x^{w-1}}^{p(x)=x}dx$

Norm $\quad \left\| y_n(x)\right\|_p^2 = \int_0^L y_n^2(x)\,x\,dx = \int_0^L \frac{1}{\not{x}}J_1^2\left(2\mu_n\sqrt{x}\right)\not{x}\,dx \quad = 2J_0^2\left(2\mu_n\sqrt{L}\right)$

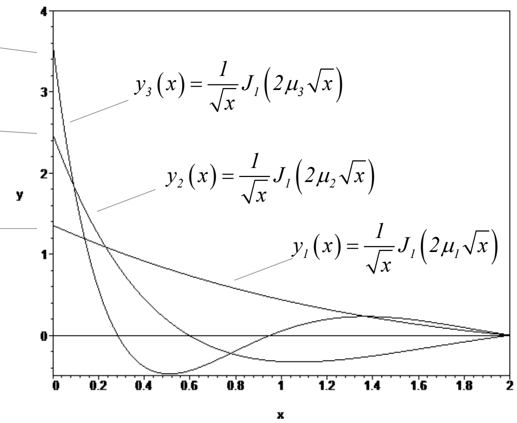
Eigenvalues are the positive roots of equation $J_1\left(2\mu\sqrt{L}\right)=0$.

Eigenfunctions $y_n(x)=\dfrac{1}{\sqrt{x}}J_1\left(2\mu_n\sqrt{x}\right)$

for $L=2$ *and* $w=2$

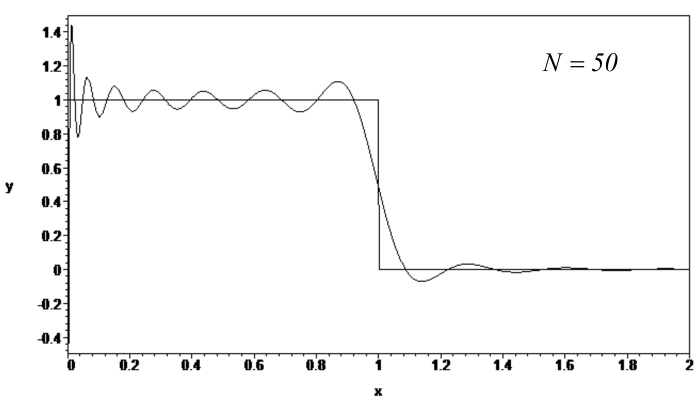


$$\lim_{x\to 0}\frac{1}{\sqrt{x}}J_1\left(2\mu_n\sqrt{x}\right)=\mu_n$$



d) Example. Generalized Fourier series representation of $h(x)=1-H(x-L/2)$

$$h(x)=\sum_{n=1}^{N}c_n\frac{y_n(x)}{\left\|y_n(x)\right\|_n^2},\qquad c_n=\int_0^L h(x)y_n(x)x\,dx$$



e) Define the Finite Integral Transform.

$$\Im\{u\}=\bar{u}_n=\int_0^L u(x)y_n(x)p(x)dx=\int_0^L u(x)\frac{1}{\sqrt{x}}J_1\left(2\mu_n\sqrt{x}\right)x\,dx=\int_0^L u(x)J_1\left(2\mu_n\sqrt{x}\right)\sqrt{x}\,dx$$

$$\Im^{-1}\{\bar{u}_n\}=u(x)=\sum_{n=1}\bar{u}_n\frac{y_n(x)}{\left\|y_n(x)\right\|_n^2}=\sum_{n=1}\bar{u}_n\frac{J_1\left(2\mu_n\sqrt{x}\right)}{2\sqrt{x}\,J_0^2\left(2\mu_n\sqrt{L}\right)}$$

f) Operational Property. Consider self-adjoint form of operator in the given differential equation

$$Lu \equiv \frac{1}{p}(ru')'$$

where functions $p(x)$ and $r(x)$ are found in part (b) following Section IX.5, p.858 (Nov 23,2019).

The boundary conditions for unknown function are given as:

$$u(0,t) < \infty \qquad t > 0$$

$$u(L,t) = f(t) \qquad t > 0$$

Let μ_n and $y_n(x)$ be solutions of the supplemental Sturm-Liouville problem in part (c). That means:

$$\frac{1}{p}(ry_n')' = -\mu_n^2 y_n \qquad (\lozenge)$$

$$y_n(0) < \infty$$

$$y_n(L) = 0$$

In part (e), the finite integral transform is defined as (eqn.(17), p.859):

$$\Im\{u\} = \overline{u}_n = \int_0^L u(x)\, y_n(x)\, p(x)\, dx$$

Then derivation of the operational problem can be performed in general (not using the particular form of y_n):

$$\Im\{Lu\} = \Im\left\{\frac{1}{p}(ru')'\right\} = \int_0^L \frac{1}{p}(ru')'\, y_n(x)\, p(x)\, dx$$

$$= \int_0^L \frac{1}{\not p}(ru')'\, y_n(x)\, \not p(\not x)\, dx \qquad = \int_0^L (ru')'\, y_n(x)\, dx$$

$$= \int_0^L y_n(x)\, d(ru')$$

$$= \overbrace{\left[y_n(x)\, ru'\right]_0^L}^{\text{show that } 0} - \int_0^L ru' y_n'(x)\, dx$$

$$= -\int_0^L r y_n'(x)\, du$$

$$= \left[r y_n'(x) u\right]_0^L - \int_0^L u\left(r y_n'(x)\right)'\, dx$$

$$= \left[r y_n'(x) u\right]_0^L - \int_0^L u \overbrace{\frac{1}{p}\left(r y_n'(x)\right)'}^{\text{use eqn. } \lozenge}\, p\, dx$$

$$= r(L)\, y_n'(L)\, u(L) - \mu_n^2 \int_0^L u y_n p\, dx$$

Boundary conditions:
singular at $x = 0$
Dirichlet at $x = L$

$$= r(L)\, y_n'(L) \overbrace{u(L)}^{f(t)} - \mu_n^2 \overline{u}_n$$

operational property is derived ■

$$\Im\{Lu\} \qquad = L^2 \frac{\mu_n}{2} J_0\left(2\mu_n\sqrt{L}\right) f(t) - \mu_n^2 \overline{u}_n, \qquad \text{where } y_n'(L) = \frac{\mu_n}{2} J_0\left(2\mu_n\sqrt{L}\right) \text{ was used}$$

g) Use the defined Finite Integral Transform and the Laplace transform to solve the given i.b.v.p.

Apply $\Im\{\ \}$

$$-\mu_n^2\bar{u}_n + L^2\frac{\mu_n}{2}J_0\left(2\mu_n\sqrt{L}\right)f(t) + \bar{S}_n(t) = \frac{1}{a^2}\frac{\partial \bar{u}_n}{\partial t}, \qquad \bar{u}_n(0) = \bar{u}_{0,n} = \Im\{u_0(x)\}$$

Apply $Laplace\{\ \}$

$$-\mu_n^2 U_n + L^2\frac{\mu_n}{2}J_0\left(2\mu_n\sqrt{L}\right)\hat{f}(s) + \hat{\bar{S}}_n(s) = \frac{1}{a^2}U_n - \frac{1}{a^2}\bar{u}_{0,n},$$

Solve for $U_n(s)$

$$U_n = \bar{u}_{0,n}\frac{1}{s+a^2\mu_n^2} + a^2\hat{\bar{S}}_n(s)\frac{1}{s+a^2\mu_n^2} + a^2L^2\frac{\mu_n}{2}J_0\left(2\mu_n\sqrt{L}\right)\hat{f}(s)\frac{1}{s+a^2\mu_n^2}, \qquad Laplace\left\{e^{-a^2\mu_n^2 t}\right\} = \frac{1}{s+a\ \mu_n^2}$$

Apply Inverse Laplace transform in terms of convolution

$$\bar{u}_n(t) = \bar{u}_{0,n}e^{-a^2\mu_n^2 t} + a^2\left[\bar{S}_n(t)*e^{-a^2\mu_n^2 t}\right] + a^2L^2\mu_n^2\frac{1}{2}J_0\left(2\mu_n\sqrt{L}\right)\left[f(t)*e^{-a^2\mu_n^2 t}\right]$$

Solution:

$$u(x,t) = \Im^{-1}\{\bar{u}_n(t)\} = \sum_{n=1}\bar{u}_n(t)\frac{y_n(x)}{\|y_n(x)\|_n^2}$$

For the case of $L=2,\ w=2,\ S(x)=S_0\delta(x-x_0),\ S_0=4,\ x_0=L/3,\ a=0.5,\ f(t)=0,\ u_0(x)=0$

$\bar{u}_n(t) = a^2\,\bar{S}_n\cdot\left(1*e^{-a^2\mu_n^2 t}\right)$, where

$$\bar{S}_n = \int_0^L S_0\delta(x-x_0)J_1\left(2\mu_n\sqrt{x}\right)\sqrt{x}\,dx = S_0J_1\left(2\mu_n\sqrt{x_0}\right)\sqrt{x_0}$$

$$\hat{\bar{S}}_n = \frac{\bar{S}_n}{s}$$

$$1*e^{-a^2\mu_n^2 t} = \int_{\tau=0}^{t} e^{-a^2\mu_n^2\tau}d\tau = \frac{1}{a^2\mu_n^2}\left(1-e^{-a^2\mu_n^2 t}\right)$$

$$\bar{u}_n(t) = +\frac{1}{\mu_n^2}S_0J_1\left(2\mu_n\sqrt{x_0}\right)\sqrt{x_0}\left(1-e^{-a^2\mu_n^2 t}\right)$$

$$u(x,t) = \sum_{n=1}\bar{u}_n(t)\frac{J_1\left(2\mu_n\sqrt{x}\right)}{2\sqrt{x}J_0^2\left(2\mu_n\sqrt{L}\right)}$$

Solution

$$u(x,t) = \frac{S_0}{2}\sqrt{\frac{x_0}{x}}\sum_{n=1} \frac{J_1\left(2\mu_n\sqrt{x_0}\right) J_1\left(2\mu_n\sqrt{x}\right)}{\mu_n^2 J_0^2\left(2\mu_n\sqrt{L}\right)}\left(1-e^{-a^2\mu_n^2 t}\right)$$

Steady state solution when $t \to \infty$

$$u(x,t) = \frac{S_0}{2}\sqrt{\frac{x_0}{x}}\sum_{n=1} \frac{J_1\left(2\mu_n\sqrt{x_0}\right) J_1\left(2\mu_n\sqrt{x}\right)}{\mu_n^2 J_0^2\left(2\mu_n\sqrt{L}\right)}$$

Limiting value of steady state at $x = 0$:

$$\lim_{x\to 0} u(x,t) = \frac{S_0}{2}\sqrt{x_0}\sum_{n=1} \frac{J_1\left(2\mu_n\sqrt{x_0}\right)}{\mu_n^2 J_0^2\left(2\mu_n\sqrt{L}\right)} \overbrace{\lim_{x\to\infty}\frac{J_1\left(2\mu_n\sqrt{x}\right)}{\sqrt{x}}}^{\mu_n} = \frac{S_0}{2}\sqrt{x_0}\sum_{n=1} \frac{J_1\left(2\mu_n\sqrt{x_0}\right)}{\mu_n J_0^2\left(2\mu_n\sqrt{L}\right)}$$

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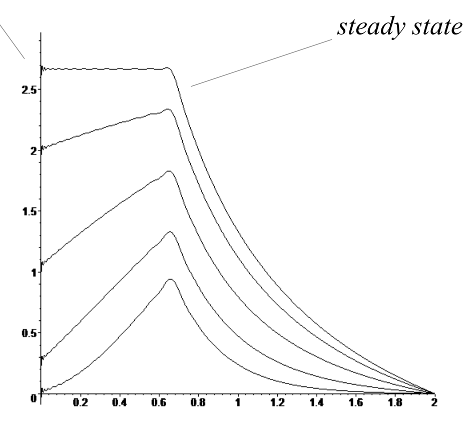

Differentiation of the Bessel Functions using the chain rule: $\frac{d}{dx}f\left[u(x)\right]=\frac{df}{du}\frac{du}{dx} \quad =\frac{df}{du}u'$

$$\frac{d}{dx}J_\nu\left[u(x)\right] \quad =\frac{d}{du}J_\nu\left[u\right]\cdot u' \quad =\left[J_{\nu-1}(u)-\frac{\nu}{u}J_\nu(u)\right]\cdot u' \quad \text{if Eqn.(49), Ch.VII, p.500 is used}$$

$$=\left[-J_{\nu+1}(u)+\frac{\nu}{u}J_\nu(u)\right]\cdot u' \quad \text{if Eqn.(50), Ch.VII, p.500 is used}$$

For example,

$$\frac{d}{dx}J_\nu\left(ax^3\right) \quad =\left[J_{\nu-1}\left(ax^3\right)-\frac{\nu}{ax^3}J_\nu\left(ax^3\right)\right]\cdot\left(ax^3\right)' \quad =\left[J_{\nu-1}\left(ax^3\right)-\frac{\nu}{ax^3}J_\nu\left(ax^3\right)\right]\cdot 3ax^2$$