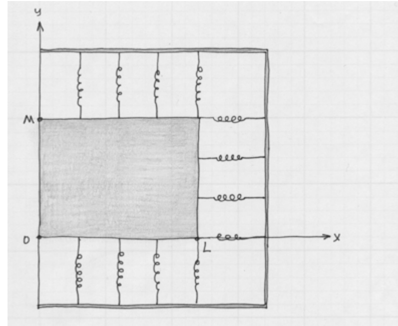
1.



Consider the IBVP describing vibration of the rectangular membrane with one fixed side and with other sides attached by the springs the frame.
Use the Integral Transform Methods to solve this problem.

a) Find the solution of the Wave Equation $u(x,y,t)$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+g(x,y,t)=a^2\frac{\partial^2 u}{\partial t^2}+b^2\frac{\partial u}{\partial t}+c^2 u \qquad t>0 \qquad (*)$$

in the domain $D=(0,L)\times(0,M)$ subject to boundary conditions:

$x=0$	$u\big\vert_{x=0}=f_3(y,t)$	$t>0$	$H_1=\dfrac{h_1}{k}$	(Dirichlet)
$x=L$	$\left[k\dfrac{\partial u}{\partial y}+h_4 u\right]\Big\vert_{x=L}=f_4(y,t)$	$t>0$	$H_2=\dfrac{h_2}{k}$	(Robin)
$y=0$	$\left[-k\dfrac{\partial u}{\partial y}+h_1 u\right]\Big\vert_{y=0}=f_1(x,t)$	$t>0$	$H_3=\dfrac{h_3}{k}$	(Robin)
$y=M$	$\left[k\dfrac{\partial u}{\partial y}+h_2 u\right]\Big\vert_{y=M}=f_2(x,t)$	$t>0$	$H_4=\dfrac{h_4}{k}$	(Robin)

and initial conditions:

$u(x,y,0)=u_0(x,y)$ (initial shape of the membrane)

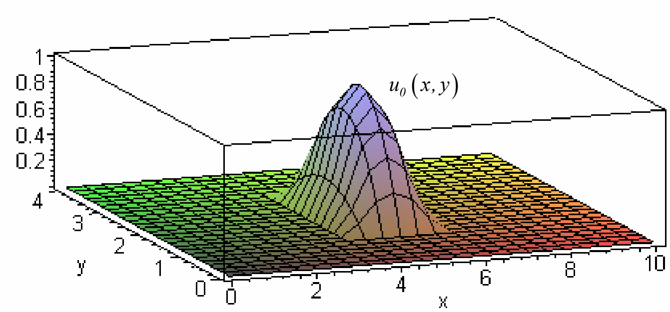
$\dfrac{\partial u(x,y,0)}{\partial t}=u_1(x,y)$ (initial velocity of the membrane)

b) Write the solution and sketch the graph for the case:

$L=10,\ M=4,$ $\qquad g(x,y,t)=0$

$a=1.0,\ b=0.8,\ c=1.0$ $\qquad f_1=f_2=f_3=f_4=0$

$k=2,\ h_1=h_2=h_3=h_4=4.0$



$$u_0(x,y)=(x-4)(6-x)(y-1)(3-y)\big[H(x-4)-H(x-6)\big]\big[H(y-1)-H(y-3)\big]$$
$$u_1(x,y)=0$$

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c) Investigate the influence of coefficients *a, b* and *c* on the vibration of the membrane.

Solution

I) Integral Transforms:

1) Finite Fourier Transform (Dirichlet-Robin) (Table FFT, p.146):

$$F_{x\ D\text{-}R} : u(x,y,t) \rightarrow \overline{u}_n(y,t) \qquad = \int_0^L u(x,y,t) KX_n(x) dx$$

$$KX_n(x) = \frac{\sin \lambda_n x}{\sqrt{\frac{2}{L} - \frac{\sin(2\lambda_n L)}{4\lambda_n}}} \qquad KX_n'(x) = \frac{\lambda_n \cos \lambda_n x}{\sqrt{\frac{2}{L} - \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$$

λ_n are positive roots of $\lambda \cos \lambda L + H_4 \sin \lambda L = 0$

$$F_{x\ D\text{-}R}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = f_3(y,t) KX_n'(0) + \frac{f_4(y,t)}{k} KX_n(L) - \lambda_n^2 \overline{u}_n(y,t)$$

$$F_{x\ D\text{-}R}\left\{f_1(x,t)\right\} = \overline{f}_{1,n}(t)$$

$$F_{x\ D\text{-}R}\left\{f_2(x,t)\right\} = \overline{f}_{2,n}(t)$$

2) Finite Fourier Transform (Robin-Robin) (Table FFT, p.146):

$$F_{y\ R\text{-}R} : u(x,y,t) \rightarrow \overline{u}_m(x,t) \qquad = \int_0^M u(x,y,t) KY_m(y) dy$$

$$KY_m(y) = \frac{\mu_m \cos \mu_m y + H_1 \sin \mu_m y}{\sqrt{\frac{\left(\mu_m^2 + H_1^2\right)}{2}\left(M + \frac{H_2}{\mu_m^2 + H_2^2}\right) + \frac{H_1}{2}}}$$

μ_m are positive roots of $\left(H_1 H_2 - \mu^2\right) \sin \mu M + \left(H_1 + H_2\right) \mu \cos \mu M = 0$

$$F_{y\ R\text{-}R}\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \frac{1}{k} f_1(x,t) KY_m(0) + \frac{f_2(x,t)}{k} KY_m(M) - \mu_m^2 \overline{u}_m(x,t)$$

$$F_{y\ R\text{-}R}\left\{f_3(y,t)\right\} = \overline{f}_{3,m}(t)$$

$$F_{y\ R\text{-}R}\left\{f_4(y,t)\right\} = \overline{f}_{4,m}(t)$$

Application of both transforms:

$$F_{y\ R\text{-}R}\left\{F_{x\ D\text{-}R}\left\{u(x,y,t)\right\}\right\} = \overline{\overline{u}}_{n,m}(t)$$

$$F_{y\ R\text{-}R}\left\{F_{x\ D\text{-}R}\left\{g(x,y,t)\right\}\right\} = \overline{\overline{g}}_{n,m}(t)$$

$$F_{y\ R\text{-}R}\left\{F_{x\ D\text{-}R}\left\{u_0(x,y)\right\}\right\} = \overline{\overline{u}}_{0,n,m}$$

$$F_{y\ R\text{-}R}\left\{F_{x\ D\text{-}R}\left\{u_1(x,y)\right\}\right\} = \overline{\overline{u}}_{1,n,m}$$

$$F_{y\ R\text{-}R}\left\{F_{x\ D\text{-}R}\left\{\frac{\partial^2 u}{\partial x^2}\right\}\right\} = F_{y\ R\text{-}R}\left\{f_3(y,t) KX_n'(0) + \frac{f_4(y,t)}{k} KX_n(L) - \lambda_n^2 \overline{u}_n(y,t)\right\} +$$

$$+ F_{x\ D\text{-}R}\left\{\frac{1}{k} f_1(x,t) KY_m(0) + \frac{f_2(x,t)}{k} KY_m(M) - \mu_m^2 \overline{u}_m(x,t)\right\}$$

$$= \overline{f}_{3,m}(t) KX_n'(0) + \frac{1}{k} \overline{f}_{4,m}(y,t) KX_n(L) - \lambda_n^2 \overline{\overline{u}}_{n,m}(t) +$$

$$+ \frac{1}{k} f_{1,n}(t) KY_m(0) + \frac{1}{k} f_{2,n}(t) KY_m(M) - \mu_m^2 \overline{\overline{u}}_{n,m}(x,t)$$

$$= \overline{f}_{3,m}(t) KX_n'(0) + \frac{1}{k} \overline{f}_{4,m}(y,t) KX_n(L) + + \frac{1}{k} f_{1,n}(t) KY_m(0) + \frac{1}{k} f_{2,n}(t) KY_m(M) - \left(\mu_m^2 + \lambda_n^2\right) \overline{\overline{u}}_{n,m}(x,t)$$

3) Laplace Transform:

$$L\left\{\overline{\overline{u}}_{n,m}(t)\right\} = U_{nm}(s)$$
$$L\left\{\overline{\overline{g}}_{n,m}(t)\right\} = G_{nm}(s)$$
$$L\left\{\overline{f}_{1,n}(t)\right\} = F_{1,n}(s)$$
$$L\left\{\overline{f}_{2,n}(t)\right\} = F_{2,n}(s)$$
$$L\left\{\overline{f}_{3,m}(t)\right\} = F_{3,m}(s)$$
$$L\left\{\overline{f}_{4,m}(t)\right\} = F_{4,m}(s)$$

II) Transformed Equation:

$$F_{y\ R\text{-}R}\left\{F_{x\ D\text{-}R}\left\{(*)\right\}\right\} \Rightarrow$$

$$\overline{f}_{3,m}(t)KX_n'(0) + \frac{1}{k}\overline{f}_{4,m}(t)KX_n(L) + \frac{1}{k}f_{1,n}(t)KY_m(0) + \frac{1}{k}f_{2,n}(t)KY_m(M) - \left(\mu_m^2 + \lambda_n^2\right)\overline{\overline{u}}_{n,m}(t) + \overline{\overline{g}}_{n,m}(t) =$$
$$= a^2\frac{\partial^2\overline{\overline{u}}_{n,m}}{\partial t^2} + b^2\frac{\partial\overline{\overline{u}}_{n,m}}{\partial t} + c^2\overline{\overline{u}}_{n,m} \qquad (**)$$

ODE subject to initial conditions: $\overline{\overline{u}}_{n,m}(0) = \overline{\overline{u}}_{0,n,m}$

$$\frac{\partial}{\partial y}\overline{\overline{u}}_{n,m}(0) = \overline{\overline{u}}_{1,n,m}$$

$$L\left\{(**)\right\} \qquad \Rightarrow$$

$$F_{3,m}(s)KX_n'(0) + \frac{1}{k}F_{4,m}(s)KX_n(L) + \frac{1}{k}F_{1,n}(s)KY_m(0) + \frac{1}{k}F_{2,n}(s)KY_m(M) - \left(\mu_m^2 + \lambda_n^2\right)U_{n,m}(s) + G_{n,m}(s) =$$
$$= a^2\left(s^2U_{nm} - s\overline{\overline{u}}_{0,n,m} - \overline{\overline{u}}_{1,n,m}\right) + b^2\left(sU_{nm} - \overline{\overline{u}}_{0,n,m}\right) + c^2U_{nm}$$

$$F_{3,m}(s)KX_n'(0) + \frac{1}{k}F_{4,m}(s)KX_n(L) + \frac{1}{k}F_{1,n}(s)KY_m(0) + \frac{1}{k}F_{2,n}(s)KY_m(M) + G_{n,m}(s) + a^2\overline{\overline{u}}_{1,n,m} + b^2\overline{\overline{u}}_{0,n,m} + sa^2\overline{\overline{u}}_{0,n,m}$$
$$= \left[a^2s^2 + b^2s + c^2 + \mu_m^2 + \lambda_n^2\right]U_{n,m}(s)$$

$$\left[a^2s^2 + b^2s + c^2 + \mu_m^2 + \lambda_n^2\right]U_{n,m}(s) = Q_{n,m}(s) + a^2\overline{\overline{u}}_{1,n,m} + b^2\overline{\overline{u}}_{0,n,m} + sa^2\overline{\overline{u}}_{0,n,m}$$

where $Q_{nm}(s) = F_{3,m}(s)KX_n'(0) + \frac{1}{k}F_{4,m}(s)KX_n(L) + \frac{1}{k}F_{1,n}(s)KY_m(0) + \frac{1}{k}F_{2,n}(s)KY_m(M) + G_{n,m}(s)$

$$U_{n,m}(s) = \frac{Q_{n,m}(s) + a^2\overline{\overline{u}}_{1,n,m} + b^2\overline{\overline{u}}_{0,n,m} + sa^2\overline{\overline{u}}_{0,n,m}}{\left[a^2s^2 + b^2s + c^2 + \mu_m^2 + \lambda_n^2\right]}$$

$$= \frac{Q_{n,m}(s)}{\left[a^2s^2 + b^2s + c^2 + \mu_m^2 + \lambda_n^2\right]} + \frac{a^2\overline{\overline{u}}_{1,n,m} + b^2\overline{\overline{u}}_{0,n,m}}{\left[a^2s^2 + b^2s + c^2 + \mu_m^2 + \lambda_n^2\right]} + \frac{s}{\left[a^2s^2 + b^2s + c^2 + \mu_m^2 + \lambda_n^2\right]}a^2\overline{\overline{u}}_{0,n,m}$$

$$a^2 s^2 + b^2 s + c^2 + \mu_m^2 + \lambda_n^2 = a^2\left[s^2 + \frac{b^2}{a^2}s + \frac{c^2+\mu_m^2+\lambda_n^2}{a^2}\right]$$

$$= a^2\left[s^2 + 2\left(\frac{b^2}{2a^2}\right)s + \left(\frac{b^2}{2a^2}\right)^2 - \left(\frac{b^2}{2a^2}\right)^2 + \frac{c^2+\mu_m^2+\lambda_n^2}{a^2}\right]$$

$$= a^2\left[\left(s+\frac{b^2}{2a^2}\right)^2 + \frac{c^2+\mu_m^2+\lambda_n^2}{a^2} - \left(\frac{b^2}{2a^2}\right)^2\right]$$

$$= a^2\left[(s-A)^2 + B^2\right] \qquad A = \left(\frac{-b^2}{2a^2}\right) \qquad B_{nm} = \frac{c^2+\mu_m^2+\lambda_n^2}{a^2} - \left(\frac{b^2}{2a^2}\right)^2$$

$$= \frac{Q_{n,m}(s)}{a^2\left[(s-A)^2+B^2\right]} + \frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m}}{a^2\left[(s-A)^2+B^2\right]} + \frac{s}{a^2\left[(s-A)^2+B^2\right]}a^2\bar{\bar{u}}_{0,n,m}$$

$$= \frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2} + \frac{B_{nm}}{\left[(s-A)^2+B_{nm}^2\right]}\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + \frac{s-A+A}{\left[(s-A)^2+B^2\right]}\bar{\bar{u}}_{0,n,m}$$

$$= \frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2} + \frac{B_{nm}}{\left[(s-A)^2+B_{nm}^2\right]}\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + \frac{s-A}{\left[(s-A)^2+B^2\right]}\bar{\bar{u}}_{0,n,m} + \frac{A}{\left[(s-A)^2+B_{nm}^2\right]}\bar{\bar{u}}_{0,n,m}$$

$$= \frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2} + \frac{B_{nm}}{\left[(s-A)^2+B_{nm}^2\right]}\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + \frac{s-A}{\left[(s-A)^2+B_{nm}^2\right]}\bar{\bar{u}}_{0,n,m} + \frac{B_{nm}}{\left[(s-A)^2+B_{nm}^2\right]}\frac{A\bar{\bar{u}}_{0,n,m}}{B_{nm}}\frac{a^2}{a^2}$$

$$= \frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2} + \frac{B_{nm}}{\left[(s-A)^2+B_{nm}^2\right]}\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2A\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + \frac{s-A}{\left[(s-A)^2+B_{nm}^2\right]}\bar{\bar{u}}_{0,n,m}$$

Inverse Laplace Transform:

$$\bar{\bar{u}}_{n,m}(t) = L^{-1}\left\{U_{nm}(s)\right\}$$

$$= L^{-1}\left\{\frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2} + \frac{B_{nm}}{\left[(s-A)^2+B_{nm}^2\right]}\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2A\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + \frac{s-A}{\left[(s-A)^2+B_{nm}^2\right]}\bar{\bar{u}}_{0,n,m}\right\}$$

$$= L^{-1}\left\{\frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2}\right\} + L^{-1}\left\{\frac{B_{nm}}{\left[(s-A)^2+B_{nm}^2\right]}\right\}\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2A\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + L^{-1}\left\{\frac{s-A}{\left[(s-A)^2+B_{nm}^2\right]}\right\}\bar{\bar{u}}_{0,n,m}$$

$$= L^{-1}\left\{\frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2}\right\} + e^{At}\sin(B_{nm}t)\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2A\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + e^{At}\cos(B_{nm}t)\bar{\bar{u}}_{0,n,m}$$

$$\bar{\bar{u}}_{n,m}(t) = L^{-1}\left\{\frac{1}{\left[(s-A)^2+B_{nm}^2\right]}\frac{Q_{n,m}(s)}{a^2}\right\} + \left[\frac{\bar{\bar{u}}_{1,n,m} + \frac{b^2}{2a^2}\bar{\bar{u}}_{0,n,m}}{B_{nm}}\sin(B_{nm}t) + \bar{\bar{u}}_{0,n,m}\cos(B_{nm}t)\right]e^{\frac{-b^2}{2a^2}t}$$

If $Q_{n,m}(s) = Q_{nm}$ (not a function of s), then

$$= L^{-1}\left\{\frac{B_{nm}}{\left[(s-A)^2 + B_{nm}^2\right]}\frac{Q_{n,m}}{a^2 B_{nm}}\right\} + e^{At}\sin(B_{nm}t)\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2 A\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + e^{At}\cos(B_{nm}t)\bar{\bar{u}}_{0,n,m}$$

$$= e^{At}\sin(B_{nm}t)\frac{Q_{n,m}}{a^2 B_{nm}} + e^{At}\sin(B_{nm}t)\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2 A\bar{\bar{u}}_{0,n,m}}{a^2 B_{nm}} + e^{At}\cos(B_{nm}t)\bar{\bar{u}}_{0,n,m}$$

$$= e^{At}\left[\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2 A\bar{\bar{u}}_{0,n,m} + Q_{nm}}{a^2 B_{nm}}\sin(B_{nm}t) + \bar{\bar{u}}_{0,n,m}\cos(B_{nm}t)\right]$$

$$= e^{\frac{-b^2}{2a^2}t}\left[\frac{a^2\bar{\bar{u}}_{1,n,m} + b^2\bar{\bar{u}}_{0,n,m} + a^2\left(\frac{-b^2}{2a^2}\right)\bar{\bar{u}}_{0,n,m} + Q_{nm}}{a^2 B_{nm}}\sin(B_{nm}t) + \bar{\bar{u}}_{0,n,m}\cos(B_{nm}t)\right]$$

$$= e^{\frac{-b^2}{2a^2}t}\left[\frac{a^2\bar{\bar{u}}_{1,n,m} + \frac{b^2}{2}\bar{\bar{u}}_{0,n,m} + Q_{nm}}{a^2 B_{nm}}\sin(B_{nm}t) + \bar{\bar{u}}_{0,n,m}\cos(B_{nm}t)\right]$$

$$= e^{\frac{-b^2}{2a^2}t}\left[\frac{a^2\bar{\bar{u}}_{1,n,m} + \frac{b^2}{2}\bar{\bar{u}}_{0,n,m} + Q_{nm}}{a^2\left(\frac{c^2 + \mu_m^2 + \lambda_n^2}{a^2} - \left(\frac{b^2}{2a^2}\right)^2\right)}\sin\left[\left(\frac{c^2 + \mu_m^2 + \lambda_n^2}{a^2} - \left(\frac{b^2}{2a^2}\right)^2\right)t\right] + \bar{\bar{u}}_{0,n,m}\cos\left[\left(\frac{c^2 + \mu_m^2 + \lambda_n^2}{a^2} - \left(\frac{b^2}{2a^2}\right)^2\right)t\right]\right]$$

$$= e^{\frac{-b^2}{2a^2}t}\left[\frac{a^2\bar{\bar{u}}_{1,n,m} + \frac{b^2}{2}\bar{\bar{u}}_{0,n,m} + Q_{nm}}{c^2 + \mu_m^2 + \lambda_n^2 - \frac{b^4}{4a^2}}\sin\left[\left(\frac{c^2 + \mu_m^2 + \lambda_n^2}{a^2} - \left(\frac{b^2}{2a^2}\right)^2\right)t\right] + \bar{\bar{u}}_{0,n,m}\cos\left[\left(\frac{c^2 + \mu_m^2 + \lambda_n^2}{a^2} - \left(\frac{b^2}{2a^2}\right)^2\right)t\right]\right]$$

$$\bar{\bar{u}}_{n,m}(t) = L^{-1}\left\{U_{nm}(s)\right\} = \left[\frac{\bar{\bar{u}}_{1,n,m} + \frac{b^2}{2a^2}\bar{\bar{u}}_{0,n,m} + \frac{Q_{nm}}{a^2}}{B_{nm}}\sin(B_{nm}t) + \bar{\bar{u}}_{0,n,m}\cos(B_{nm}t)\right]e^{\frac{-b^2}{2a^2}t}$$

Inverse Finite Fourier Transforms – Solution of the IBVP:

$$u(x,y,t) = F^{-1}{}_{y\ R\text{-}R}\left\{F^{-1}{}_{x\ D\text{-}R}\left\{\bar{\bar{u}}_{n,m}(t)\right\}\right\}$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{\bar{u}}_{nm}(t)KX_n(x)KY_m(y)$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{\bar{\bar{u}}_{1,n,m} + \frac{b^2}{2a^2}\bar{\bar{u}}_{0,n,m} + \frac{Q_{nm}}{a^2}}{B_{nm}}\sin(B_{nm}t) + \bar{\bar{u}}_{0,n,m}\cos(B_{nm}t)\right]e^{\frac{-b^2}{2a^2}t}KX_n(x)KY_m(y)$$