ME-505	**FINAL EXAM**	FALL 2025
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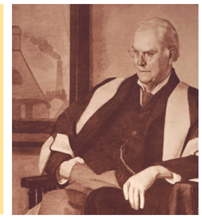
TEAMMATE:	Joseph Morrell	

You may work in teams of two students. However, you must submit your **own** test report and indicate the name of your teammate.

Your submitted solutions should include all your work.
The report must be well-organized, clean, and readable.
This is a firm requirement, and it will be considered in the evaluation and grading of the exam.

1. Submit your solutions through Learning Suite **before midnight on Wednesday, December 17**. Graphics and animations may be submitted separately.
2. **You may not discuss any part of the exam** with anyone except your teammate.
3. Computers and standard math software may be used **only as supportive tools**.
4. Class notes, the class website, and regular math books may be used.
5. No set of rules can cover every situation—please be reasonable and do only what you believe is proper.

> *"One of the most interesting and successful applications of hydrodynamical theory is to the small oscillations, under gravity, of a liquid having a free surface. "*
>
> Sir Horace Lamb *Hydrodynamics*

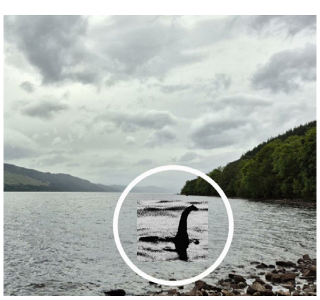


AQUARIUM

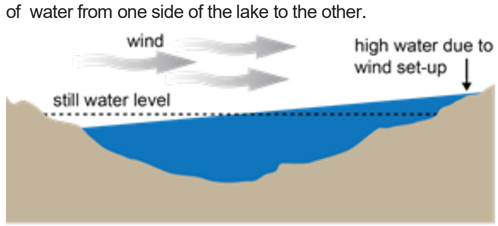
Physical model: Consider a finite volume of water in a container with a rectangular cross-section (like an aquarium). Our object of study is a plane sheet of water of a uniform depth z0. We intend to investigate small oscillations of a liquid with a free surface under gravity. The mathematical description of this motion is similar to the modeling of transverse vibrations of a uniformly stretched membrane by the wave equation, with the parameter $w^2 = gz_0$ and a damping coefficient γ. We casually observe such usual phenomena in everyday life: the bouncing of a water surface in a glass or a bottle; water motion in a shaking bucket (why is it so easy to spill it onto the floor?); radially propagating waves produced by raindrops on a surface of puddles in the street or water pools in the backyard; or, on a more dramatic scale, seiches in lakes or lochs caused by wind forcing or the gravitational influence of the Moon (where, in Loch Ness, some monster might appear).

Yet even in these familiar situations, in many cases it is not obvious to intuitively describe the shape of free surface even approximately: are we seeing running waves or standing ones; is it simply a bouncing of a flat plane (as in one example in Lamb's book), or is the surface evolving into a more complicated shape?

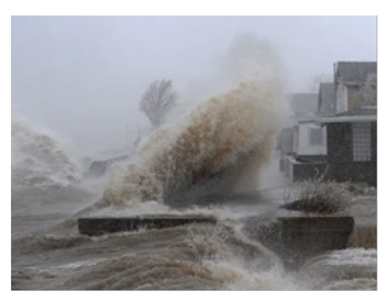
Let us therefore try to formulate and solve a mathematical initial-boundary problem describing the evolution of the shape of the free surface produced by a particular initial deflection from equilibrium.



Lake Erie Seiches: Once the winds relax, the water sloshes back to the end it came from, leading to a back-and-forth movement of water from one side of the lake to the other.



Wind setup is a local rise in water level caused by wind.



Let the elevation of the surface above the plane bottom be $\eta(x,y,t) = z_0 + u(x,y,t)$, where $u(x,y,t)$ represents the local deflection of the free surface from the mean depth z_0 (sufficiently deep).

Use the **Integral Transforms** technique to solve the following initial-boundary value problem for the

unknown function $\qquad u(x,y,t), \quad 0<x<L,\ 0<y<M,\ t>0$

governing equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{w^2}\left(\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t}\right) \qquad \begin{matrix} 0<x<L \\ 0<y<M \end{matrix} \qquad t>0 \qquad (1)$$

Boundary conditions

$$\left[-\frac{\partial u}{\partial x} + hu\right]_{x=0} = f_0(y,t) \qquad 0<y<M \qquad t>0 \qquad (2)$$

$$\left[+\frac{\partial u}{\partial x} + hu\right]_{x=L} = f_L(y,t) \qquad 0<y<M \qquad t>0 \qquad (3)$$

$$\left[-\frac{\partial u}{\partial y} + hu\right]_{y=0} = g_0(x,t) \qquad 0<x<L \qquad t>0 \qquad (4)$$

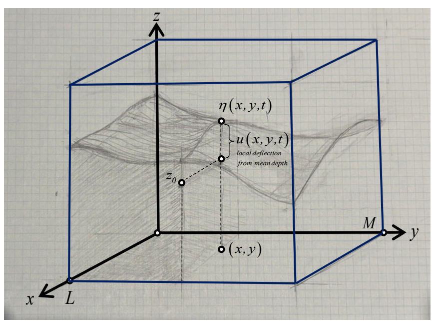
$$\left[+\frac{\partial u}{\partial y} + hu\right]_{y=M} = g_M(x,t) \qquad 0<x<L \qquad t>0 \qquad (5)$$

Initial conditions

$$u(x,y,0) = u_0(x,y) \qquad 0 \le x \le L, \quad 0 \le y \le M \qquad (6)$$

$$\frac{\partial}{\partial t} u(x,y,0) = u_1(x,y) \qquad 0 \le x \le L, \quad 0 \le y \le M \qquad (7)$$

$\eta(x,y,t) = z_0 + u(x,y,t)$ *elevation of the surface above the bottom*



z_0 *is the mean depth*

1) **Identify the differential operators** in the governing equation and choose the appropriate integral transforms to eliminate them. You may use the standard finite Fourier Transform with its already established eigenvalues and eigenfunctions (verify that they are applicable for representing functions by a Fourier series), as well as the Laplace transform. Apply these transforms to reduce the differential equation to an algebraic equation for the transformed unknown function. ***Write this intermediate result.***

2) **Simplify the transformed equation** using the following assumptions:

$$f_0(y,t) = f_L(y,t) = g_0(y,t) = g_M(y,t) = u_1(x,y) = 0$$

and solve for the transformed unknown function. ***Write it.***

3) **Use the inverse integral transforms** to obtain the solution of the given initial value problem. ***Write it***.

4) **Visualize the solution** $u(x, y, t)$ for $L = 3.0, M = 2.0, h = 0.1, w = 0.01, \gamma = 0.001$, $u_0(x,y) = S_0 \dfrac{x+y}{L+M}$, $S_0 = 0.2$.

5) **Sketch the plot** of $u(x, M/2, t)$ at the moments of time $t = 100,\ 200,\ 500,\ 2000$ using 20 terms in the series.

6). **Exercise your creativity** by modifying some aspects of the problem and visualize your new solution.

7) **Make observations and comments** regarding the modelling of tis physical process using this IBVP.

Philippe de Champaigne   Portrait of two men

Portrait of René Descartes

after Frans Hals

Portrait of Blaise Pascal

Philippe de Champaigne

ME EN 505 Final

Benjamin Diehl and Joseph Morrell

Part 1: Intermediate Result

1. Governing equation, domain, boundary/initial conditions

On $0 < x < L$, $0 < y < M$, $t > 0$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{w^2}\left(\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t}\right). \tag{1}$$

Nonhomogeneous Robin boundary conditions:

$$\left[-\frac{\partial u}{\partial x} + hu\right]_{x=0} = f_0(y,t), \qquad 0 < y < M,\ t > 0, \tag{2}$$

$$\left[+\frac{\partial u}{\partial x} + hu\right]_{x=L} = f_L(y,t), \qquad 0 < y < M,\ t > 0, \tag{3}$$

$$\left[-\frac{\partial u}{\partial y} + hu\right]_{y=0} = g_0(x,t), \qquad 0 < x < L,\ t > 0, \tag{4}$$

$$\left[+\frac{\partial u}{\partial y} + hu\right]_{y=M} = g_M(x,t), \qquad 0 < x < L,\ t > 0. \tag{5}$$

Initial conditions:

$$u(x,y,0) = u_0(x,y), \qquad 0 \le x \le L,\ 0 \le y \le M, \tag{6}$$

$$\frac{\partial u}{\partial t}(x,y,0) = u_1(x,y), \qquad 0 \le x \le L,\ 0 \le y \le M. \tag{7}$$

2. Finite Fourier Transform in x (Robin–Robin table entry)

We use the Robin–Robin eigenproblem in x (FFT table, pg. 759 in textbook):

$$X_n''(x) + \mu_n^2 X_n(x) = 0, \qquad \big[-X_n'(0) + hX_n(0)\big] = 0, \qquad \big[+X_n'(L) + hX_n(L)\big] = 0. \tag{8}$$

From the table (specialized to $H_1 = H_2 = h$), the eigenvalues $\mu_n > 0$ satisfy

$$(h^2 - \mu^2)\sin(\mu L) + 2h\mu\cos(\mu L) = 0. \tag{9}$$

The associated eigenfunction may be written as

$$X_n(x) = \mu_n\cos(\mu_n x) + h\sin(\mu_n x). \tag{10}$$

The norm (as recorded from the table) is

$$\|X_n\|^2 = \frac{\mu_n^2 + h^2}{2}\left(L + \frac{h}{\mu_n^2 + h^2}\right) + \frac{h}{2}. \tag{11}$$

Define the finite Fourier transform (FinFT) in x:

$$\bar{u}_n(y,t) = \mathcal{F}_x\{u\} = \int_0^L u(x,y,t)\,X_n(x)\,dx, \qquad u(x,y,t) = \sum_{n=1}^{\infty}\bar{u}_n(y,t)\,\frac{X_n(x)}{\|X_n\|^2}. \tag{12}$$

Operational property (2nd derivative in x)

Using the tabulated operational property (and keeping nonhomogeneous BC forcing):

$$\int_0^L \frac{\partial^2 u}{\partial x^2}(x,y,t)\, X_n(x)\, dx = -\mu_n^2\, \bar{u}_n(y,t) + f_0(y,t)\, X_n(0) + f_L(y,t)\, X_n(L). \tag{13}$$

Here the boundary terms match the nonhomogeneous Robin data in (2)–(3). This is simplified from the table because h $= k_1 = k_2 = 1$.

Also,

$$\int_0^L \frac{\partial^2 u}{\partial y^2}(x,y,t)\, X_n(x)\, dx = \frac{\partial^2 \bar{u}_n}{\partial y^2}(y,t), \qquad \int_0^L \frac{\partial u}{\partial t}(x,y,t)\, X_n(x)\, dx = \frac{\partial \bar{u}_n}{\partial t}(y,t), \tag{14}$$

and similarly for $\partial^2/\partial t^2$ and $\partial/\partial t$.

Applying $\mathcal{F}_x$ to (1) yields a PDE in (y,t):

$$-\mu_n^2 \bar{u}_n(y,t) + f_0(y,t) X_n(0) + f_L(y,t) X_n(L) + \frac{\partial^2 \bar{u}_n}{\partial y^2}(y,t) = \frac{1}{w^2}\left(\frac{\partial^2 \bar{u}_n}{\partial t^2}(y,t) + 2\gamma \frac{\partial \bar{u}_n}{\partial t}(y,t)\right). \tag{15}$$

Transforming boundary and initial conditions with a FinFT in x

Define the x-Finite Fourier Transform (FinFT) with Robin–Robin eigenfunctions $X_n(x)$:

$$\bar{u}_n(y,t) = \mathcal{F}_x\{u\} = \int_0^L u(x,y,t)\, X_n(x)\, dx. \tag{16}$$

Because the transform is taken in x only, any condition that is a function of (x,t) at fixed y (such as the y-boundary data at $y=0$ and $y=M$) transforms directly by integrating against $X_n(x)$.

Initial conditions. Applying $\mathcal{F}_x$ to the initial conditions (6)–(7) gives

$$\bar{u}_n(y,0) = \int_0^L u_0(x,y)\, X_n(x)\, dx \;\equiv\; \bar{u}_{n,0}(y), \tag{17}$$

$$\left.\frac{\partial \bar{u}_n}{\partial t}(y,t)\right|_{t=0} = \int_0^L u_1(x,y)\, X_n(x)\, dx \;\equiv\; \bar{u}_{n,1}(y). \tag{18}$$

y-boundary conditions. For the Robin conditions (4)–(5),

$$\left[-\frac{\partial u}{\partial y} + hu\right]_{y=0} = g_0(x,t), \qquad \left[+\frac{\partial u}{\partial y} + hu\right]_{y=M} = g_M(x,t), \tag{19}$$

their x-FinFTs are

$$g_{n,0}(t) \equiv \mathcal{F}_x\{g_0(\cdot,t)\} = \int_0^L g_0(x,t)\, X_n(x)\, dx, \tag{20}$$

$$g_{n,M}(t) \equiv \mathcal{F}_x\{g_M(\cdot,t)\} = \int_0^L g_M(x,t)\, X_n(x)\, dx. \tag{21}$$

Equivalently, transforming the *left-hand sides* of the boundary operators (valid since $\partial/\partial y$ commutes with $\int_0^L (\cdot)\, X_n\, dx$),

$$\int_0^L \left[-\frac{\partial u}{\partial y}(x,0,t) + hu(x,0,t) \right] X_n(x)\, dx = g_{n,0}(t), \tag{22}$$

$$\int_0^L \left[+\frac{\partial u}{\partial y}(x,M,t) + hu(x,M,t) \right] X_n(x)\, dx = g_{n,M}(t). \tag{23}$$

These transformed functions $g_{n,0}(t)$ and $g_{n,M}(t)$ appear as forcing terms in the y-transformed operational property for $\partial^2/\partial y^2$.

3. Finite Fourier Transform in y (Robin–Robin table entry)

We now apply a FinFT in y using the Robin–Robin eigenproblem on $0 < y < M$:

$$Y_m''(y) + \lambda_m^2 Y_m(y) = 0, \qquad \left[-Y_m'(0) + hY_m(0) \right] = 0, \qquad \left[+Y_m'(M) + hY_m(M) \right] = 0. \tag{24}$$

Eigenvalues $\lambda_m > 0$ satisfy

$$(h^2 - \lambda^2)\sin(\lambda M) + 2h\lambda\cos(\lambda M) = 0, \tag{25}$$

with eigenfunction

$$Y_m(y) = \lambda_m \cos(\lambda_m y) + h\sin(\lambda_m y), \tag{26}$$

and norm (table form)

$$\|Y_m\|^2 = \frac{\lambda_m^2 + h^2}{2}\left(M + \frac{h}{\lambda_m^2 + h^2} \right) + \frac{h}{2}. \tag{27}$$

Define the y-FinFT of $\bar{u}_n(y,t)$:

$$\bar{u}_{n,m}(t) = \mathcal{F}_y\{\bar{u}_n\} = \int_0^M \bar{u}_n(y,t)\, Y_m(y)\, dy, \qquad \bar{u}_n(y,t) = \sum_{m=1}^{\infty} \bar{u}_{n,m}(t)\, \frac{Y_m(y)}{\|Y_m\|^2}. \tag{28}$$

Operational property (2nd derivative in y)

The table operational property gives, including the nonhomogeneous Robin data in y:

$$\int_0^M \frac{\partial^2 \bar{u}_n}{\partial y^2}(y,t)\, Y_m(y)\, dy = -\lambda_m^2\, \bar{u}_{n,m}(t) + g_{n,0}(t)\, Y_m(0) + g_{n,M}(t)\, Y_m(M), \tag{29}$$

where the transformed boundary data are

$$g_{n,0}(t) = \int_0^L g_0(x,t)\, X_n(x)\, dx, \qquad g_{n,M}(t) = \int_0^L g_M(x,t)\, X_n(x)\, dx. \tag{30}$$

Similarly define the y-transforms of the x-boundary data:

$$f_{0,m}(t) = \int_0^M f_0(y,t)\, Y_m(y)\, dy, \qquad f_{L,m}(t) = \int_0^M f_L(y,t)\, Y_m(y)\, dy. \tag{31}$$

Applying $\mathcal{F}_y$ to (15) gives a forced ODE in time for $\bar{u}_{n,m}(t)$:

$$-\mu_n^2\, \bar{u}_{n,m}(t) \;+\; X_n(0)\, f_{0,m}(t) + X_n(L)\, f_{L,m}(t) \;-\; \lambda_m^2\, \bar{u}_{n,m}(t) \;+\; Y_m(0)\, g_{n,0}(t) + Y_m(M)\, g_{n,M}(t)$$
$$= \frac{1}{w^2}\left(\frac{d^2\bar{u}_{n,m}}{dt^2} + 2\gamma \frac{d\bar{u}_{n,m}}{dt} \right). \tag{32}$$

Define

$$\sigma_{nm} = \mu_n^2 + \lambda_m^2. \tag{33}$$

Transforming initial conditions with a FinFT in y

After applying $\mathcal{F}_x$ one obtains $\bar{u}_n(y,t)$. The y-Finite Fourier Transform (FinFT) with Robin–Robin eigenfunctions $Y_m(y)$ is defined by

$$\bar{u}_{n,m}(t) = \mathcal{F}_y\{\bar{u}_n\} = \int_0^M \bar{u}_n(y,t)\, Y_m(y)\, dy. \tag{34}$$

Applying $\mathcal{F}_y$ to the x-transformed initial conditions (17)–(18) yields the modal initial data in (n,m):

$$\bar{u}_{n,m}(0) = \int_0^M \bar{u}_n(y,0)\, Y_m(y)\, dy = \int_0^M \left[\int_0^L u_0(x,y)\, X_n(x)\, dx\right] Y_m(y)\, dy \;\equiv\; \bar{u}_{n,m,0}, \tag{35}$$

$$\left.\frac{d\bar{u}_{n,m}}{dt}\right|_{t=0} = \int_0^M \left.\frac{\partial \bar{u}_n}{\partial t}(y,t)\right|_{t=0} Y_m(y)\, dy = \int_0^M \left[\int_0^L u_1(x,y)\, X_n(x)\, dx\right] Y_m(y)\, dy \;\equiv\; \bar{u}_{n,m,1}. \tag{36}$$

These coefficients provide the initial conditions for the Laplace-transformed ODE in t for each mode (n,m).

4. Laplace transform in t

Let

$$U(s) = \mathcal{L}\{\bar{u}_{n,m}(t)\} = \int_0^\infty \bar{u}_{n,m}(t) e^{-st}\, dt. \tag{37}$$

Also use

$$\mathcal{L}\left\{\frac{d\bar{u}_{n,m}}{dt}\right\} = sU(s) - \bar{u}_{n,m}(0), \qquad \mathcal{L}\left\{\frac{d^2\bar{u}_{n,m}}{dt^2}\right\} = s^2U(s) - s\bar{u}_{n,m}(0) - \dot{\bar{u}}_{n,m}(0). \tag{38}$$

Denote

$$\bar{u}_{n,m,0} = \bar{u}_{n,m}(0), \qquad \bar{u}_{n,m,1} = \left.\frac{d\bar{u}_{n,m}}{dt}\right|_{t=0}. \tag{39}$$

Taking the Laplace transform of (32) results in

$$-\mu_n^2\, U(s) \;-\; \lambda_m^2\, U(s) \;+\; X_n(0)\, F_{0,m}(s) \;+\; X_n(L)\, F_{L,m}(s) \;+\; Y_m(0)\, G_{n,0}(s) \;+\; Y_m(M)\, G_{n,M}(s)$$
$$= \frac{1}{w^2}\Big(s^2U(s) - s\,\bar{u}_{n,m,0} - \bar{u}_{n,m,1} + 2\gamma\,[\,sU(s) - \bar{u}_{n,m,0}\,]\Big). \tag{40}$$

and solving algebraically for $U(s)$ gives

$$U(s) = \frac{w^2\Big(X_n(0)\, F_{0,m}(s) + X_n(L)\, F_{L,m}(s) + Y_m(0)\, G_{n,0}(s) + Y_m(M)\, G_{n,M}(s)\Big) + \bar{u}_{n,m,0}(s+2\gamma) + \bar{u}_{n,m,1}}{w^2\sigma_{nm} + s^2 + 2\gamma s}, \tag{41}$$

where $F_{0,m}(s) = \mathcal{L}\{f_{0,m}(t)\}$, $F_{L,m}(s) = \mathcal{L}\{f_{L,m}(t)\}$, and similarly for $G_{n,0}(s)$ and $G_{n,M}(s)$. σ is defined in (33).

Equation (40) or (41) are the solution to Part 1.

Part 2: Homogeneous Robin boundary conditions and zero initial time-derivative

We now impose *homogeneous* Robin boundary conditions on all sides:

$$f_0 = f_L = g_0 = g_M = 0 \quad \Longrightarrow \quad F_{0,m} = F_{L,m} = G_{n,0} = G_{n,M} = 0. \tag{42}$$

In addition, we assume a zero initial time-derivative,

$$u_1(x, y) = 0 \quad \Longrightarrow \quad \bar{u}_{n,m,1} = 0. \tag{43}$$

Under these assumptions, the general Laplace-domain expression reduces to

$$U(s) = \bar{u}_{n,m,0} \, \frac{s + 2\gamma}{s^2 + 2\gamma s + w^2 \sigma_{nm}}, \qquad \sigma_{nm} = \mu_n^2 + \lambda_m^2 \tag{44}$$

where $\bar{u}_{n,m,0}$ is defined in (35).

This is the solution for Part 2.

Part 3: Inverse Integral Transforms

1. Inverse Laplace Transform

Completing the square in the denominator of (44) gives

$$s^2 + 2\gamma s + w^2 \sigma_{nm} = (s + \gamma)^2 + \left(w^2 \sigma_{nm} - \gamma^2\right). \tag{45}$$

Define the constant

$$A_{nm} = \sqrt{w^2 \sigma_{nm} - \gamma^2}. \tag{46}$$

Then (44) may be rewritten as

$$U(s) = \bar{u}_{n,m,0} \left[\frac{s + \gamma}{(s + \gamma)^2 + A_{nm}^2} + \frac{\gamma}{(s + \gamma)^2 + A_{nm}^2} \right]. \tag{47}$$

Using the standard Laplace transform pairs

$$\mathcal{L}^{-1}\left\{ \frac{s}{s^2 + A^2} \right\} = \cos(At), \qquad \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + A^2} \right\} = \frac{1}{A} \sin(At),$$

together with the shift theorem

$$\mathcal{L}^{-1}\{F(s + \gamma)\} = e^{-\gamma t} f(t),$$

the inverse Laplace transform yields

$$\bar{u}_{n,m}(t) = \bar{u}_{n,m,0} \, e^{-\gamma t} \left[\cos(A_{nm} t) + \frac{\gamma}{A_{nm}} \sin(A_{nm} t) \right]. \tag{48}$$

2. Inverse Finite Fourier Transforms

The physical solution is recovered by successively inverting the finite Fourier transforms in y and x. First, invert the transform in y:

$$\bar{u}_n(y,t) = \sum_{m=1}^{\infty} \bar{u}_{n,m}(t)\, \frac{Y_m(y)}{\|Y_m\|^2}, \tag{49}$$

followed by inversion in x:

$$u(x,y,t) = \sum_{n=1}^{\infty} \bar{u}_n(y,t)\, \frac{X_n(x)}{\|X_n\|^2}. \tag{50}$$

Substituting the modal time dependence

$$\bar{u}_{n,m}(t) = \bar{u}_{n,m,0}\, e^{-\gamma t} \left[\cos(A_{nm}t) + \frac{\gamma}{A_{nm}} \sin(A_{nm}t)\right], \qquad A_{nm} = \sqrt{w^2(\mu_n^2 + \lambda_m^2) - \gamma^2},$$

into (49)–(50) yields the final double-series representation

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{u}_{n,m,0}\, e^{-\gamma t} \left[\cos(A_{nm}t) + \frac{\gamma}{A_{nm}} \sin(A_{nm}t)\right] \frac{Y_m(y)}{\|Y_m\|^2} \frac{X_n(x)}{\|X_n\|^2}, \tag{51}$$

again, where $\bar{u}_{n,m,0}$ is defined by (35).

This is the solution for Part 3.

Part 4: Visualize the Solution

MATLAB was used to visualize the solution in (51) for the following parameters: $L = 3.0, M = 2.0, h = 0.1, w = 0.01, \gamma = 0.001, u_0(x,y) = S_0 \frac{x+y}{L+M}, S_0 = 0.2$. 20 terms were used for n and m.

The file *Part_4_visualized.gif* shows a 3D surface from time $t = 0$ to $t = 2000$ with 200 time steps. Here the axes are allowed independent scaling to show how much the wave changes in the u-axis.

Part 5: Plot of $u(x, M/2, t)$

A plot of $u(x, M/2, t)$ for $t = 100, 200, 500, 2000$ using 20 terms for n and m is shown in Figure 1. A line for $t = 0$ was also shown.

Part 6: Creative changes to the Problem & Visualization

We changed the initial condition to be a gaussian centered at $x = L/2$ and $y = M/2$, where L was changed 10, and M stayed at 2.

The initial condition is therefore

$$u(x,y,0) = S_0 \exp -\frac{(x - L/2)^2 + (y - M/2)^2}{2 * (0.05 * L)^2}, \tag{52}$$

where S_0 was increased to 1.5.

The file *Part_6_visualized_2.gif* shows the surface for these conditions using 20 terms for n and m from time $t = 0$ to $t = 2000$ with 200 time steps. The x, y, and u axes were set to the same scale to see it better.

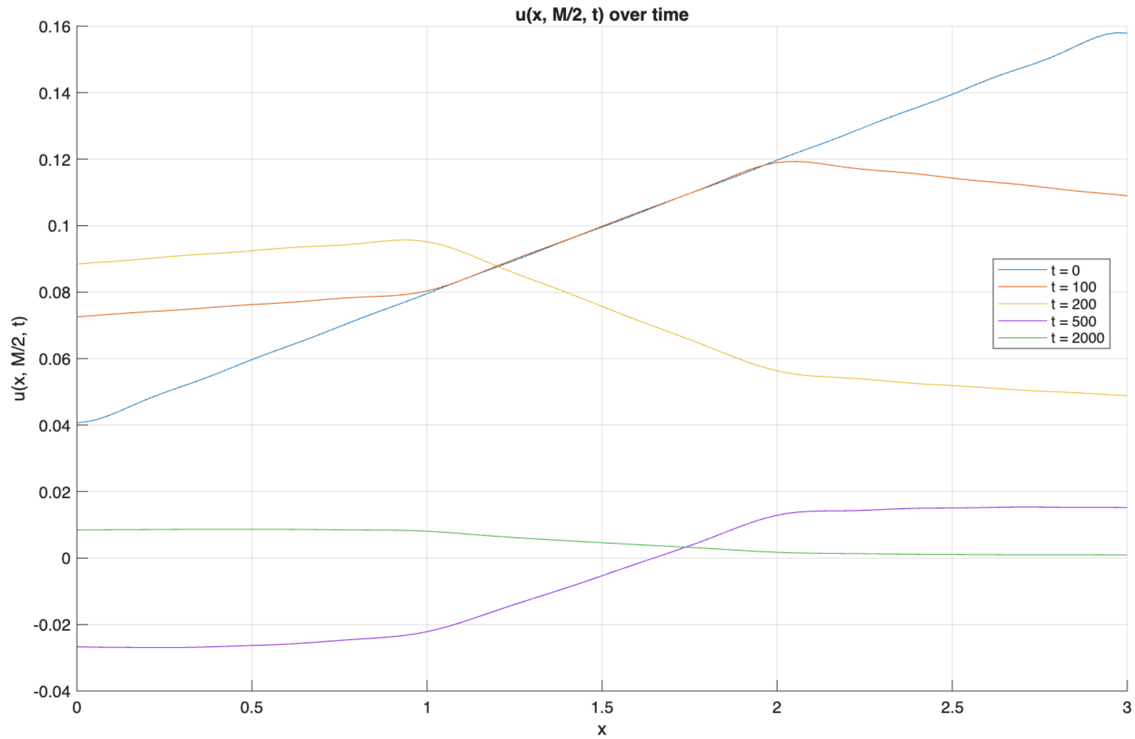


Figure 1: A plot of $u(x, M/2, t)$ for $t = 0, 100, 200, 500, 2000$ using 20 terms for n and m.

Part 7: Observations and Comments

Comments on Original IBV Problem

For Parts 1-3, it would have become near impossible to solve for $u(x, y, t)$ if simplifications of $f_0 = f_L = g_0 = g_M = 0$ were not made. Taking a Finite Fourier Transform in x and y was pretty straight-forward, with the biggest technical challenge being solving for the eigenvalues of the functions, as it is important to not skip any eigenvalues. If $L = M$, further simplifications could be made as $\mu_n = \lambda_m$.

Even with simplifications, inverting the Laplace space equation $U(s)$ was difficult, and took up a lot of scratch paper to find the right method. From there, the inverse Finite Fourier Transforms were easy to write.

For Part 4, at $x = 0$ and $y = 0$, $u = 0$, and then increases linearly in x and y to a maximum value of 0.2 at $(x = L, y = M)$. When time starts, this raised portion of the surface slides down the sides of the container. The wave propagates, and reflects off the walls of the square containers, creating an interesting wavefront, parallel to the container walls in all dimensions.

As time increases, there were times where the entire surface had a negative value for u (remembering $u(x, y, t)$ represents the local deflection of the free surface from the mean depth z_0). This is physically impossible for water, as water is an incompressible fluid. Perhaps further constraints could be added to ensure conservation of mass.

Surprisingly, 20 terms in each series appears to be sufficient to accurately simulate the IBVP.

Comments on Creative IBV Problem

Implementing a gaussian centered around $L/2$ and $M/2$ was simple, all it required was changing the initial conditions as given in (52). We decided to center this gaussian to test our code, as each quadrant of the container should be symmetric. We found this was the case.

By lengthening the container, we were able to easily see the wave propagate the length of the container, and reflect off the end walls. All the while, the wave reflected off the two longer walls, and continually made some interesting shapes in the center.

It is probable that the same surface-volume limitation of this model exists in our modified IBVP, but it was less obvious as the x-y space was bigger, and the wave deflections were more complex.

Conclusion

Overall, this was an exciting problem to solve, and a very fun to play with. We were able to optimize the code in a way where we could made complex changes and it would only take a few seconds to run and generate an animation.