ME-505 FALL 2020 **FINAL EXAM**

NAME: 1.

2.

You can work in teams of two students. However, you should turn in your own test, indicating the name of your teammate.

1. Submit in Learning Suite before Midnight on **Thursday, December 17.**
2. *Please write neatly and show all your work*.
3. Any content of the exam cannot be discussed with anyone except of your teammate.
4. Computer and standard math software can be used only as the supportive tools.
5. Class notes, class web-site, and regular math books can be used.
6. No set of rules can cover all possible situations – be reasonable – do only what you believe is proper.

Seagull Problem

Consider the following initial boundary value problem for the function $u(x,t)$:

$$e^{-2x}\frac{\partial^2 u}{\partial x^2}+\frac{1}{k}S(x,t)=\frac{1}{\alpha}\frac{\partial u}{\partial t}, \qquad x_1<x<x_2, \qquad t>0$$

$$u(x_1,t)=f_1(t) \qquad t>0$$

$$u(x_2,t)=f_2(t) \qquad t>0$$

$$u(x,0)=u_0(x) \qquad x_1\le x\le x_2$$

1) Analyze the differential operator L with respect to variable x.

Rewrite it in self-adjoint form and find the weight function $p(x)$.

2) Set the supplemental Sturm-Liouville problem.

Find the general solution of the differential equation in the Sturm-Liouville problem

(hint: consider the equations solvable in terms of Bessel functions).

Apply boundary conditions to generate eigenvalues and corresponding eigenfunctions

Write the first five eigenvalues and sketch the first five eigenfunctions for $x_1=1.0,\ x_2=2.0$.

Define the weighted inner product.

Define the square of the norm of eigenfunctions.

3) Use the found eigenfunctions to represent the function $h(x)=1-H(x-x_0)$ by the Generalized Fourier series in the interval $x_1\le x\le x_2$. Sketch the graph of $h(x)$ and truncated Fourier series (with 20 terms). Use $x_0=5/3$.

4) Define the Finite Integral Transform pair based on the found eigenfunctions.

5) Derive the operational property of the defined integral transform: apply transform to operator L subject to non-homogeneous boundary conditions.

6) Use the defined Finite Integral Transform and the Laplace transform to solve the given i.b.v.p.

Write the formal solution for the transformed function $\overline{u}_n(t)$ using the convolution theorem.

7) Find the solution for the case

$$S(x)=S_0\delta(x-x_0)\sin(wt),\ f_1(t)=0,\ f_2(t)=0,\ u_0(x)=0$$

$$x_0=5/3,\ x_1=1.0,\ x_2=2.0,\ S_0=4,\ w=1.0,\ \alpha=0.5,\ k=0.5$$

Sketch the solution curves for the given moments of time: $t=5,\ \ t=12$.

1) Operator L .

$$Lu \equiv e^{-2x}\frac{\partial^2 u}{\partial x^2} \qquad p(x) = \frac{1}{e^{-2x}}e^{\int 0dx} = e^{2x}$$

Operator is already in self-adjoint form

$$r(x) = 1\ , \quad q(x) = 0$$

2) Sturm-Liouville Problem

$$e^{-2x}y'' = \lambda y \qquad \lambda = -\mu^2$$

$$y(x_1) = 0$$

$$y(x_2) = 0$$

$$y'' + \mu^2 e^{2x} y = 0$$

This is the equation solution of which can be expressed in terms of Bessel functions.

See equation c) on p.505 $\quad y'' + \left(a^2 e^{2x} - p^2\right)y = 0$

Identify: $\quad p = 0\ , \quad a = \mu\ ,$ therefore, the general solution can be written as

$$y(x) = c_1 J_0\left(\mu e^x\right) + c_2 Y_0\left(\mu e^x\right)$$

Apply boundary conditions:

$$y(x_1) = c_1 J_0\left(\mu e^{x_1}\right) + c_2 Y_0\left(\mu e^{x_1}\right) = 0$$

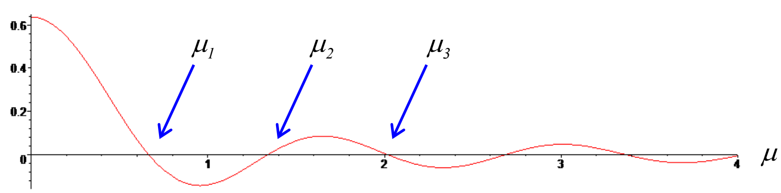
$$y(x_2) = c_1 J_0\left(\mu e^{x_2}\right) + c_2 Y_0\left(\mu e^{x_2}\right) = 0$$

This system of algebraic equations for the constants c_1 and c_2 has non-trivial solution only if the determinant of the matrix of coefficients is equal to zero, i.e.

Eigenvalues

$$J_0\left(\mu e^{x_1}\right)Y_0\left(\mu e^{x_2}\right) - J_0\left(\mu e^{x_2}\right)Y_0\left(\mu e^{x_1}\right) = 0$$

That yields the characteristic equation for the eigenvalues μ_n



$$\mu_1 = 0.66458$$
$$\mu_2 = 1.34079$$
$$\mu_3 = 2.01480$$
$$\mu_4 = 2.68814$$
$$\mu_5 = 3.36119$$

Therefore, the corresponding eigenfunctions are defined by

$$y_n(x) = c_{1,n} J_0\left(\mu_n e^x\right) + c_{2,n} Y_0\left(\mu_n e^x\right)$$

They still satisfy the homogeneous boundary conditions, for instance, at $x = x_1$

$$c_{1,n} J_0\left(\mu_n e^{x_1}\right) + c_{2,n} Y_0\left(\mu_n e^{x_1}\right) = 0$$

From this equation, the relationship between the arbitrary coefficients can be established as

$$c_{1,n} = -\,c_{2,n}\frac{Y_0\left(\mu_n e^{x_1}\right)}{J_0\left(\mu_n e^{x_1}\right)}, \quad \text{then} \quad y_n(x) = -\,c_{2,n}\frac{Y_0\left(\mu_n e^{x_1}\right)}{J_0\left(\mu_n e^{x_1}\right)} J_0\left(\mu_n e^x\right) + c_{2,n} Y_0\left(\mu_n e^x\right)$$

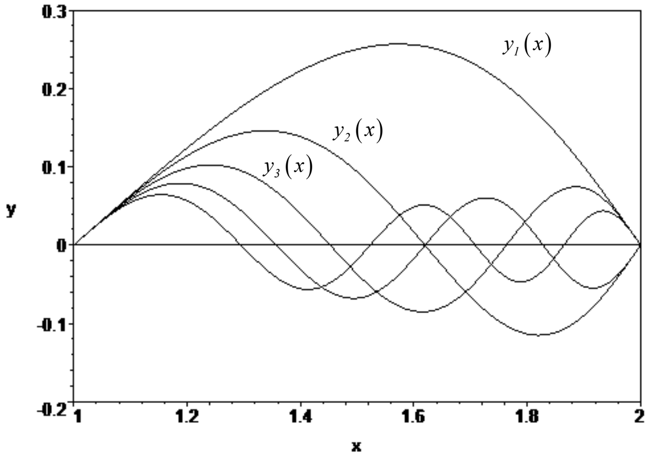
If we choose the arbitrary coefficients $c_{2,n}$ as $c_{2,n} = J_0\left(\mu_n e^{x_1}\right)$, then eigenfunctions can be written as

Eigenfunctions

$$y_n(x) = J_0\left(\mu_n e^{x_1}\right)Y_0\left(\mu_n e^x\right) - Y_0\left(\mu_n e^{x_1}\right)J_0\left(\mu_n e^x\right)$$

Eigenfunctions

$$y_n(x) = J_0(\mu_n e^{x_1}) Y_0(\mu_n e^x) - Y_0(\mu_n e^{x_1}) J_0(\mu_n e^x)$$



Egenfunctions can look different depending on how the coefficients $c_{1,n}$ and $c_{2,n}$ in the solution $y_n(x) = c_{1,n} J_0(\mu_n e^x) + c_{2,n} Y_0(\mu_n e^x)$ are chosen.
But they have obey the property f) of the Sturm-Liouville theorem (p.439)

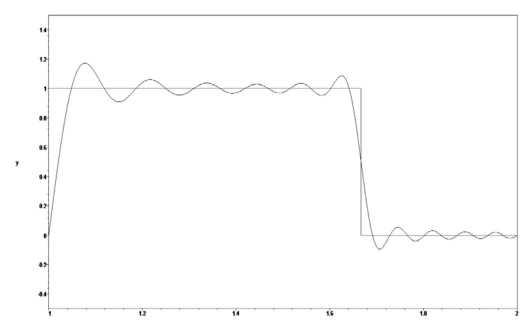
The square of the norm of eigenfunctions is defined by the weighted inner product:

Norm

$$\left\| y_n(x) \right\|_{p=e^{2x}} = (y_n, y_n)_p = \int_{x_1}^{x_2} y_n^2(x) e^{2x} dx$$

3) Representation of

$$h(x) = 1 - H(x - x_0) \quad = \sum_{n=1}^{20} b_n \frac{y_n(x)}{\left\| y_n(x) \right\|_p^2}, \text{ where } b_n = (h, y_n)_p == \int_{x_1}^{x_2} h(x) y_n(x) e^{2x} dx$$



4) **Define**

Direct Transform

$$\Im\{u(x)\} = \quad \bar{u}_n \quad = (u, y_n)_p \quad = \int_{x_1}^{x_2} u(x) y_n(x) p(x) dx \qquad (17)$$

Inverse Transform

$$\Im^{-1}\{\bar{u}_n\} = \quad u(x) \quad = \sum_{n=1}^{\infty} \bar{u}_n \frac{y_n(x)}{\left\| y_n \right\|_p^2}, \qquad p(x) = e^{2x} \qquad (18)$$

5) **Operational property** **D-D** (p.860)

$$\Im\{Lu\} \quad = \ - \ \mu_n^2 \ \bar{u}_n \ - y_n'(x_2) r(x_2) f_2(t) + y_n'(x_1) r(x_1) f_1(t) \qquad (20)$$

$$= \ - \ \mu_n^2 \ \bar{u}_n \ - y_n'(x_2) f_2(t) + y_n'(x_1) f_1(t)$$

6) Apply the Finite Integral Transform $\Im\{u(x,t)\} = \bar{u}_n(t)$ **to the given equation**

$$-\mu_n^2\ \bar{u}_n\ \ - y_n'(x_2) f_2(t) + y_n'(x_1) f_1(t) + \frac{1}{k} S_n(t) = \frac{1}{\alpha}\frac{\partial \bar{u}_n}{\partial t}$$

$$S_n(t) = \int_{x_1}^{x_2} S(x,t)\, y_n(x)\, p(x)\, dx$$

$$u_{n,0} = \int_{x_1}^{x_2} u_0(x)\, y_n(x)\, p(x)\, dx \qquad \text{transformed initial condition}$$

$$-\alpha\mu_n^2\ \bar{u}_n\ \ - \alpha y_n'(x_2) f_2(t) + \alpha y_n'(x_1) f_1(t) + \frac{\alpha}{k} S_n(t) = \frac{\partial \bar{u}_n}{\partial t}$$

Apply the Laplace transform $U_n(s) = \mathscr{L}\{\bar{u}(t)\}$

$$-\alpha\mu_n^2\ U_n\ \ - \alpha y_n'(x_2)\mathscr{L}\{f_2(t)\} + \alpha y_n'(x_1)\mathscr{L}\{f_1(t)\} + \frac{\alpha}{k}\mathscr{L}\{S_n(t)\} = sU_n - u_{n,0} \qquad U_n = \{\bar{u}_n\} = \int_0^\infty \bar{u}_n(t) e^{-st} dt$$

$$U_n\ \ = -\alpha y_n'(x_2)\mathscr{L}\{f_2(t)\}\frac{1}{s+\alpha\mu_n^2} + \alpha y_n'(x_1)\mathscr{L}\{f_1(t)\}\frac{1}{s+\alpha\mu_n^2} + \frac{\alpha}{k}\mathscr{L}\{S_n(t)\}\frac{1}{s+\alpha\mu_n^2} + u_{n,0}\frac{1}{s+\alpha\mu_n^2}$$

$$U_n\ \ = -\alpha y_n'(x_2)\mathscr{L}\{f_2(t)\}\mathscr{L}\left\{e^{-\alpha\mu_n^2 t}\right\} + \alpha y_n'(x_1)\mathscr{L}\{f_1(t)\}\mathscr{L}\left\{e^{-\alpha\mu_n^2 t}\right\} + \frac{\alpha}{k}\mathscr{L}\{S_n(t)\}\mathscr{L}\left\{e^{-\alpha\mu_n^2 t}\right\} + u_{n,0}\mathscr{L}\left\{e^{-\alpha\mu_n^2 t}\right\}$$

Apply the inverse Laplace transform (use convolution theorem):

$$\bar{u}_n(t)\ \ = -\alpha y_n'(x_2)\int_0^t f_2(t-\tau) e^{-\alpha\mu_n^2\tau} d\tau + \alpha y_n'(x_1)\int_0^t f_1(t-\tau) e^{-\alpha\mu_n^2\tau} d\tau + \frac{\alpha}{k}\int_0^t S_n(t-\tau) e^{-\alpha\mu_n^2\tau} d\tau + u_{n,0} e^{-\alpha\mu_n^2 t}$$

7) Find the solution for the case

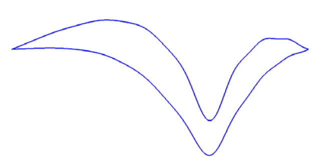
$$S(x,t) = S_0\delta(x-x_0)\sin(wt), \qquad S_n(t) = \int_{x_1}^{x_2} S_0\delta(x-x_0)\, sin(wt)\, y_n(x)\, p(x)\, dx = S_0\, sin(wt)\, y_n(x_0)\, p(x_0)$$

$$u_0(x) = 0,\ f_1(t) = 0,\ f_2(t) = 0$$

$$x_0 = 5/3,\ x_1 = 1.0,\ x_2 = 2.0,\ S_0 = 4,\ w = 1.0,\ \alpha = 0.5,\ k = 0.5$$

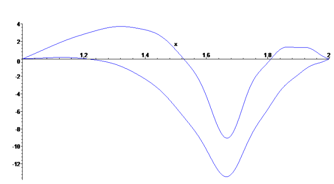
Solution:

FP-02 - SEAGULL F20 Final.mws



$$u(x,t) = \sum_{n=1}^{\infty} \bar{u}_n(t)\frac{y_n(x)}{\|y_n\|_p^2}, \qquad \text{where}$$

$$\bar{u}_n(t)\ \ = \frac{\alpha}{k} y_n(x_0)\, p(x_0)\int_0^t sin[w(t-\tau)]\, e^{-\alpha\mu_n^2\tau} d\tau$$



Convolution integral: $\dfrac{e^{(-a\,\mu_n^2\,t)}\,w - w\cos(w\,t) + a\,\mu_n^2\sin(w\,t)}{a^2\,\mu_n^4 + w^2}$, and $p(x_0) = e^{2x_0}$

Sketch the solution for $t = 5,\ \ t = 12$

Philippe de Champaigne   Portrait of two men

Portrait of René Descartes

after Frans Hals

Portrait of Blaise Pascal

Philippe de Champaigne