ME-505	**FINAL EXAM**	FALL 2022

You can work in teams of two students.

However, you should turn in your own test report, indicating the name of your teammate.

YOUR NAME:
TEAMMATE:

1. Submit solutions through the Learning Suite (**single file**) before Midnight on **Friday, December 16.** Graphics and animations can be submitted separately.
2. ***Please write neatly and show all your work.***
3. Any content of the exam cannot be discussed with anyone except of your teammate.
4. Computer and standard math software can be used only as the supportive tools.
5. Class notes, class web-site, and regular math books can be used.
6. No set of rules can cover all possible situations – be reasonable – do only what you believe is proper.

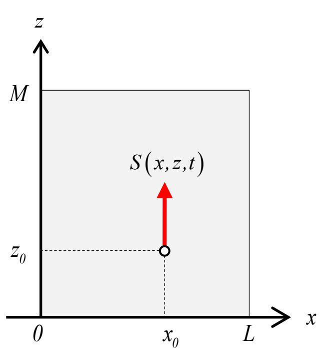
A FLYING CARPET AND A FIREBIRD

Use the Integral Transforms Technique to solve the following initial boundary value problem for the unknown function

$$u(x,z,y), \quad 0<x<L,\ 0<z<M,\ t>0$$

Governing Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{1}{z^2}\frac{\partial^2 u}{\partial z^2}+\frac{2}{z}\frac{\partial u}{\partial z}+S(x,z,t)=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad \begin{matrix} 0<x<L \\ 0<z<M \\ t>0 \end{matrix} \qquad (1)$$



Boundary conditions:

$$\left[-\frac{\partial u}{\partial x}+hu\right]_{x=0}=f_0 \qquad 0<z<M, \qquad t>0 \qquad (2)$$

$$\left[+\frac{\partial u}{\partial x}+hu\right]_{x=L}=f_L \qquad 0<z<M, \qquad t>0 \qquad (3)$$

$$u(x,0,t)<\infty \qquad 0<x<L, \qquad t>0 \qquad (4)$$

$$\left[+\frac{\partial u}{\partial z}+hu\right]_{z=M}=g_M \qquad 0<x<L, \qquad t>0 \qquad (5)$$

Initial conditions:

$$u(x,z,0)=0 \qquad 0\le x\le L, \qquad 0\le z\le M \qquad (6)$$

$$\frac{\partial}{\partial t}u(x,z,0)=0 \qquad 0\le x\le L, \qquad 0\le z\le M \qquad (7)$$

Source term:

$$S(x,z,t)=S_0\,\delta(x-x_0)\,\delta(z-z_0)\,\delta(t-t_0) \qquad (8)$$

Identify the integral transforms needed to eliminate differential operators with respect to the *x, z,* and *t* variables.

A. Differential operator L_z in the z – variable. Construction of the Integral Transform.

1) ***Differential operator*** with respect to the *z*-variable requires construction of the special integral transform.

Follow the Outline of the Finite Integral Transform method (Section IX.5.5, p.862).

Analyze the differential operator *L* with respect to the variable *z*.

Rewrite the operator *L* in self-adjoint form.

2) Set ***the supplemental Eigenvalue Problem*** for the operator *L*.

Find the general solution (you may consider the equations solvable in terms of Bessel functions).

Apply the boundary conditions to generate ***eigenvalues*** and corresponding ***eigenfunctions***, using the following parameters: $M = 3.0$ and $h = 1.5$. Note that this Sturm-Liouville problem is singular (similar to *IX.5 Chain*).

Write explicitly ***the first three eigenvalues*** and sketch ***the first three eigenfunctions*** in $0 \le z \le M$.

Define the ***weighted inner product.*** Define ***the square of the norm*** of eigenfunctions (do not calculate).

3) ***Validate*** the found solution of the Eigenvalue problem by application of the found eigenfunctions to represent the step-wise constant function $f(z) = Heaviside(z - M/2)$ by the Generalized Fourier series in the interval $0 \le z \le M$. Sketch the graph of $f(z)$ and of truncated Fourier series (with 20 terms).

4) Define the ***Finite Integral Transform*** pair.

5) Derive the ***operational property*** of the defined integral transform: apply transform to the differential operator *L* subject to the boundary conditions (4-5). See notes IX.4, p.860.

B. Differential operator L_x in the *x* – variable. Corresponding Integral Transform.

6) Identify the integral transform applicable to differential operator in the *x*-variable.

Write explicitly ***the first three eigenvalues*** and sketch t***he first three eigenfunctions*** in $0 \le x \le L$, using the following parameters: $L = 4.0$ and $h = 1.5$.

7) ***Validate*** the found eigenfunctions to represent the step-wise constant function $f(x) = Heaviside(x - L/2)$ by the Generalized Fourier series in the interval $0 \le x \le L$. Sketch the graph of $f(x)$ and of the truncated Fourier series.

C. Solution of the given Initial-Boundary Value Problem

6) Use the defined Integral Transforms in the *x* and *z* variables, and the Laplace transform to solve the given i.b.v.p.

Write the transformed solution in the case of $f_0 = f_L = g_M = 0$, $t_0 = 0$.

Use the inverse Finite Integral Transforms to write ***the solution for*** $u(x,z,t)$.

7) Visualize the solution for the case of $L = 4.0,\ M = 3.0,\ x_0 = 2.5,\ z_0 = 2.0,\ t_0 = 0.0,\ S_0 = 10.0,\ v = 0.05$

8) Sketch the contour plot of solution $u(x,\ z,\ t)$ for the moments of time $t = 20.0,\ 30.0,\ 40.0$ (using 20 contous).

D. Exercise your creativity to modify the problem and to visualize your solution.

Philippe de Champaigne   Portrait of two men

Portrait of René Descartes

after Frans Hals

Portrait of Blaise Pascal

Philippe de Champaigne