FINAL EXAM ME 505

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The water depth (ordinate of the water free surface) is $h(x,t)=u(x,t)+h_0$
where h_0 represents the depth in the undisturbed state, and the variable u(x,t) is described by the following IBVP:

$x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = \frac{1}{w^2}\left[\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t}\right]$		$0<x<L,\ t>0$
BOUNDARY CONDITIONS	$u(0,t)<\infty$	$t>0$
	$u(L,t)=g(t)$	$t>0$
INITIAL CONDITIONS	$u(x,0)=f(x)$	$0\le x\le L$
	$\frac{\partial}{\partial t}u(x,0)=0$	$0\le x\le L$

A. Differential Operator L in the x-variable (construct the Integral Transform)

$$Lu \equiv x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}$$

A1. Analysis of L in the x-variable

$$Lu \equiv x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}$$

$a_0(x)=x \quad a_1(x)=2 \quad a_2(x)=0$

$$p(x)=\frac{1}{a_0(x)}e^{\int\frac{a_1(x)}{a_0(x)}dx} = \frac{1}{x}e^{\int\frac{2}{x}dx} = \frac{1}{x}e^{\log x^2} = \frac{1}{x}x^2 = x$$

$$\boxed{p(x)=x}$$

$q(x)=a_2(x)p(x) = 0\,p(x)=0$

$r(x)=a_0(x)p(x) = x(x) = x^2$

Write in self-adjoint form

$$Lu(x) \equiv \frac{1}{p}\left[(ru')'+qu\right]$$

$$\boxed{Lu(x) \equiv \frac{1}{x}\left[(x^2u')'\right]}$$

A2. Supplemental Eigenvalue Problem

$$Ly=\lambda y,\quad y(x)$$

$$\frac{1}{x}\left[(x^2y')'\right]=\lambda y$$

Boundary conditions (homogenous)

$y(0)<\infty,\quad y(L)=0$

Sturm-Liouville Form

$$(ry')'+(q-\lambda\cdot p)y=0$$

$$(x^2y')'+(-\lambda x)y=0 \quad \text{Let } \lambda=-\mu^2$$

$$(x^2y')'+(\mu^2x)y=0$$

$$x^2y''+2xy'+\mu^2xy=0$$

$$\boxed{y''+\frac{2}{x}y'+\frac{\mu^2}{x}y=0}$$

GENERALIZED BESSEL EQUATION

$$y''+\left[\frac{1-2m}{x}-2\alpha\right]y'+\left[p^2a^2x^{2p-2}+\alpha^2+\frac{\alpha(2m-1)}{x}+\frac{m^2-p^2\nu^2}{x^2}\right]y=0$$

$1-2m=2 \quad -2\alpha=0$
$2m=-1 \quad \alpha=0$
$m=-1/2$

$2p-2=-1$
$2p=1$
$p=\frac{1}{2}$

$p^2a^2=\mu^2$
$\frac{1}{4}a^2=\mu^2$
$a^2=4\mu^2$
$a=2\mu$

$m^2-p^2\nu^2=0$
$\frac{1}{4}-\frac{1}{4}\nu^2=0$
$\frac{1}{4}\nu^2=\frac{1}{4}$
$\nu^2=1$
$\nu=1$

$$p^2a^2x^{2p-2}+\frac{m^2-p^2\nu^2}{x^2}=\frac{\mu^2}{x}$$

The solution takes the form

$$\boxed{y = x^m e^{\alpha x}\left[c_1 J_\nu(ax^p) + c_2 Y_\nu(ax^p)\right]}$$

$$y = x^{-\frac{1}{2}} e^0 \left[c_1 J_1(2\mu x^{1/2}) + c_2 Y_1(2\mu x^{1/2})\right]$$

$$y = \frac{1}{\sqrt{x}}\left[c_1 J_1(2\mu\sqrt{x}) + c_2 Y_1(2\mu\sqrt{x})\right]$$

APPLY BOUNDARY CONDITIONS.

1. $y(0) < \infty$

$$y(0) = \frac{1}{\sqrt{0}}\left[c_1 J_1(2\mu(0)) + c_2 Y_1(2\mu(0))\right]$$

$$y(0) = \frac{1}{\sqrt{0}}\left[c_1 J_1(0) + c_2 Y_1(0)\right] < \infty$$

$$\lim_{a\to 0^+} \frac{J_1(a)}{\sqrt{a}} = \frac{0}{0} \qquad \overset{\text{L'hopital}}{\lim_{a\to 0}} \frac{J_1(a)}{\sqrt{a}} = \lim_{a\to 0} \frac{\frac{1}{2}\left[J_0(a) - J_2(a)\right]}{\frac{1}{2}\frac{1}{\sqrt{a}}} = \lim_{a\to 0} -\sqrt{a}\left[J_0(a) - J_2(a)\right] = 0$$

$$\lim_{a\to 0^+} \frac{Y_1(a)}{\sqrt{a}} = \infty$$

Since $y(0) < \infty$ and $\lim_{x\to 0} \frac{1}{\sqrt{x}} J_1(2\mu\sqrt{x})$ is finite and $\lim_{x\to 0} \frac{1}{\sqrt{x}} Y_1(2\mu\sqrt{x}) = \infty$,

c_2 must equal 0.

$$y = \frac{1}{\sqrt{x}} c_1 J_1(2\mu\sqrt{x}) \qquad \text{Let } c_1 = 1$$

2. $y(L) = 0$

$$y(L) = 0 = \frac{1}{\sqrt{L}} J_1(2\mu\sqrt{L})$$

$$0(\sqrt{L}) = J_1(2\mu\sqrt{L})$$

$$J_1(2\mu_n\sqrt{L}) = 0$$

Characteristic equation for μ_n:

$$\boxed{J_1(2\mu_n\sqrt{L}) = 0} \qquad \text{using } L = 5,\ J_1(2\mu_n\sqrt{5}) = 0$$

Eigenfunctions:

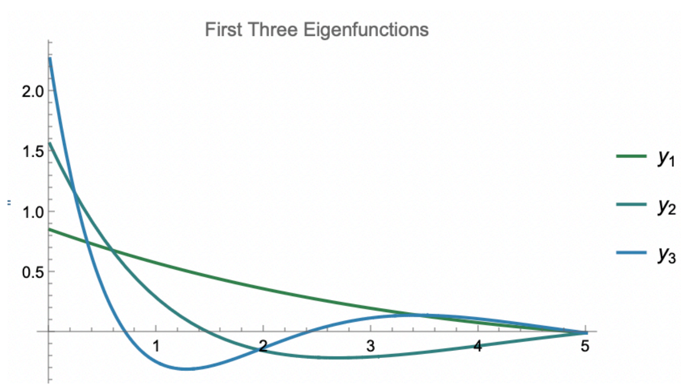
$$\boxed{y_n(x) = \frac{1}{\sqrt{x}} J_1(2\mu_n\sqrt{x})}$$

For the case L=5, explicitly list the first 3 eigenvalues and sketch 3 eigenfunctions on [0,L].

$$J_1(2\mu_n\sqrt{L}) = 0$$

$\mu_1 \approx 0.856796$, $\mu_2 \approx 1.568733$,

$\mu_3 \approx 2.274857$



INNER PRODUCT VECTOR SPACE

define the inner product for $v(x), w(x) \in L^2(0,L)$ recall $p(x) = x$

$$\boxed{(v,w)_p = \int_0^L v(x)\,w(x)\,x\,dx}$$

NORM

define the norm of functions $v(x) \in L^2(0,L)$

$$\|v(x)\|_p^2 = (v,v)_p = \int_0^L v^2 x\,dx$$

Square of norm of eigenfunctions

$$\|y_n(x)\|_p^2 = \int_0^L \left[\frac{1}{\sqrt{x}} J_1(2\mu_n\sqrt{x})\right]^2 x\,dx = \int_0^L x\,\frac{1}{x} J_1(2\mu_n\sqrt{x})^2\,dx = \int_0^L J_1(2\mu_n\sqrt{x})^2\,dx$$

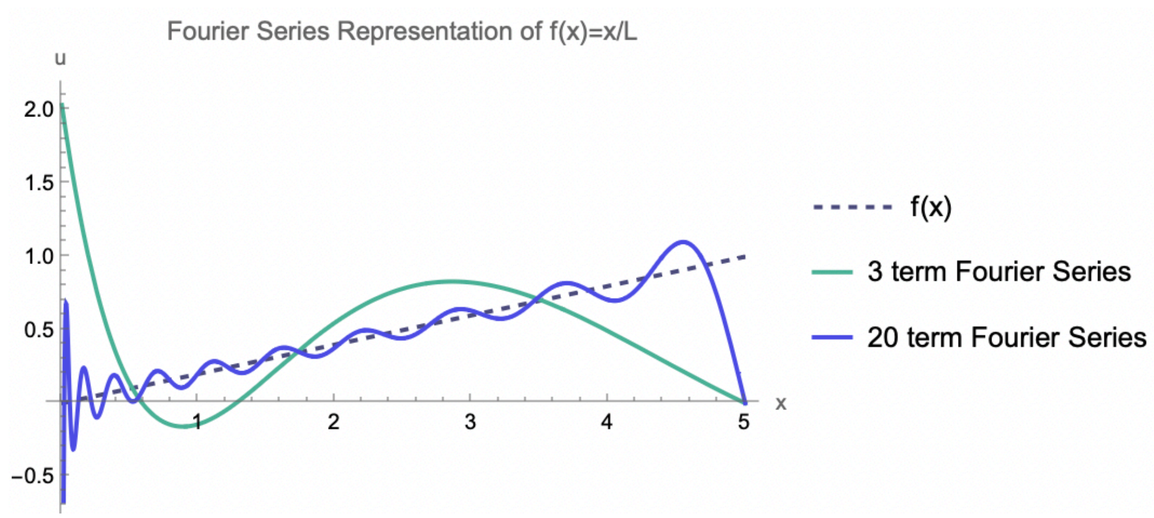
$$\boxed{\|y_n(x)\|_p^2 = \int_0^L J_1(2\mu_n\sqrt{x})^2\,dx}$$

VALIDATION. Represent $f(x) = \frac{x}{L}$ over $[0,L]$

$$\bar{a}_n = \int_0^L f(x)\,y_n(x)\,p(x)\,dx$$

$$f(x) = \sum_{n=1}^{\infty} \bar{a}_n \frac{y_n(x)}{\|y_n\|^2}$$

$$f(x) = \sum_{n=1}^{\infty} \int_0^L f(x)\,y_n(x)\,p(x)\,dx\,\frac{y_n(x)}{\|y_n\|^2}$$



Define the Finite Integral Transform Pair

Direct Transform $\mathcal{J}\{w(x)\} = (w, y_n)_p = \int_0^L w(x) y_n(x)\, x\, dx = \overline{w}_n$

Inverse Transform $\mathcal{J}^{-1}\{\overline{w}_n\} = \sum_{n=1}^{\infty} \overline{w}_n \frac{y_n(x)}{\|y_n\|^2} = w(x)$

where $y_n(x) = \frac{1}{\sqrt{x}} J_1(2\mu_n\sqrt{x})$,

$\|y_n\|^2 = \int_0^L J_1(2\mu_n\sqrt{x})^2\, dx$

Operational Property of the Integral Transform

$$\mathcal{J}\{Lu\} = \mathcal{J}\{x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\}$$

Boundary Conditions $u(0,t) < \infty$ I

$u(L,t) = g(t)$ I

Operational Property for I-I, derived p. 860 of notes $x_1 = 0$, $x_2 = L$

$$\mathcal{J}\{Lu\} = -\mu_n^2\,\overline{u}_n - y_n'(L)r(L)u(L) + y_n'(0)r(0)u(0)$$

$$= -\mu_n^2\,\overline{u}_n - y_n'(L)(L^2)u(L) + y_n'(0)(0^2)u(0) \qquad r(x) = x^2$$

$$= -\mu_n^2\,\overline{u}_n - L^2 y_n'(L)u(L) + 0$$

$$= -\mu_n^2\,\overline{u}_n - L^2 y_n'(L)g(t) \qquad u(L,t) = g(t)$$

$$\boxed{\mathcal{J}\{Lu\} = -\mu_n^2\,\overline{u}_n - L^2 y_n'(L)g(t)}$$

B. Differential Operator t-variable: Recall Laplace Transform

The Laplace Transform:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

C. Solution of the Given Initial-Boundary Value Problem

$$x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = \frac{1}{w^2}\left[\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t}\right] \qquad 0<x<L,\ t>0$$

BOUNDARY CONDITIONS	$u(0,t)<\infty$	$t>0$
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INITIAL CONDITIONS	$u(x,0)=f(x)$	$0\le x\le L$
	$\frac{\partial}{\partial t}u(x,0)=0$	$0\le x\le L$

$$\mathcal{J}\left\{x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\right\} = \mathcal{J}\left\{\frac{1}{w^2}\left[\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t}\right]\right\}$$

$$-\mu_n^2\bar{u}_n - L^2 y_n'(L)g(t) = \frac{1}{w^2}\left[\frac{\partial^2}{\partial t^2}\bar{u}_n + 2\gamma\frac{\partial}{\partial t}\bar{u}_n\right]$$

$$-w^2\mu_n^2\bar{u}_n - w^2L^2 y_n'(L)g(t) = \frac{\partial^2}{\partial t^2}\bar{u}_n + 2\gamma\frac{\partial}{\partial t}\bar{u}_n$$

$$\mathcal{J}\{u(x,t)\} = \bar{u}_n(t)$$

$$\mathcal{J}\{u(x,0)\} = \bar{u}_n(0) = \mathcal{J}\{f(x)\} = \bar{f}_n$$

$$\mathcal{J}\left\{\frac{\partial}{\partial t}u(x,0)\right\} = \frac{\partial}{\partial t}\bar{u}_n(0) = \mathcal{J}\{0\} = 0$$

APPLY THE LAPLACE TRANSFORM

$$\mathcal{L}\{\bar{u}_n\} = U_n \qquad \mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{-w^2\mu_n^2\bar{u}_n - w^2L^2y_n'(L)g(t)\} = \mathcal{L}\left\{\frac{\partial^2\bar{u}_n}{\partial t^2} + 2\gamma\frac{\partial\bar{u}_n}{\partial t}\right\}$$

$$-w^2\mu_n^2 U_n - w^2L^2y_n'(L)G(s) = s^2U(s) - s\bar{u}_n(0) - \frac{\partial}{\partial t}\bar{u}_n(0) + 2\gamma[sU(s) - \bar{u}_n(0)]$$

$$-w^2\mu_n^2 U_n - w^2L^2y_n'(L)G(s) = s^2U(s) - s\bar{f}_n - 0 + 2\gamma[sU(s) - \bar{f}_n]$$

$$-w^2\mu_n^2 U_n - w^2L^2y_n'(L)G(s) = s^2U_n - s\bar{f}_n + 2\gamma sU_n - 2\gamma\bar{f}_n$$

$$s^2U_n + 2\gamma sU_n + w^2\mu_n^2U_n = -w^2L^2y_n'(L)G(s) + s\bar{f}_n + 2\gamma\bar{f}_n$$

$$(s^2 + 2\gamma s + w^2\mu_n^2)U_n = -w^2L^2y_n'(L)G(s) + (s+2\gamma)\bar{f}_n$$

$$U_n = \frac{-w^2L^2y_n'(L)G(s) + (s+2\gamma)\bar{f}_n}{(s^2+2\gamma s+w^2\mu_n^2)}$$

$$s^2+2\gamma s+w^2\mu_n^2 = s^2+2\gamma s+\gamma^2-\gamma^2+w^2\mu_n^2$$
$$= (s+\gamma)^2 + \left(\sqrt{w^2\mu_n^2-\gamma^2}\right)^2$$
$$= (s+\gamma)^2 + \beta_n^2$$

$$\beta_n = \sqrt{w^2\mu_n^2-\gamma^2}$$

$$U_n = \frac{-w^2L^2y_n'(L)G(s) + (s+2\gamma)\bar{f}_n}{(s+\gamma)^2+\beta_n^2}$$

$$U_n = -w^2L^2y_n'(L)\frac{1}{(s+\gamma)^2+\beta_n^2}G(s) + \bar{f}_n\frac{s+\gamma}{(s+\gamma)^2+\beta_n^2} + \bar{f}_n\gamma\frac{1}{(s+\gamma)^2+\beta_n^2}$$

$$U_n = \frac{-w^2L^2y_n'(L)}{\beta_n}\frac{\beta_n}{(s+\gamma)^2+\beta_n^2}G(s) + \bar{f}_n\frac{s+\gamma}{(s+\gamma)^2+\beta_n^2} + \frac{\bar{f}_n\gamma}{\beta_n}\frac{\beta_n}{(s+\gamma)^2+\beta_n^2}$$

$$U_n = \frac{-w^2L^2y_n'(L)}{\beta_n}\mathcal{L}\{e^{-\gamma t}\sin\beta_n t\}\mathcal{L}\{g(t)\} + \bar{f}_n\mathcal{L}\{e^{-\gamma t}\cos\beta_n t\} + \frac{\bar{f}_n\gamma}{\beta_n}\mathcal{L}\{e^{-\gamma t}\sin\beta_n t\}$$

INVERSE LAPLACE TRANSFORM

$$\bar{u}_n = \frac{-w^2L^2y_n'(L)}{\beta_n}\left(e^{-\gamma t}\sin\beta_n t * g(t)\right) + \bar{f}_n e^{-\gamma t}\cos\beta_n t + \frac{\bar{f}_n\gamma}{\beta_n}e^{-\gamma t}\sin\beta_n t$$

$$\bar{u}_n = \frac{-w^2L^2y_n'(L)}{\beta_n}\int_0^t e^{-\gamma(t-\tau)}\sin\beta_n(t-\tau)g(\tau)\,d\tau + \bar{f}_n e^{-\gamma t}\cos\beta_n t + \frac{\bar{f}_n\gamma}{\beta_n}e^{-\gamma t}\sin\beta_n t$$

INVERSE $\mathcal{J}$

$$u(x,t) = \sum_{n=1}^{\infty}\bar{u}_n\frac{y_n(x)}{\|y_n\|^2}$$

Final Expression for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} \bar{u}_n(t) \frac{y_n(x)}{\|y_n\|^2}$$

where $\bar{u}_n(t) = \frac{-w^2 L^2 y_n'(L)}{\beta_n} \int_0^t e^{-\gamma(t-\tau)} \sin\beta_n(t-\tau) g(\tau)\, d\tau + \bar{f}_n e^{-\gamma t} \cos\beta_n t + \frac{\bar{f}_n \gamma}{\beta_n} e^{-\gamma t} \sin\beta_n t,$

$y_n(x) = \frac{1}{\sqrt{x}} J_1(2\mu_n\sqrt{x})$, $\quad\mu_n$: positive roots of $J_1(2\mu_n\sqrt{L})$,

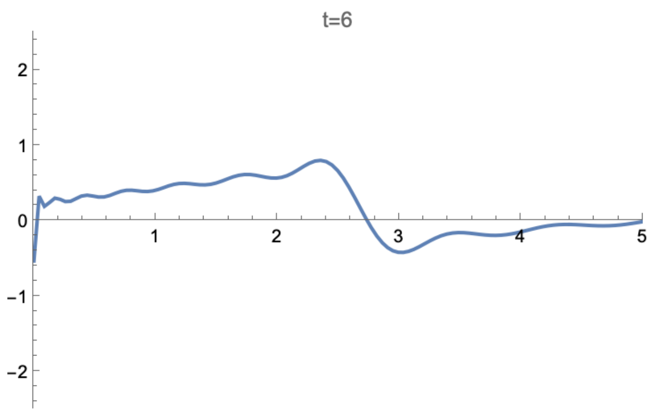
$\beta_n = \sqrt{w^2\mu_n^2 - \gamma^2}$,

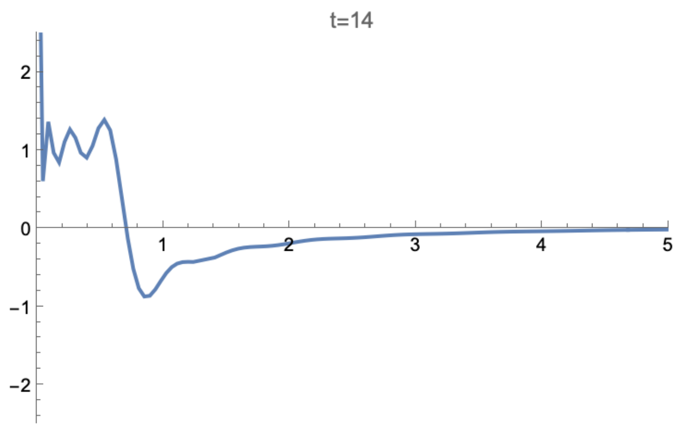
$\bar{f}_n = \int_0^L f(x) y_n(x)\, x\, dx$

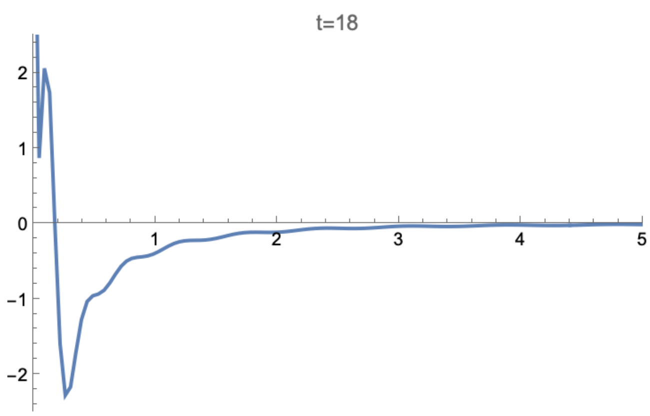
Visualize the solution for the case

$u(x,0) = f(x) = \frac{x}{L}$, $g(t) = 0$, $w = 0.2$, and $\gamma = 0.05$

Sketch the profiles of $u(x,t)$ at $t = 6, 14$, and 18. Use 20 terms in the series solution.

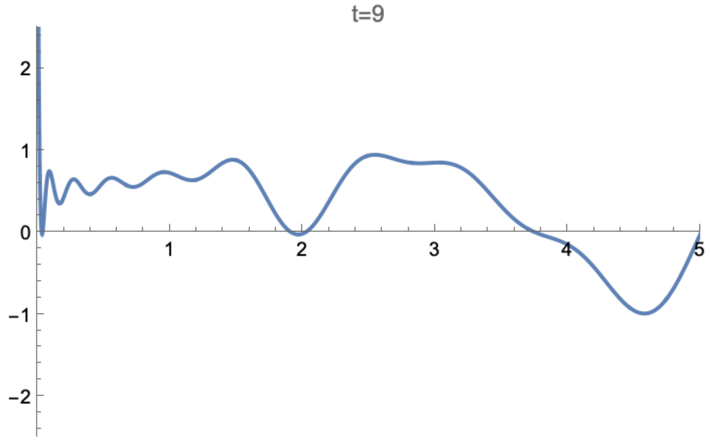


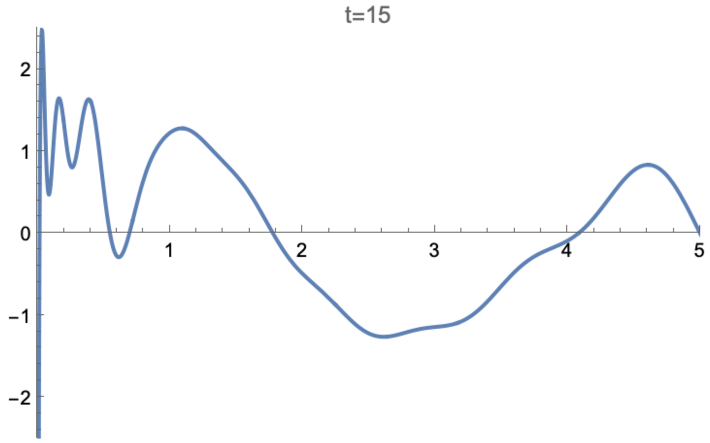




Other cases

$l = 5$
$w = 0.2$
$\gamma = 0.05$
$f(x) = x/L$
$g(t) = \sin(t/2)$





original parameters with custom calculations to show a boat go across a crest