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unknown function $u(x,y,t), \qquad 0 < x < L, 0 < y < M, t > 0$

governing equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{w^2}\left(\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t}\right) \qquad 0 < x < L,\quad 0 < y < M \qquad t > 0 \qquad (1)$$

Boundary conditions

$$\left[-\frac{\partial u}{\partial x} + hu\right]_{x=0} = f_0(y,t) \qquad 0 < y < M \qquad t > 0 \qquad (2)$$

$$\left[+\frac{\partial u}{\partial x} + hu\right]_{x=L} = f_L(y,t) \qquad 0 < y < M \qquad t > 0 \qquad (3)$$

$$\left[-\frac{\partial u}{\partial y} + hu\right]_{y=0} = g_0(x,t) \qquad 0 < x < L \qquad t > 0 \qquad (4)$$

$$\left[+\frac{\partial u}{\partial y} + hu\right]_{y=M} = g_M(x,t) \qquad 0 < x < L \qquad t > 0 \qquad (5)$$

Initial conditions

$$u(x,y,0) = u_0(x,y) \qquad 0 \le x \le L, \qquad 0 \le y \le M \qquad (6)$$

$$\frac{\partial}{\partial t}u(x,y,0) = u_1(x,y) \qquad 0 \le x \le L, \qquad 0 \le y \le M \qquad (7)$$

1) **Identify the differential operators** in the governing equation and choose the appropriate integral transforms to eliminate them. You may use the standard finite Fourier Transform with its already established eigenvalues and eigenfunctions (verify that they are applicable for representing functions by a Fourier Series), as well as the Laplace transform. Apply these transforms to reduce the differential equation to an algebraic equation for the transformed unknown function. ***Write this intermediate result***.

The two differential operators used in the governing equation are the Laplacian (∇^2) in the spatial domain:

$$L_x u \equiv \frac{\partial^2 u}{\partial x^2}, \qquad and \qquad L_y u \equiv \frac{\partial^2 u}{\partial y^2}$$

and a time operator on the right-hand side of the equation.

Since there are two spatial variables I will use the Finite Fourier Transform twice to eliminate both x and y.

The Sturm-Liouville problem for the first operator is:

$$X'' + \lambda^2 X = 0$$

With the boundary conditions:

$$[-X' + hX]_{x=0} = 0$$

$$[+X' + hX]_{x=L} = 0$$

These boundary conditions are Robin-Robin, so the corresponding eigenvalues (μ_n) are the positive roots of

$$(h^2 - \lambda^2)\sin\lambda L + 2h\lambda\cos\lambda L = 0$$

with eigenfunctions:

$$X_n = \lambda_n \cos\lambda_n x + h\sin\lambda_n x$$

and operational property:

$$\int_0^L \left[\frac{\partial^2 u(x,y,t)}{\partial x^2}\right] X_n(x)dx = f_L X_n(L) + f_0 X_n(0) - \lambda_n^2\, \bar{u}_n(y,t)$$

The Sturm-Liouville problem for the second operator is of the same form:

$$Y'' + \mu^2 Y = 0$$

With the boundary conditions:

$$[-Y' + hY]_{y=0} = 0$$

$$[+Y' + hY]_{y=M} = 0$$

These boundary conditions are Robin-Robin, so the corresponding eigenvalues (μ_n) are the positive roots of

$$(h^2 - \mu^2)\sin\mu M + 2h\mu\cos\mu M = 0$$

with eigenfunctions:

$$Y_m = \mu_m \cos \mu_m y + h \sin \mu_m y$$

and operational property:

$$\int_0^M \left[\frac{\partial^2 u(x,y,t)}{\partial y^2}\right] Y_m(y)dy = g_M Y_m(M) + g_0 Y_m(0) - \mu_m^2\, \bar{u}_m(x,t)$$

Combining the two operation properties we get the double-transformed equation:

$$\iint\limits_{0\,0}^{M\,L} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] X_n(x)Y_m(y)dxdy = \bar{\bar{u}}_{n,m}(t)$$

Applying these transforms to the governing equation gives:

$$\frac{1}{w^2}\left(\frac{\partial^2 \bar{\bar{u}}_{n,m}}{\partial t^2} + 2\gamma \frac{\partial \bar{\bar{u}}_{n,m}}{\partial t}\right) + (\lambda_n^2 + \mu_m^2)\bar{\bar{u}}_{n,m} =$$

$$X_n(0)\bar{f}_{0,m}(t) + X_n(L)\bar{f}_{L,m}(t) + Y_m(0)g_{0,t}(t) + Y_m(M)g_{M,t}(t)$$

Justify using it or something…

Now I'll apply the Laplace transform to eliminate the time derivatives.

Starting on the left-hand side:

$$\mathcal{L}\left\{\frac{1}{w^2}\left(\frac{\partial^2 \bar{\bar{u}}_{n,m}}{\partial t^2} + 2\gamma \frac{\partial \bar{\bar{u}}_{n,m}}{\partial t}\right) + (\lambda_n^2 + \mu_m^2)\bar{\bar{u}}_{n,m}\right\}$$
$$= \frac{1}{w^2}\left[\left(s^2 U_{n,m} - s\bar{\bar{u}}_{n,m}(0) - \bar{\bar{u}}'_{n,m}(0)\right) + 2\gamma\left(sU_{n,m} - \bar{\bar{u}}_{n,m}(0)\right)\right] + (\lambda_n^2 + \mu_m^2)U_{n,m}$$

And the right-hand side:

$$\mathcal{L}\{X_n(0)\bar{f}_{0,m}(t) + X_n(L)\bar{f}_{L,m}(t) + Y_m(0)g_{0,t}(t) + Y_m(M)g_{M,t}(t)\}$$
$$= X_n(0)\hat{\bar{f}}_{0,m}(s) + X_n(L)\hat{\bar{f}}_{L,m}(s) + Y_m(0)\hat{g}_{0,n}(s) + Y_m(M)\hat{g}_{M,n}(s)$$

Solving for the Transformed function, $U_{n,m}$, gives:

$$U_{n,m} = \frac{(s+2\gamma)\bar{\bar{u}}_{n,m}(0)}{s^2 + 2\gamma s + w^2(\lambda_n^2 + \mu_m^2)}$$
$$+ \frac{w^2\left[X_n(0)\hat{\bar{f}}_{0,m}(s) + X_n(L)\hat{\bar{f}}_{L,m}(s) + Y_m(0)\hat{g}_{0,n}(s) + Y_m(M)\hat{g}_{M,n}(s)\right]}{s^2 + 2\gamma s + w^2(\lambda_n^2 + \mu_m^2)}$$

2) Simplify the transformed equation using the following assumptions:

$f_0(y,t) = f_L(y,t) = g_0(y,t) = g_M(y,t) = u_1(x,y) = 0$

And solve for the transformed unknown function. ***Write it***.

Substituting 0 for the above functions simplifies the transformed equation to:

$$U_{n,m} = \bar{\bar{u}}_{n,m}(0)\frac{s+2\gamma}{s^2+2\gamma s+w^2\left(\mu_n^2+\mu_m^2\right)}$$

3) Use the inverse integral transforms to obtain the solution of the given initial value problem. ***Write it***.

First, I'll find the inverse Laplace transform by completing the square in the denominator and separating the numerator using partial fractions:

$$\bar{\bar{u}}_{n,m}(0)\frac{s+2\gamma}{s^2+2\gamma s+w^2\left(\lambda_n^2+\mu_m^2\right)} = \bar{\bar{u}}_{n,m}(0)\left[\frac{s+\gamma}{(s+\gamma)^2+\beta_{n,m}^2}+\frac{\gamma}{(s+\gamma)^2+\beta_{n,m}^2}\right]$$

where

$$\beta_{n,m}^2 = w^2(\lambda_n^2+\mu_m^2)-\gamma^2$$

This gives us the inverse Laplace transform of

$$\mathcal{L}^{-1}\{U_{n,m}(s)\} = \bar{\bar{u}}_{n,m}(0)\left[e^{-\gamma t}\cos(\beta_{n,m}t)+\frac{\gamma}{\beta_{n,m}}e^{-\gamma t}\sin(\beta_{n,m}t)\right]$$

Now I'll take the inverse Finite Fourier Transform in y:

$$\begin{aligned}\bar{u}_n(y,t) &= \sum_{m=1}^{\infty}\bar{\bar{u}}_{n,m}(t)\frac{Y_m(y)}{\|Y_m(y)\|^2}\\ &= \sum_{m=1}^{\infty}\bar{\bar{u}}_{n,m}(0)\left[e^{-\gamma t}\cos(\beta_{n,m}t)+\frac{\gamma}{\beta_{n,m}}e^{-\gamma t}\sin(\beta_{n,m}t)\right]\frac{Y_m(y)}{\|Y_m(y)\|^2}\end{aligned}$$

And the inverse Finite Fourier Transform in x:

$$u(x,y,t) = \sum_{n=1}^{\infty}\bar{u}_n(t)\frac{X_n(x)}{\|X_n(x)\|^2}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{\bar{u}}_{n,m}(0)\left[e^{-\gamma t}\cos(\beta_{n,m}t)+\frac{\gamma}{\beta_{n,m}}e^{-\gamma t}\sin(\beta_{n,m}t)\right]\frac{X_n(x)}{\|X_n(x)\|^2}\frac{Y_m(y)}{\|Y_m(y)\|^2}$$

which can be simplified to:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{n,m}X_n(x)Y_m(y)e^{-\gamma t}\left[\cos(\beta_{n,m}t)+\frac{\gamma}{\beta_{n,m}}\sin(\beta_{n,m}t)\right]$$

where

$$A_{n,m} = \frac{\bar{\bar{u}}_{n,m}(0)}{\|X_n(x)\|^2 \|Y_m(y)\|^2},$$

$$\bar{\bar{u}}_{n,m}(0) = \int_0^M \int_0^L u_0(x,y) X_n(x) Y_m(y) dx\, dy,$$

$$\beta_{n,m} = \sqrt{w^2(\lambda_n^2 + \mu_m^2) - \gamma^2}$$

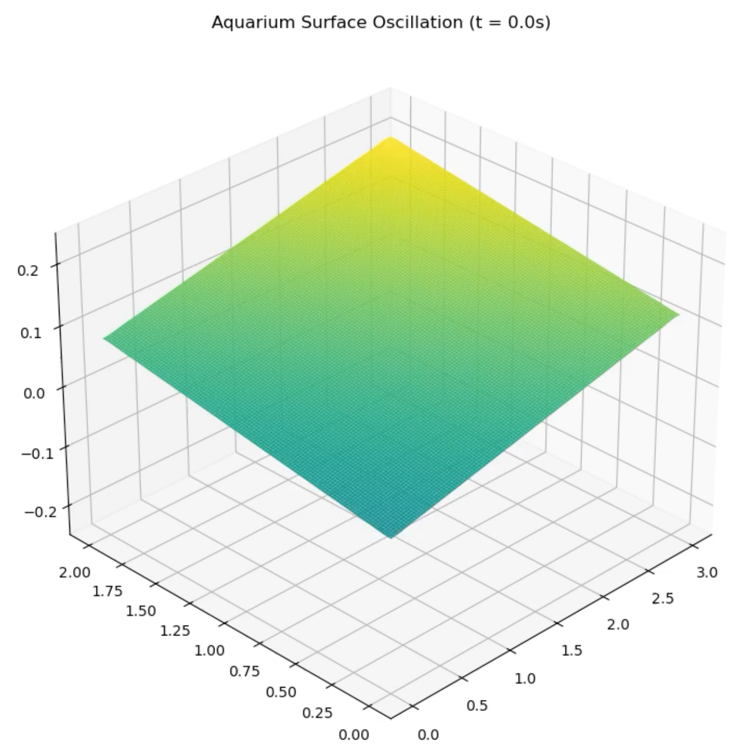
λ_n and μ_m are the positive roots of the characteristic equations:

$$(h^2 - \lambda^2)\sin(\lambda L) + 2h\lambda\cos(\lambda L) = 0$$

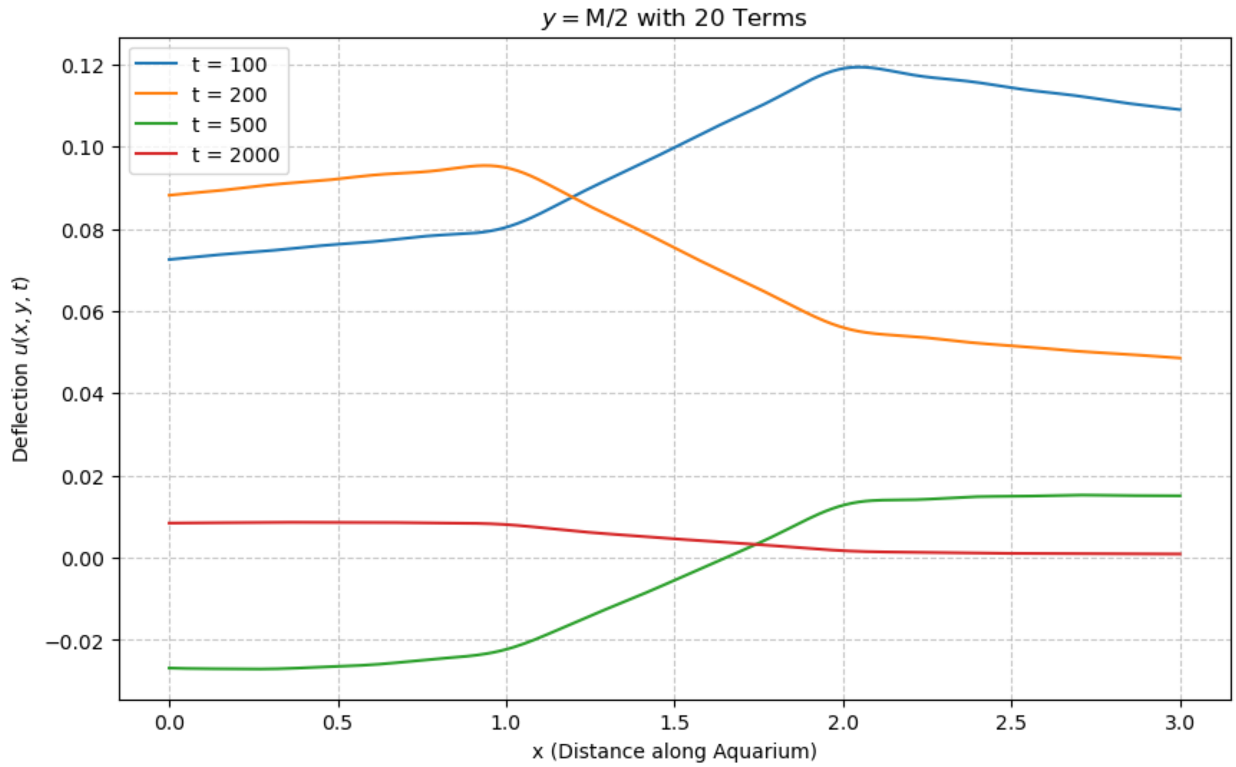
and

$$(h^2 - \mu^2)\sin(\mu M) + 2h\mu\cos(\mu M) = 0$$

4) **Visualize the solution** $u(x,y,t)$ for $L = 3.0, M = 2.0, h = 0.1, w = 0.01, \gamma = 0.001$, $u_0(x,y) = S_0 \frac{x+y}{L+M}, S_0 = 0.2$.

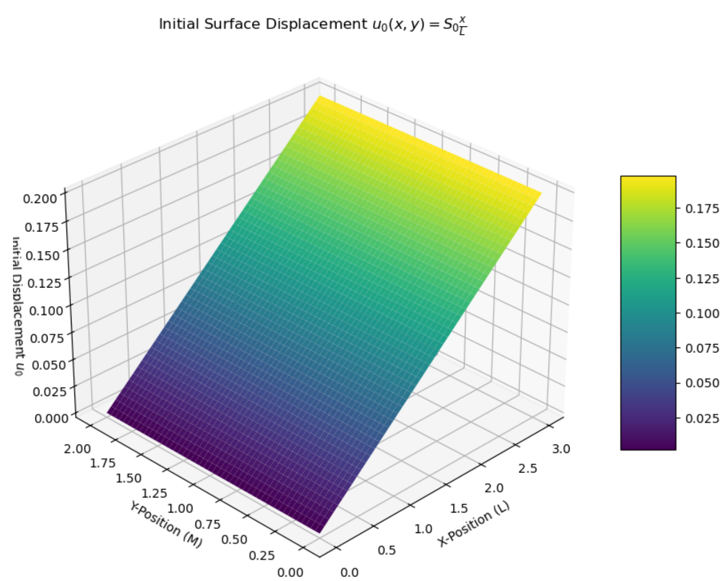


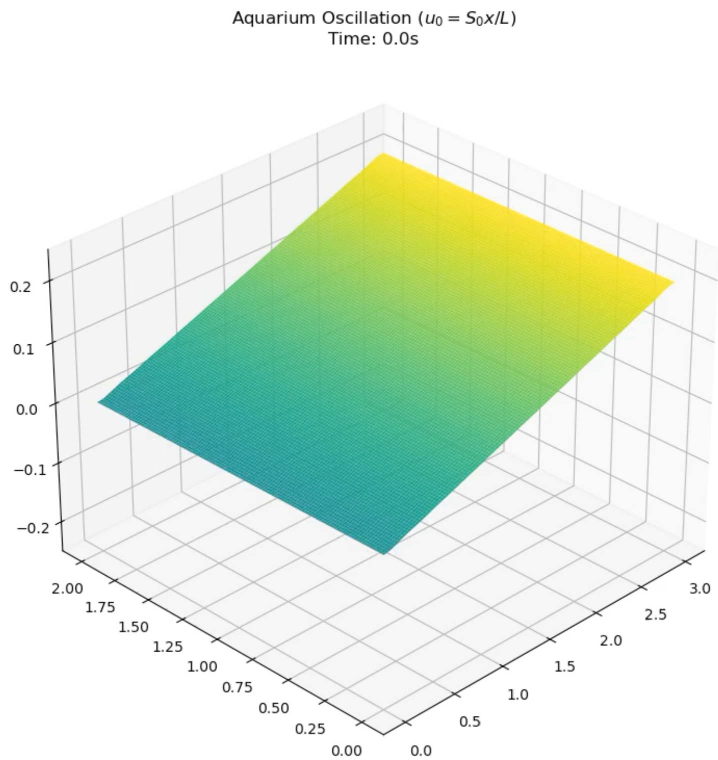
5) **Sketch the plot** of $u(x, M/2, t)$ at the moments of time $t = 100, 200, 500, 2000$ using 20 terms in the series.



6) **Exercise your creativity** by modifying some aspects of the problem and visualize your new solution.

This new solution has the initial condition $u_0(x, y) = S_0 \frac{x}{L}$, as if the tank had been tilted at an angle and suddenly returned to a flat surface.





7) **Make observations and comments** regarding the modelling of this physical process using this IBVP.

From my results it appears the surface oscillates until it ends at a displacement of zero, but that doesn't match the conservation of mass for this problem. If the initial displacement is measured as $u_0(x, y) = \frac{x+y}{5}$, then the displacement everywhere is positive. Since water is incompressible, the total volume should not change, so the surface would come to rest above zero displacement. In my modified scenario, I essentially mad the problem 1 dimensional by eliminating the y component of the initial displacement. I think in a true tank there would be some displacement in the y-direction even if the original displacement did not vary with y.