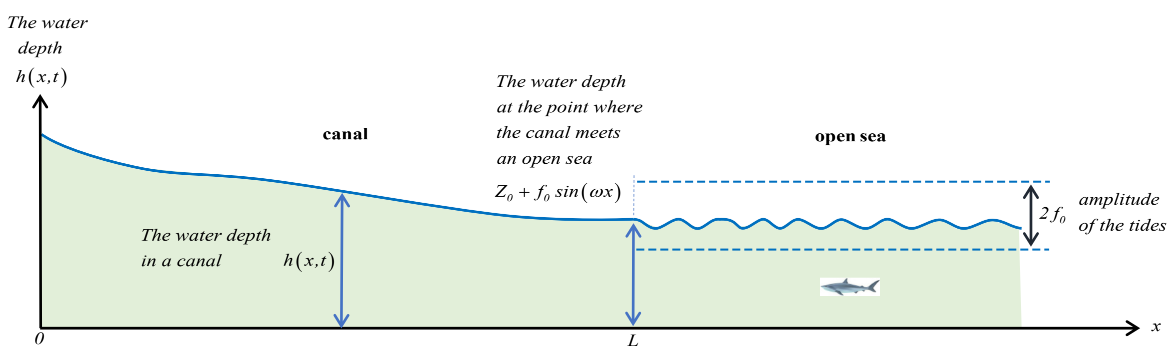
TIDAL WAVES IN A CANAL



The water depth in a canal is $h(x,t) = u(x,t) + Z_0$, where Z_0 is the mean depth at the point $x = L$, where the canal meets an open sea, and $u(x,t)$ is described by the following Initial Boundary Value Problem:

$x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x} = \dfrac{1}{w^2}\dfrac{\partial^2 u}{\partial t^2}$	$0 < x < L, \quad t > 0$
$u(0,t) < \infty$	$t > 0$
$u(L,t) = f_0 \sin(\omega t)$	$t > 0$
$u(x,0) = 0$	$0 \le x \le L$
$\dfrac{\partial}{\partial t} u(x,0) = 0$	$0 \le x \le L$

Construct the integral transform $\Im_x$ for the differential operator with respect to the variable x (follow the outline on p.862).

1. Operator

$$Lu \equiv x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

Rewrite operator in self-adjoint form (6),

$$Lu \equiv \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$$

in which we can identify

$p(x) = 1,\ r(x) = x,\quad q(x) = 0$

2. EVP/SLP

$$\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) = \lambda y \quad \Rightarrow \quad (xy')' + \Big[0 + \overbrace{(-\lambda)}^{\mu^2}\Big] y = 0 \quad \Rightarrow \quad \lambda = -\mu^2$$

$$\boxed{(xy_n')' = -\mu_n^2 y_n}$$

$y(0) < \infty$

$y(L) = 0$

$$xy'' + y' + \mu^2 y = 0$$

$$y'' + \frac{1}{x}y' + \frac{\mu^2}{x}y = 0$$

This is the Generalized Bessel equation, VII.6.11, p.503,

with parameters $m = 0,\ \alpha = 0,\ p = \frac{1}{2},\ a = 2\mu,\ \nu = 0$, then

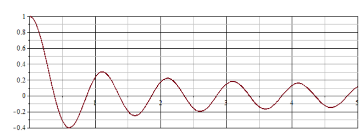
$y(x) = c_1 J_0\left(2\mu\sqrt{x}\right) + c_2 Y_0\left(2\mu\sqrt{x}\right)$ ***general solution***

$y(0) < \infty \quad \Rightarrow \quad c_2 = 0$ for bounded solution

$y(L) = 0 \quad \Rightarrow \quad J_0\left(2\mu\sqrt{L}\right) = 0$ characteristic equation for eigenvalues μ_n

Eigenvalues

μ_n are the roots of $J_0\left(2\mu\sqrt{L}\right) = 0$



0.3802363069

0.8728009845

1.368274523

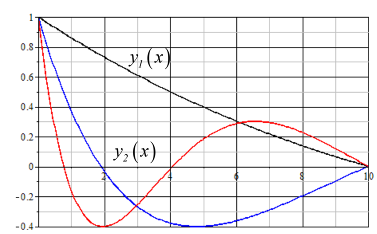
Eigenfunctions

$$y_n(x) = J_0\left(2\mu_n\sqrt{x}\right)$$

$$\left\|y_n(x)\right\|^2 = \int_0^L J_0^2\left(2\mu_n\sqrt{x}\right)dx$$

$$= L J_0^2\left(2\mu_n\sqrt{L}\right) + L J_1^2\left(2\mu_n\sqrt{L}\right)$$

$$= L J_1^2\left(2\mu_n\sqrt{L}\right)$$



3. Define the Finite Integral Transform pair

Direct Transform	$\Im_x\{u(x)\}$	$=(u, y_n)_{p=1}$	$= \int_0^L u(x)y_n(x)dx$	$= \bar{u}_n$	(17)
Inverse Transform	$\Im_x^{-1}\{\bar{u}_n\}$		$= \sum_{n=1}^{\infty} \bar{u}_n \frac{y_n(x)}{\left\|y_n\right\|^2}$	$= u(x)$	(18)

4. Derive operational property of integral transform $\Im_x$

$$\Im_x\left\{\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial u}\right)\right\} = \int_0^L \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial u}\right) y_n(x)\,dx \quad \overset{\text{intergration by parts}}{=} \cancel{\left[x\frac{\partial u}{\partial x}(x)\,y_n(x)\right]_{x=0}^{x=L}} - \int_0^L \left(x\frac{\partial u}{\partial u}\right) y_n'(x)\,dx$$

$$= -\int_0^L \left(\frac{\partial u}{\partial u}\right) x\,y_n'(x)\,dx$$

$$= -\left[u(x)\,x\,y_n'\right]_{x=0}^{x=L} + \int_0^L u(x)\left(xy_n'\right)'\,dx$$

$$= -L\overbrace{u(L)}^{f_L}\,y_n'(L) + \int_0^L u\ \overbrace{\left(xy_n'\right)'}^{(xy_n')'=-\mu_n^2 y_n}\ dx$$

$$= -L\,f_L(t)\,y_n'(L) - \mu_n^2\,\overline{u}_n$$

Differentiation of the Bessel Functions using the chain rule: $\quad \dfrac{d}{dx}f\left[u(x)\right] = \dfrac{df}{du}\dfrac{du}{dx} = \dfrac{df}{du}u'$

$$\frac{d}{dx}J_\nu\left[u(x)\right] = \frac{d}{du}J_\nu[u]\cdot u' = \left[J_{\nu-1}(u) - \frac{\nu}{u}J_\nu(u)\right]\cdot u' \quad \text{if Eqn.(49), Ch.VII, p.500 is used}$$

$$= \left[-J_{\nu+1}(u) + \frac{\nu}{u}J_\nu(u)\right]\cdot u' \quad \text{if Eqn.(50), Ch.VII, p.500 is used}$$

For example,

$$\frac{d}{dx}J_\nu\left(ax^3\right) = \left[J_{\nu-1}\left(ax^3\right) - \frac{\nu}{ax^3}J_\nu\left(ax^3\right)\right]\cdot\left(ax^3\right)' = \left[J_{\nu-1}\left(ax^3\right) - \frac{\nu}{ax^3}J_\nu\left(ax^3\right)\right]\cdot 3ax^2$$

$$\frac{d}{dx}J_0\left(2\mu_n\sqrt{x}\right) = \left[-J_1\left(2\mu_n\sqrt{x}\right)\right]\cdot\left(2\mu_n\sqrt{x}\right)' = \left[-J_1\left(2\mu_n\sqrt{x}\right)\right]\cdot\frac{\mu_n}{\sqrt{x}}$$

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6. The Laplace transform $\mathcal{L}$

$$\mathcal{L}\{u(t)\} = \hat{u}(s)$$

$$\mathcal{L}\left\{\frac{\partial}{\partial t}u(t)\right\} = s\hat{u}(s) - u(0)$$

$$\mathcal{L}\left\{\frac{\partial^2}{\partial t^2}u(t)\right\} = s^2\hat{u}(s) - su(0) - u'(0)$$

5. Apply integral transform $\Im_x$ to the equation and initial conditions.

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x} = \frac{1}{w^2}\frac{\partial^2 u}{\partial t^2}$$

$$-L\,f_L(t)\,y_n'(L)-\mu_n^2\,\overline{u}_n = \frac{1}{w^2}\frac{\partial^2\overline{u}_n}{\partial t^2}$$

Initial conditions: $\Im_x\{u(x,0)\}=0$

$$\Im_x\left\{\frac{\partial}{\partial t}u(x,0)\right\}=0$$

Apply Laplace transform $\mathcal{L}\{...\}$

$$-w^2L\,y_n'(L)\cdot\{f_L(t)\}-w^2\mu_n^2\,\hat{\overline{u}}_n = s^2\hat{\overline{u}}_n$$

$$y_n'(L)=\left[-J_1\left(2\mu_n\sqrt{L}\right)\right]\cdot\frac{\mu_n}{\sqrt{L}}$$

$$\left(s^2+w^2\mu_n^2\right)\hat{\overline{u}}_n=-w^2L\,y_n'(L)\{f_L(t)\}$$

$$\hat{\overline{u}}_n=\frac{-w\,L\,y_n'(L)}{\mu_n}\{f_L(t)\}\,\frac{w\mu_n}{s^2+w^2\mu_n^2}$$

$$\hat{\overline{u}}_n=\frac{-wL\,y_n'(L)}{\mu_n}\{f_L(t)\}\,\left\{\sin\left(w\mu_n t\right)\right\}$$

Apply inverse Laplace transform using convolution

$$\overline{u}_n=\frac{-wL\,y_n'(L)}{\mu_n}\,f_0\sin(\omega t)*\sin\left(w\mu_n t\right)$$

$$-\ \frac{-\sin(\omega\,t)\ w\,\mu+\sin(w\,\mu\,t)\ \omega}{(-\omega+w\,\mu)\ (\omega+w\,\mu)}$$

$$\overline{u}_n = wf_0\sqrt{L}\,J_1\left(2\mu_n\sqrt{L}\right)\frac{w\mu_n\sin(\omega t)-\omega\sin\left(w\mu_n t\right)}{w^2\mu_n^2-\omega^2}$$

$$u(x,t)=\sum_{n=1}^{\infty}\overline{u}_n(t)\frac{y_n(x)}{\left\|y_n\right\|^2}$$

Apply inverse transform $\Im_x^{-1}\{\ \overline{u}_n\ \}$

$$h(x,t)=u(x,t)+Z_0 = Z_0+\sum_{n=1}^{\infty}\overline{u}_n(t)\frac{y_n(x)}{\left\|y_n\right\|^2}$$

$$\left\|y_n\right\|^2=L\,J_1^2\left(2\mu_n\sqrt{L}\right)$$

Solution:

$$h(x,t)= Z_0+\frac{wf_0}{\sqrt{L}}\sum_{n=1}^{\infty}\frac{w\mu_n\sin(\omega t)-\omega\sin\left(w\mu_n t\right)}{w^2\mu_n^2-\omega^2}\frac{J_0\left(2\mu_n\sqrt{x}\right)}{J_1\left(2\mu_n\sqrt{L}\right)}$$

Compare to Lamb's solution

$$\eta=C\frac{J_0\left(2\kappa^{\frac{1}{2}}x^{\frac{1}{2}}\right)}{J_0\left(2\kappa^{\frac{1}{2}}L^{\frac{1}{2}}\right)}\cos(\omega t+\varepsilon)$$

[Lamb, p.276]

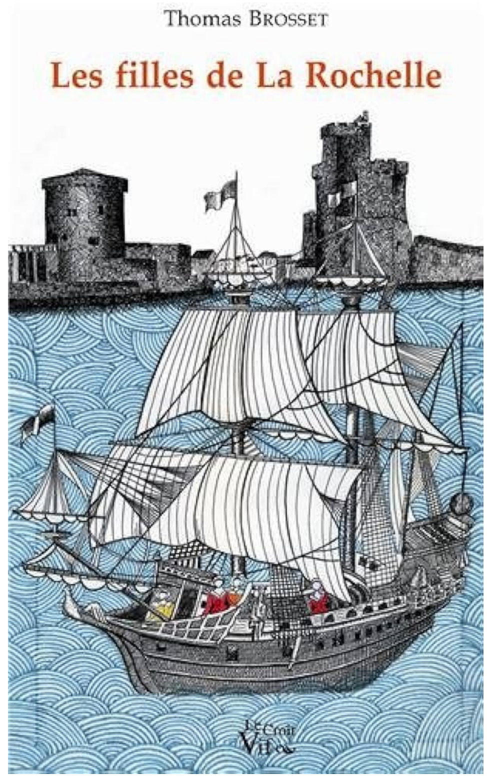
Les filles de La Rochelle

1.

Ce sont les filles de La Rochelle,
Ont armé un bâtiment,
Pour aller faire la course
Dedans les mers du Levant.

Refrain

Ah ! La feuille s'envole, s'envole,
Ah ! La feuille s'envole au vent.



2.

La grand-vergue est en ivoire,
Les poulies en diamant ;
La grand-voile est en dentelle,
La misaine en satin blanc ;

3.

Les cordages du navire
Sont de fils d'or et d'argent,
Et la coque est en bois rouge
Travaillé fort proprement ;

4.

L'équipage du navire,
C'est tout filles de quinze ans ;
Le capitaine qui les commande
Est le roi des bons enfants.

5.

Hier, faisant sa promenade
Dessus le gaillard d'avant,
Aperçut une brunette
Qui pleurait dans les haubans :

6.

Qu'avez-vous, gentille brunette,
Qu'avez-vous à pleurer tant ?
Av'vous perdu père et mère
Ou quelqu'un de vos parents ?

7.

J'ai cueilli la rose blanche,
Qui s'en fut la voile au vent :
Elle est partie vent arrière,
Reviendra-z-en louvoyant…