3. (10) The Running on the Waves. Use integral transform technique to solve the following IBVP:

Governing equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+S_0\delta\left(x-x_0\right)\delta\left(y-a\cdot t\right)=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}+2\gamma^2\frac{\partial u}{\partial t}, \quad 0<x<L,\ 0<y<M,\ t>0$$

Boundary conditions

$x=0$	$-k_1\dfrac{\partial u}{\partial x}\left(0,y,t\right)+h_1 u\left(0,y,t\right)=0$	$0<y<M$,	$t>0$	**R**
$x=L$	$+k_2\dfrac{\partial u}{\partial x}\left(L,y,t\right)+h_2 u\left(L,y,t\right)=0$	$0<y<M$,	$t>0$	**R**
$y=0$	$u\left(x,0,t\right)=f_0\cdot\left(1+\sin\omega t\right)$	$0<x<L$,	$t>0$	**D**
$y=M$	$\dfrac{\partial u\left(x,M,t\right)}{\partial y}=0$	$0<x<L$,	$t>0$	**N**

Initial conditions

$u\left(x,y,0\right)=0$	$0\le x\le L$,	$0\le y\le M$
$\dfrac{\partial}{\partial t}u\left(x,y,0\right)=0$	$0\le x\le L$,	$0\le y\le M$

Solve for: $L=1$, $M=4$, $x_0=0.5$, $k_1=k_2=2$, $h_1=2$, $h_2=3$, $f_0=0.3$, $S_0=0.8$, $\omega=3.0$, $v=0.5$, $\gamma=1.63$

Visualize $u\left(x,y,t\right)$ with the help of the contour plot (use 10 contours) for $t=2$, $t=5$, $t=8$, $t=11$

Use other form of visualization.

Speed of waves v and the source speed a can be equal or different.

The Finite Fourier Transforms corresponding to given differential operators and boundary conditions (IX.3 new):

$\Im_x$

$$\overline{u}_n = \int_0^L u(x) X_n(x)\,dx$$

operational property

R $-k_1 u'(0) + h_1 u(0) = 0$

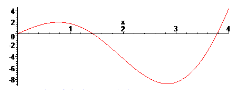
λ_n *are positive roots of*

$-\lambda_n^2 \overline{u}_n$

R $k_2 u'(L) + h_2 u(L) = 0$

$$\left(H_1 H_2 - \lambda^2\right)\sin\lambda L + \left(H_1 + H_2\right)\lambda\cos\lambda L = 0$$

$X_n(0) = \mu_n$



1.423771955

3.778256980

6.654072636

$$X_n = \lambda_n \cos\lambda_n x + H_1 \sin\lambda_n x$$

$$\left\|X_n\right\|^2 = \frac{\left(\lambda_n^2 + H_1^2\right)}{2}\left(L + \frac{H_2}{\lambda_n^2 + H_2^2}\right) + \frac{H_1}{2}$$

$\Im_y$

$$\overline{u}_m = \int_0^M u(y) Y_m(y)\,dy$$

operational property

D $u(0) = f_0$

$$\mu_m = \left(m + \frac{1}{2}\right)\frac{\pi}{M}, \quad n = 0, 1, \ldots \qquad \sin(\mu_m x) \qquad \frac{M}{2}$$

$-\mu_m^2 \overline{u}_m + f_0 Y_m'(0)$

N $u'(M) = 0$

$Y_m(0) = \mu_m$

i) Apply $\Im_x \Im_y$ Note the order of transforms applied to the first two terms!!

$$\Im_y \Im_x \left\{\frac{\partial^2 u}{\partial x^2}\right\} + \Im_x \Im_y \left\{\frac{\partial^2 u}{\partial y^2}\right\} + \Im_y \Im_x \left\{S_0 \delta\left(x - x_0\right)\delta\left(y - a\cdot t\right)\right\} = \Im_y \Im_x \left\{\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + 2\gamma^2 \frac{\partial u}{\partial t}\right\}$$

$$-\lambda_n^2 \overline{\overline{u}}_{n,m} - \mu_m^2 \overline{\overline{u}}_{n,m} + f_0(t)\cdot\mu_m \cdot \Im_x\{1\} + S_0 X_n\left(x_0\right)\sin\left(\mu_m \cdot a \cdot t\right) = \frac{1}{v^2}\frac{\partial^2 \overline{\overline{u}}_{n,m}}{\partial t^2} + 2\gamma^2 \frac{\partial \overline{\overline{u}}_{n,m}}{\partial t}$$

transformed eqn

$$\textit{where}\ \ \Im_x\{1\} = \int_0^L X_n(x)\,dx = \sin\lambda_n L - \frac{H_1}{\lambda_n}\cos\lambda_n L + \frac{H_1}{\lambda_n}$$

ii) Apply Laplace transform $L\{\ \}$

$$-v^2 \lambda_n^2 \hat{\overline{\overline{u}}}_{n,m} - v^2 \mu_m^2 \hat{\overline{\overline{u}}}_{n,m} + v^2 L\left\{f_0(t)\right\}\cdot\mu_m \cdot \Im_x\{1\} + v^2 S_0 X_n\left(x_0\right) L\left\{\sin\left(\mu_m \cdot a \cdot t\right)\right\} = s^2 \hat{\overline{\overline{u}}}_{n,m} + 2v^2\gamma^2 s \hat{\overline{\overline{u}}}_{n,m}$$

$$\left(s^2 + 2v^2\gamma^2 s + v^2\lambda_n^2 + v^2\mu_m^2\right)\hat{\overline{\overline{u}}}_{n,m} = v^2 L\left\{f_0(t)\right\}\cdot\mu_m \cdot \Im_x\{1\} + v^2 S_0 X_n\left(x_0\right) L\left\{\sin\left(\mu_m \cdot a \cdot t\right)\right\}$$

$$\left(\left(s + v^2\gamma^2\right)^2 + \underbrace{v^2\lambda_n^2 + v^2\mu_m^2 - v^4\gamma^4}_{b_{n,m}^2}\right)\hat{\overline{\overline{u}}}_{n,m} = v^2 L\left\{f_0(t)\right\}\cdot\mu_m \cdot \Im_x\{1\} + v^2 S_0 X_n\left(x_0\right) L\left\{\sin\left(\mu_m \cdot a \cdot t\right)\right\}$$

$$\hat{\overline{\overline{u}}}_{n,m} = \frac{1}{b_{n,m}} v^2 \cdot \mu_m \cdot \Im_x\{1\} L\left\{f_0(t)\right\}\frac{b_{n,m}}{\left(s + v^2\gamma^2\right)^2 + b_{n,m}^2} + \frac{1}{b_{n,m}} v^2 S_0 X_n\left(x_0\right) L\left\{\sin\left(\mu_m \cdot a \cdot t\right)\right\}\frac{b_{n,m}}{\left(s + v^2\gamma^2\right)^2 + b_{n,m}^2}$$

Transformed solution :

$$\hat{\overline{\overline{u}}}_{n,m}(s) = \frac{1}{b_{n,m}} v^2 \cdot \mu_m \cdot \Im_x\{1\} L\left\{f_0(t)\right\} L\left\{e^{-v^2\gamma^2 t}\sin\left(b_{n,m} t\right)\right\} + \frac{1}{b_{n,m}} v^2 S_0 X_n\left(x_0\right) L\left\{\sin\left(\mu_m \cdot a \cdot t\right)\right\} L\left\{e^{-v^2\gamma^2 t}\sin\left(b_{n,m} t\right)\right\}$$

iii) Apply inverse Laplace transform in convolution form:

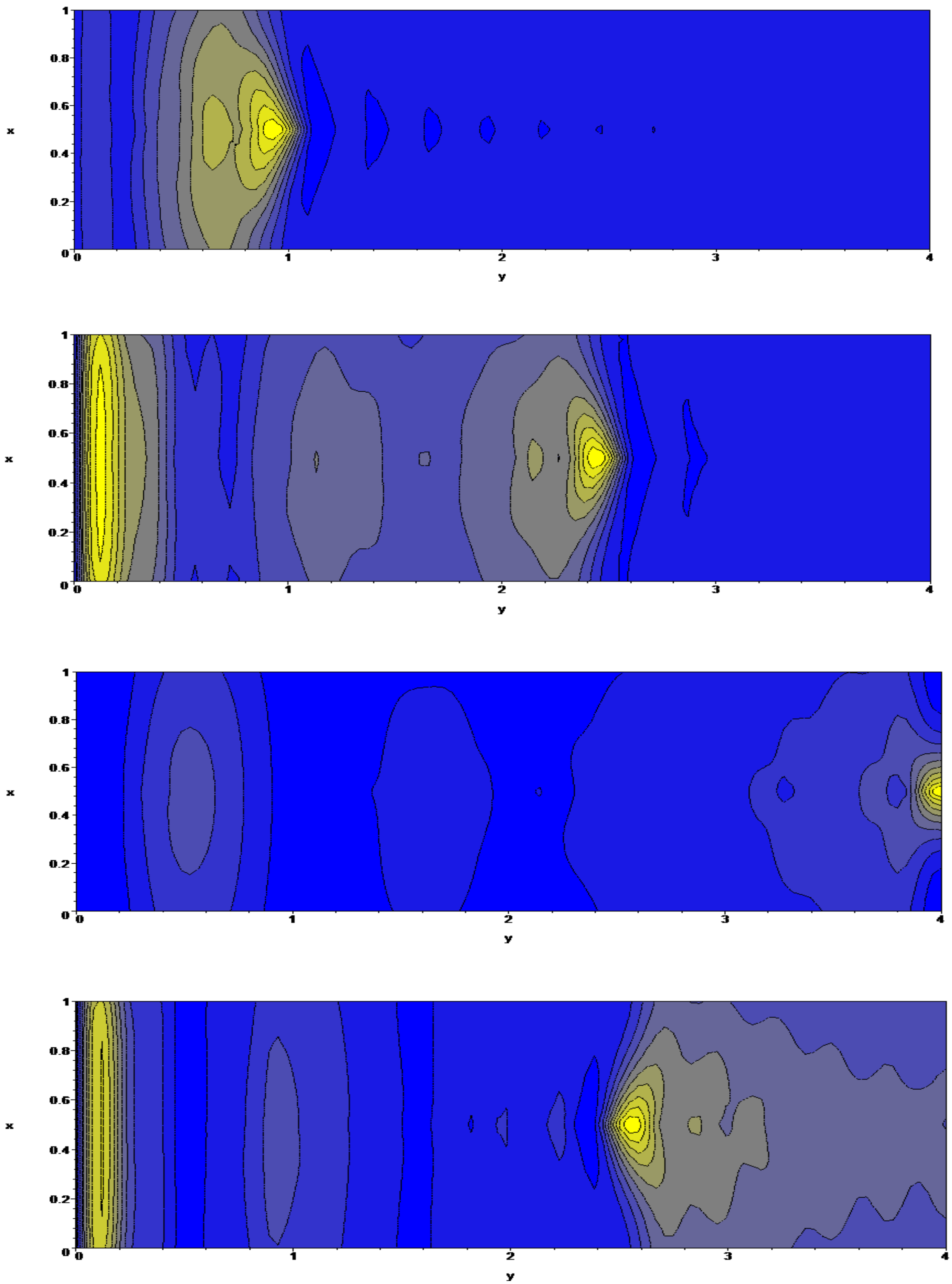
$$\bar{\bar{u}}_{n,m}(t)=\frac{1}{b_{n,m}}v^2\cdot\mu_m\cdot\Im_x\{1\}\left[f_0(t)*e^{-v^2\gamma^2 t}\sin\left(b_{n,m}t\right)\right]+\frac{1}{b_{n,m}}v^2 S_0 X_n(x_0)\left[\sin\left(\mu_m\cdot a\cdot t\right)*e^{-v^2\gamma^2 t}\sin\left(b_{n,m}t\right)\right]$$

iv) Apply inverse Finite Fourier Transforms:

$$u(x,y,t)=\sum_{n=1}\sum_{m=0}\bar{\bar{u}}_{n,m}(t)\,\frac{X_n(x)Y_m(y)}{\|X_n\|^2\|Y_m\|^2}$$

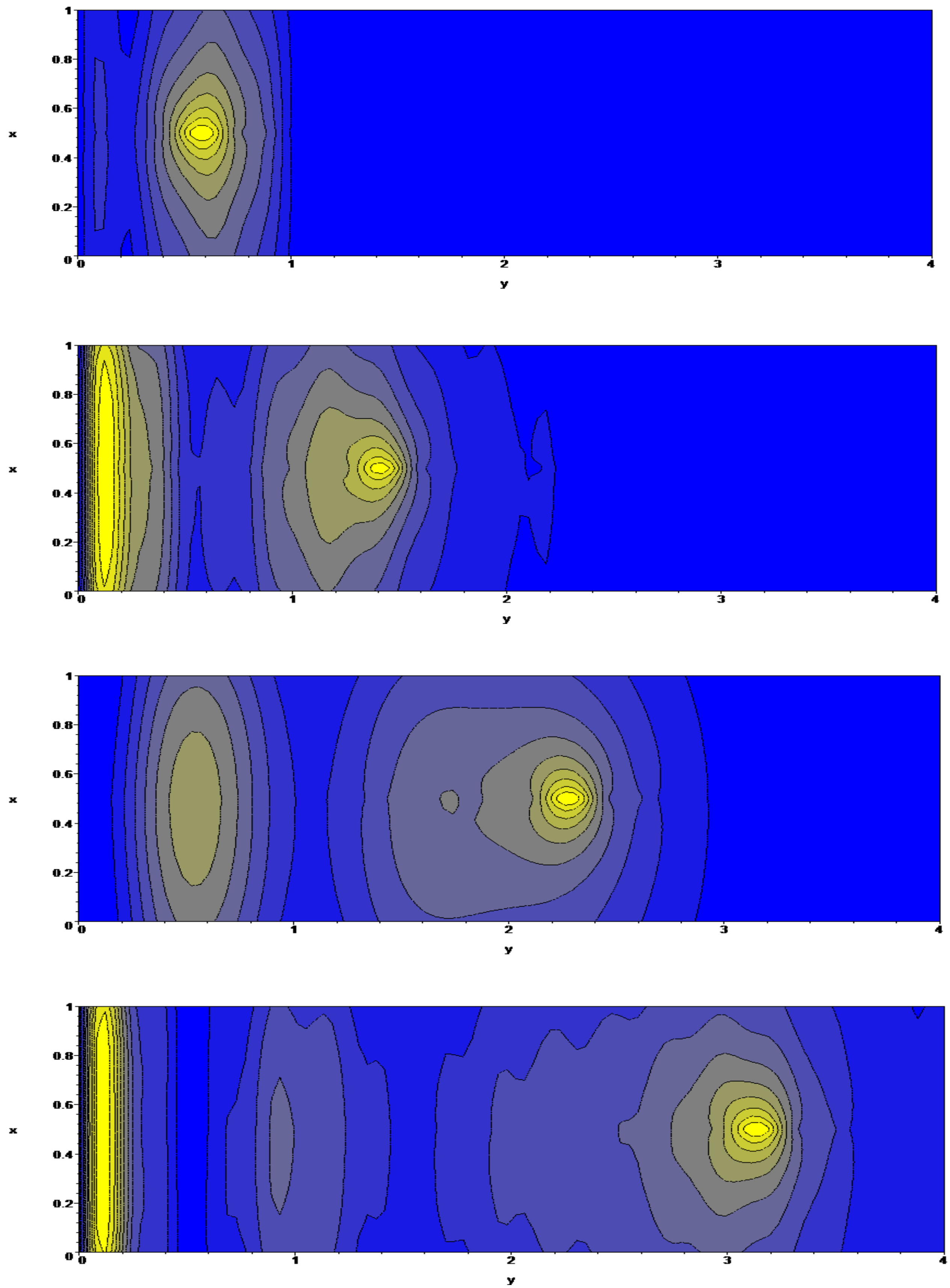
$a = v = 0.5$

$t = 2\ ,\ t = 5\ ,\ t = 8\ ,\ t = 11$

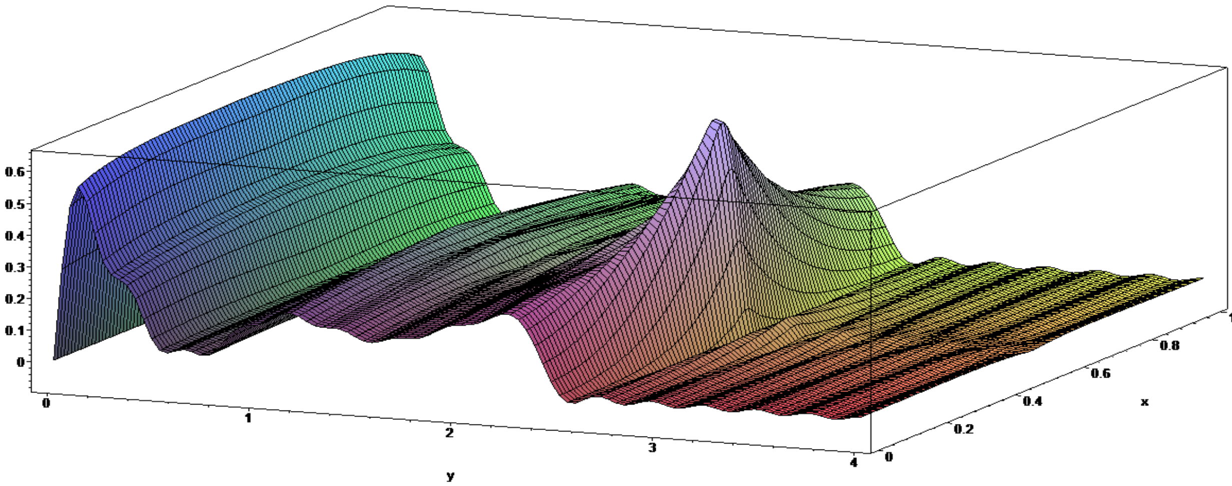


$a = 0.29, \quad v = 0.5$

$t = 2\ ,\ t = 5\ ,\ t = 8\ ,\ t = 11$



t=5 a=v=0.5



t=8 $a = 0.29, \quad v = 0.5$