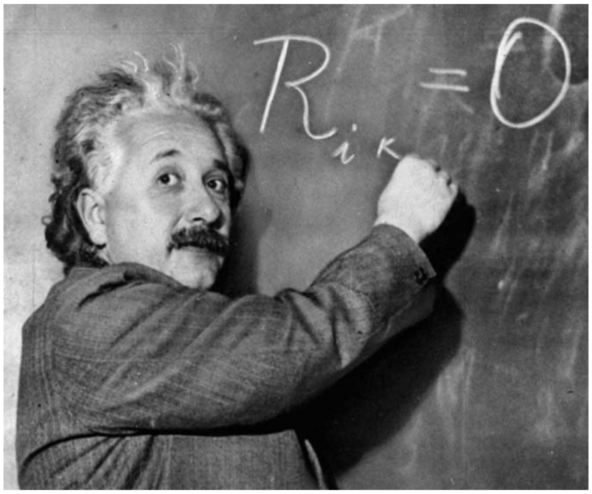
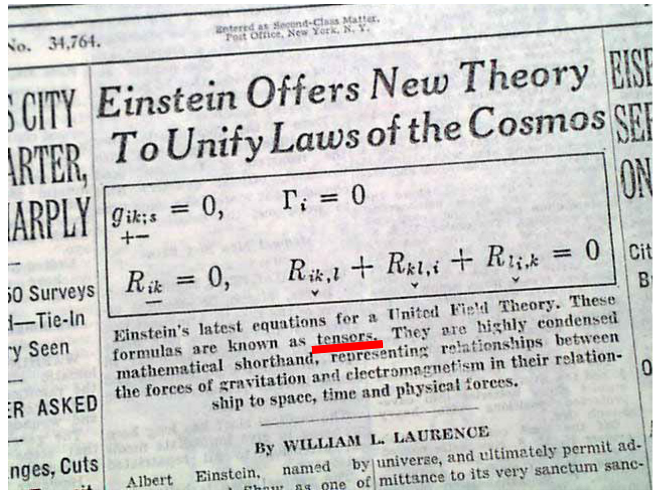
IV.1 VECTORS AND TENSORS





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Einstein Offers New Theory To Unify Laws of the Cosmos

$g_{ik;s} = 0, \quad \Gamma_i = 0$

$R_{ik} = 0, \quad R_{ik,l} + R_{kl,i} + R_{li,k} = 0$

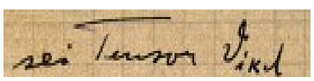
Einstein's latest equations for a United Field Theory. These formulas are known as tensors. They are highly condensed mathematical shorthand, representing relationships between the forces of gravitation and electromagnetism in their relationship to space, time and physical forces.

By WILLIAM L. LAURENCE

Albert Einstein, named by ... universe, and ultimately permit admittance to its very sanctum sanc-

IV.1.1. INTRODUCTION

In mathematics and mechanics, various quantities require different mathematical representations. Some, like temperature, density, and mass, are described using a sigle numerical value in appropriate unit. Others, such as velocity, acceleration, and force, possess both magnitude and direction. There are also more complicated situations when for some physical quantities we need to describe their distributions in different directions (for example, shear stress). The common mathematical objects used for this purpose are ***scalars, vectors, matrices*** etc. However, their application can become challenging when a ***change of coordinate system*** is needed.

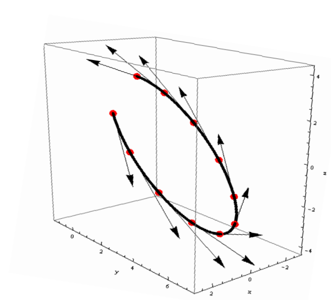


For more convenient and universal description that is independent of the coordinate system, more general mathematical objects are employed. They are called ***tensors***.

The tensors can be of different order. A ***zero order tensor*** which is characterized by a single real number corresponds to a scalar. A ***tensor of the first order*** is defined by a triple of real numbers and it corresponds to a vector.
A ***second order tensor*** defined by nine real numbers corresponds to a matrix. In general, an ***n^{th} order tensor*** is characterized by 3^n components.

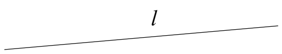
The primary purpose of tensor notations is to provide a specific organization of their components which obey the so called transformation laws of its components under the change of the coordinate system. Operations with these objects are studied by tensor analysis.

We will restrict our study mainly to 3-dimensional tensors which are used for description of the physical quantities in Euclidean space E_3.

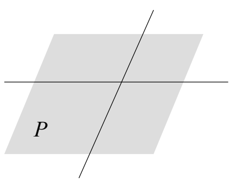


IV.1.2. EUCLIDIAN SPACE E_3 We assume that Euclidian 3-dimensional space E_3 consists of geometrical points; and that in this space we can draw lines and curves, planes and surfaces which obey the requirements of elementary Euclidian geometry. Also, we assume that we are able to perform with the help of ruler and compass the construction of segments and angles, drawing of rays, parallel lines etc., and that we can measure the distance between points in terms of the defined unit length. Recall the basic definitions (which are more intuitive than rigorous) of the geometrical objects and their symbolic visualization and notations:

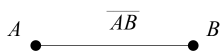
Point defines the position in space but has *"no part"*.



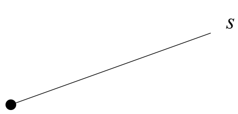
Line is a set of points which can be treated as a translation of a point – an unbounded straight line. The intersection of two lines yields a point. A line can be defined by two points in space (there is only one line which passes through two fixed points).



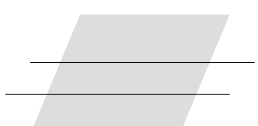
Plane is a set of points obtained by translation of one line along another line. The intersection of two planes yields a line. A plane can be defined by two intersecting lines.



Segment is a line bounded on both sides (a line connecting two points). Any fixed segment can be chosen as the unit for measurement for the lengths between points.

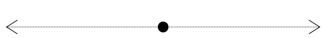


Ray is a line bounded on one side. A ray defines a direction.

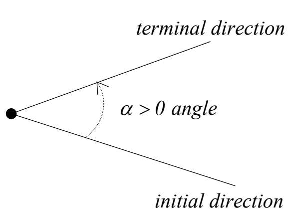


Direction is defined by a ray.

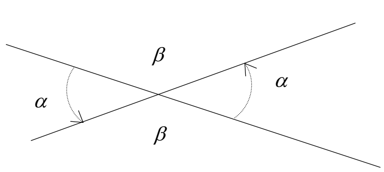
If two lines lying in the same plane are parallel we say that they have the same direction.



Each line decomposed into two rays defines two opposite directions.



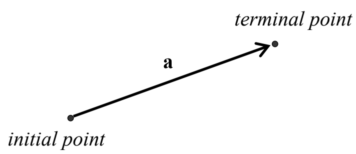
Angle is formed by two rays; one ray determines the *terminal* direction and the other ray determines the *initial* direction (measuring of the angle from initial to terminal direction ***ccw*** yields *positive* angles; ***cw*** – *negative* angles)



Two lines with a common point define ***two pairs of angles***. (*conjugate?)*

IV.1.3. FREE GEOMETRIC VECTORS IN EUCLIDIAN SPACE E_3

We define a ***geometric vector*** (***free vector*** or just a ***vector***) as a directed segment in the Euclidian space E_3. It can be visualized as a segment with arrows indicating its direction.

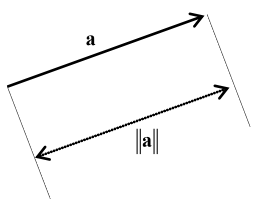


Vectors usually are designated by the lower-case bold letters $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, ...$ or by letters with arrows above $\vec{a}, \vec{b}, \vec{u}, ...$
There is also a special designation for the unit basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ or $\hat{i}, \hat{j}, \hat{k}$.

The arrowed end of a vector indicates the direction and is called a *terminal point,* the other end is called an *initial point.*

Vectors can be placed in any location of Euclidian space. There is no need of coordinate system for their definition (although the coordinate system may be helpful for operation with vectors and for other types of vectors which will be defined later (*position vectors*)).

norm (***magnitude***)

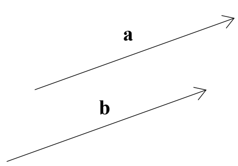


The distance between the initial and the terminal points of a vector (the length of the segment) is said to be the ***norm*** (***absolute value, magnitude*** or ***modulus***).

It is denoted in one of the following ways

$$a = \| \mathbf{a} \| \qquad \textit{norm of vector } \mathbf{a}$$

equality



a = b

We say that two vectors are ***equal*** if they have the same direction and norm.
It means that geometric vectors are not associated with a particular position in the space, and they can be moved to any location without loosing their identity (that is why they are also called ***free vectors***).

Any vector is a representative of a whole family of all vectors with the same norm and direction. If vector $\mathbf{a}$ can be obtained by a parallel translation of another vector $\mathbf{b}$ then it is the same vector. In engineering, comparison of vectors can be performed only if their norms are measured with the same units.

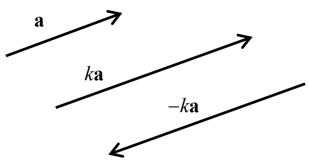
zero vector $\mathbf{0}$

A ***zero vector*** $\mathbf{0}$ is a vector with a zero norm. The direction of such a vector looses its sense, because the terminal point coincides with the initial point. Any point in space is representative of a unique zero vector.

OPERATIONS WITH GEOMETRIC VECTORS (VECTOR ALGEBRA)

multiplication by a scalar

After multiplying the vector $\mathbf{a}$ by a positive scalar $k > 0$ the resulting vector $k\mathbf{a}$ has the direction of the vector $\mathbf{a}$ and norm $k\|\mathbf{a}\|$.



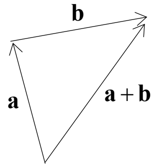
After multiplying the vector $\mathbf{a}$ by a negative scalar $k < 0$ the resulting vector $k\mathbf{a}$ has the direction opposite to the direction of the vector $\mathbf{a}$ (the terminal and the initial points are interchanged) and the same norm $\|\mathbf{a}\|$.

Therefore, $\|k\mathbf{a}\| = |k|\|\mathbf{a}\| \quad k \in \mathbb{R}$

Vectors which are scalar multiples of each other are called ***collinear***.

Multiplication by $k = 0$ turns any vector to a zero vector, $0\mathbf{a} = \mathbf{0}$.
A zero vector is collinear to any vector.

sum

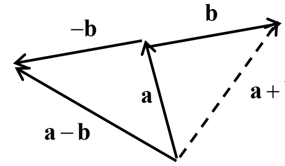


The sum of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a}+\mathbf{b}$ determined by the following rule: place the initial point of vector $\mathbf{b}$ to the terminal point of the vector $\mathbf{a}$; then the vector $\mathbf{a}+\mathbf{b}$ has the initial point of vector $\mathbf{a}$ and the terminal point of vector $\mathbf{b}$ (it is called ***the triangle rule***).

subtraction of vectors

Define formally subtraction of two vectors by addition of the negative vector:

$$\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b})$$



Abelian group

Defined in this way geometric vectors with the operation addition form an ***abelian group*** with the zero vector as a neutral element (it means that they really are vectors).

Indeed, using elementary geometric construction, it can be shown that the associative rule is valid

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$$

The neutral element is a zero vector

$$\mathbf{a}+\mathbf{0} = \mathbf{0}+\mathbf{a}$$

The inverse to a vector $\mathbf{a}$ is a vector with the same norm and opposite direction

$$(-1)\mathbf{a} = -\mathbf{a}$$

And finally, the operation addition is commutative

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$$

"... to devote all my life to the cultivation of my reason, and to progress as much as possible in the knowledge of truth..."

René Descartes

Lange Bisschopstraat, Deventer – one of the places where Descartes lived in the Netherlands

"Our heart is full of warmth, yet we no longer feel it, for we have grown accustomed to it."

Here lived René Descartes (1596–1650)

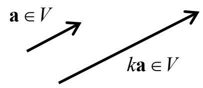
Established in the Netherlands,
the French philosopher resided in this house
during his Parisian visits of 1644, 1647 and 1648

"Taking myself as I am, with one foot in one country and the other in another, I find my condition very happy, for it is free."

IV.1.4. VECTOR SPACES

Consider a set of all geometric (free) vectors $V = \{\mathbf{a}\}$ uniquely represented by the position vectors with the operation of addition of vectors $\mathbf{a}+\mathbf{b}$ and with the operation of multiplication of a vector by a scalar $k\mathbf{a}$.
Let us verify that $(V,+,\cdot)$ satisfies the axioms of a vector space (Section 3.1):

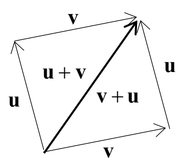
*The **closure axioms***



i) For any $\mathbf{a},\mathbf{b} \in V$ there is the vector $\mathbf{a}+\mathbf{b} \in V$

ii): For any $\mathbf{a} \in V$ and $k \in \mathbb{R}$ there is the vector $k\mathbf{a} \in V$
These axioms are the corollaries of the axiomatic properties of the geometrical Euclidian space: that any two points of the Euclidian space can be connected by a segment, and that any segment can be elongated by any factor or reduced by any factor.

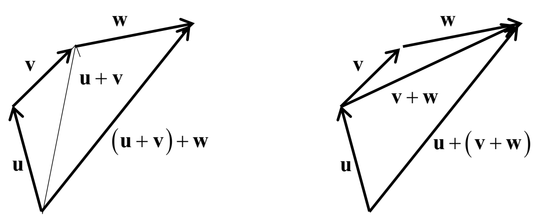
*The **vector axioms***



1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ *commutative law*

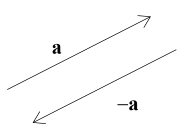
From geometrical construction, it is seen the result of summation is the same diagonal of the parallelogram. It also yields the other equivalent definition of the summation rule called ***the parallelogram rule***. This rule is used for summation of the position vectors.

2) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ *associative rule*
Verification of this axiom also can be performed by geometrical constructions yielding the same resulting vector.



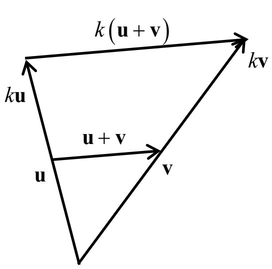
3) The neutral element is a zero vector

$$\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}$$



4) The inverse to a vector $\mathbf{a}$ is a vector $-1\cdot\mathbf{a}=-\mathbf{a}$
(the vector with the same norm and the opposite direction).

The simple geometrical considerations yield the remaining properties:



5) If $\mathbf{u} \in V$ and $a,b \in \mathbb{R}$, then $a(b\mathbf{u})=(ab)\mathbf{u}$ *associative law*

6) If $\mathbf{u},\mathbf{v} \in V$ and $k \in \mathbb{R}$, then $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$ *distributive law*

7) If $\mathbf{u} \in V$ and $a,b \in \mathbb{R}$, then $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$ *distributive law*

8) If $\mathbf{u} \in V$, then $(1)\mathbf{u}=\mathbf{u}$

The properties 5,7, and 8 are the properties of collinear vectors (vectors lying on the same line are called collinear).
Therefore, we verified that

Vector Space $(V,+,\cdot)$

The set of all geometric vectors V with operations of addition of vectors and multiplication of vectors by a scalar, form a ***vector space*** $(V,+,\cdot)$

The general facts and properties of vector spaces (considered in Chapter III) can be applied to the vector space of geometrical vectors. Now we want to find a way for representation of geometrical vectors, namely we need to determine the dimension of the vector space and construct the basis of the vector space.
Recall some definitions concerning the vector spaces from Section III.2 and formulate them in terms of geometric vectors.

Linear combination is a finite sum of the form

$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \ldots + \alpha_n\mathbf{a}_n \quad = \sum_{i=1}^{n} \alpha_i \mathbf{a}_i \qquad \alpha_i \in \mathbb{R},\ \mathbf{a}_i \in V$$

Linear independence

The set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is ***linearly independent*** if their linear combination is equal to a zero vector if and only if all coefficients are equal to zero. Therefore,

$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \ldots + \alpha_n\mathbf{a}_n = \mathbf{0} \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

If a finite set of vectors is not linearly independent then it is said to be ***linearly dependent***. Therefore, it is possible to construct a linear combination of linearly dependent vectors equal to a zero vector with the coefficients not all equal to zero.

If in a set of vectors, one of them can be represented as a linear combination of other vectors, then they are linearly dependent.

Also, in a set of linearly dependent vectors, one of them can be represented as a linear combination of other vectors.

If a set of vectors includes a zero vector, then it is linearly dependent.

If two vectors are linearly dependent, they are collinear (lie on the same line).

Any three linearly dependent vectors are coplanar (lie in the same plane).

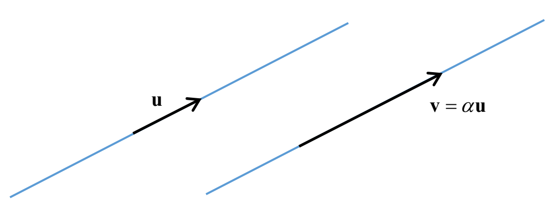
VECTORS ON THE LINE

1-dimensional vector space $\mathbb{R}$ ***(collinear vectors)***:

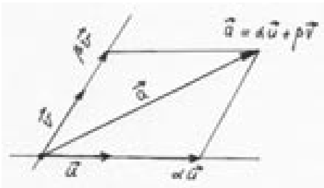
If two vectors $\mathbf{u}$ and $\mathbf{v}$ lie on the same line or on parallel lines, then one of the vectors can be represented as the scalar multiple of the other

$$\mathbf{u} = \alpha \mathbf{v} \qquad (1)$$

Conclusion: any two collinear vectors are linearly dependent.



VECTORS ON THE PLANE ***2-dimensional vector space*** $\mathbb{R}_2$



Let $\mathbf{u}$ and $\mathbf{v}$ be two linearly independent vectors. If vector $\mathbf{a}$ is coplanar with $\mathbf{u}$ and $\mathbf{v}$, then it can be uniquely represented as a linear combination

$$\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v} \qquad (2)$$

Geometricly, this fact can be easily confirmed. Set all three vectors to the same initial point. Build a parallelogram with vector $\mathbf{a}$ as a diagonal and with two sides on the lines along vectors $\mathbf{u}$ and $\mathbf{v}$. Then scale vectors $\mathbf{u}$ and $\mathbf{v}$ to vectors $\alpha\mathbf{u}$ and $\beta\mathbf{v}$ which coincide with the sides of the parallelogram. Then, obviously, $\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v}$.

To see that this expansion is unique, assume that there exists the other expansion

$$\mathbf{a} = \alpha'\mathbf{u} + \beta'\mathbf{v}$$

Subtract this equation from the previous one, then

$$\mathbf{0} = (\alpha - \alpha')\mathbf{u} + (\beta - \beta')\mathbf{v}$$

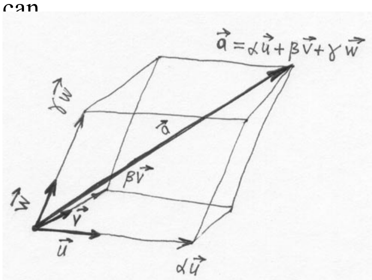
Hence vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, coefficients in this expansion should be equal to zero, and therefore

$$\alpha = \alpha' \qquad \beta = \beta'$$

Conclusion: any three coplaner vectors are linearly dependent.

VECTORS IN 3-D SPACE ***3-dimensional vector space*** $\mathbb{R}_3$

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three linearly independent vectors. Then any vector $\mathbf{a}$ can be uniquely represented as a linear combination

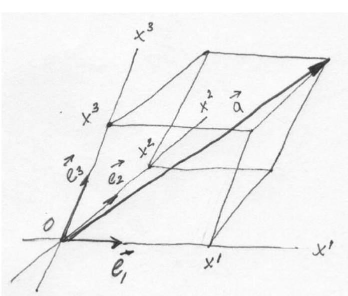


$$\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \qquad (3)$$

Again, as a proof, consider the following geometric construction. Place all four vectors at the same initial point. Pairs of vectors $\mathbf{uv}$, $\mathbf{vw}$, and $\mathbf{wu}$ define three nes in the space. Through the terminal point of vector $\mathbf{a}$ draw three more nes which are parallel to them. Then intersections of the six planes form a allelepiped with the vector $\mathbf{a}$ as a diagonal. Scale vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ to tors $\alpha\mathbf{u}$, $\beta\mathbf{v}$ and $\gamma\mathbf{w}$ which coincide with the edges of the parallelepiped. Then from geometric consideration it is obvious that $\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$. Uniqueness of this expansion can be checked similarly to the previous case.

Conclusion: any four vectors are linearly dependent.

BASIS



Because any vector $\mathbf{a}$ in the set of all geometric vectors E_3 can be represented by a linear combination of any three linear independent vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

$$\mathbf{a} = \sum_{i=1}^{3} \alpha_i \mathbf{e}_i \quad = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 \qquad (4)$$

the span of the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ generates the ***Euclidian vector space*** E_3 .

A set of any three linear independent vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a ***basis*** of E_3 .

Therefore, a vector space of geometric vectors E_3 is 3-dimensional.

COORDINATE SYSTEM

Place the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ at the same initial point O called the *origin*, and draw lines along the vectors $\mathbf{e}_i$ called Ox^1, Ox^2, Ox^3. Associate each of these lines with the real axis which directions coincide with the direction of vectors $\mathbf{e}_i$ Then they will form an ***oblique coordinate system***, and the coefficients in the expansion (4) are called the coordinates of vector $\mathbf{a}$. Denote them by

$$\mathbf{a} = \sum_{i=1}^{3} x^i \mathbf{e}_i \quad = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \tag{5}$$

with the **upper indices**.

If vectors in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are mutually **orthogonal** (see definition below)

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{for } i \neq j \qquad i,j = 1,2,3$$

Orthonormal coordinates

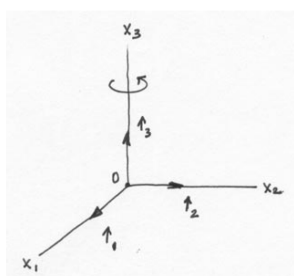
then they form an ***orthogonal coordinate system***. If in addition, the orthogonal basis vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ are of unit length and

$$\mathbf{i}_i \cdot \mathbf{i}_j = \delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad i,j = 1,2,3$$

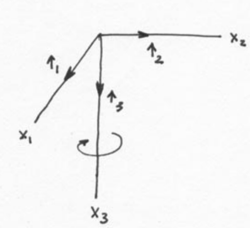
then the basis is called ***orthonormal*** (where δ_{ij} is called the ***Kronecker delta***). The coordinate system formed by the orthogonal (or orthonormal) basis is called a ***rectangular coordinate system*** (or the ***Carthesian coordinate system***). We will use two notations for the Cartesian coordinate system: $0xyz$ and $0x_1x_2x_3$.

Expansion in the rectangular coordinate system uses coefficients with the **lower indices**:

$$\mathbf{a} = \sum_{i=1}^{3} x_i \mathbf{i}_i = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{6}$$



Right rectangular coordinate system



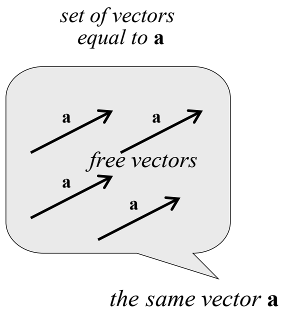
Left rectangular coordinate system

The ***right rectangular coordinate system*** is preferred in mathematical modeling in engineering. The other standard notation for the vector components is

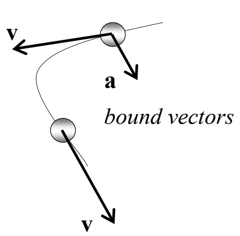
$$\mathbf{a} \quad = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \;\; = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{6b}$$

Therefore, any vector $\mathbf{a} = \sum_{i=1}^{3} x_i \mathbf{i}_i$ uniquely defines a point in the Euclidean space with the coordinates (x_1, x_2, x_3) or (x, y, z). Therefore, alternatively to coordinates, vectors can be used for specifications of points, and instead of functions of three variables $f(x,y,z)$ the vector functions $f(\mathbf{r})$ can be used.

FREE VECTORS, BOUND VECTORS, POSITION VECTORS



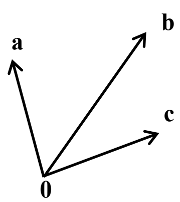
The equality of the geometrical vectors (***free vectors***) was defined through the equality of their direction and length. According to this definition, any vector **a** has infinitely many vectors equal to it obtained by parallel translation of **a** in Euclidian space. We will treat them as a class of vectors represented by any one of them – it means that all of them are just the same vector. It will provide us a uniqueness of the result of operations with vectors; but we still have the flexibility with handling the vectors – we can associate it with any convenient location for analysis. In three dimensions, it is uniquely represented by three numbers.



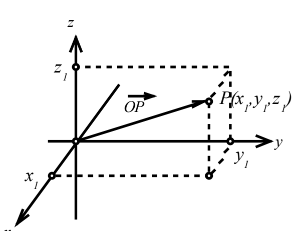
There are also situations in mechanics when vectors are referred to a specified point in space, for example, they can be associated with the velocity and the acceleration of moving particles or with the velocity field of fluid flow, or with the forces acting on bodies, or gradients and fluxes etc. These vectors are called the ***bound vectors***. For their definition we also need specification of a position in space; in 3 dimensions, bound vectors are given by six numbers.

Free vectors are the most general kind of vectors. The handling bound vectors always can be reduced to operations with free vectors.

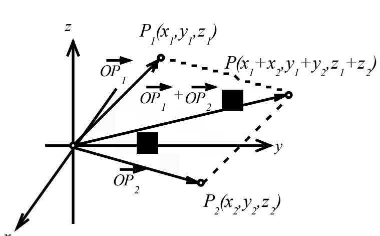
Position vectors



There is also a special case of bound vectors – ***position vectors*** – which all refer to a fixed point uniquely defining the zero position vector **0**. Therefore, they need only three numbers for their definition, but the operations with the bound and position vectors should be modified in such a way that the result is also a bound or position vector. The comparison of the free vectors and position vectors is demonstrated in the Table "Vectors in Euclidean Space." This table also includes definition of vectors as the 1st order tensors which will be studied in Section IV.1.7.



Position vectors are the subset of all geometric vectors. Position vectors are all vectors with the initial point at the same fixed point 0 called the origin and some terminal point P. A position vector is denoted by $\vec{0}P$. The definition of a position vector does not require the introduction of a coordinate system, however, description of position vectors is more convenient if a coordinate system is introduced.



Operations with position vectors are similar to operations with free vectors with some modifications:

- Two position vectors are equal if their terminal points are the same.
- A zero vector is represented only by the origin.
- Scalar multiplication is equivalent to scalar multiplication of free vectors.
- The sum of two position vectors is determined by the parallelogram rule.
- The set of all position vectors with the operation vector summation and operation of scalar multiplication form a vector space.

Coordinate vectors

Because all position vectors have the same initial point, they are completely determined only by their terminal point. It means that a different identification of the terminal point can also form a vector space. If the rectangular coordinate system $Oxyz$ is set to the origin, then such identification of terminal points can be performed by it's coordinates and operations with position vectors can be expressed in terms of the coordinates of the terminal points. It is not exactly a set of the position vectors because they are not geometric objects (directed segments) but rather the ordered triple of real numbers, but they will be completely identical vector spaces, and allows them to be used interchangeably.

Denote position vectors by the coordinates of the terminal point (x_1, x_2, x_3).

Two coordinate vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are ***equal***,

$$\mathbf{a} = \mathbf{b} \quad \text{if} \quad a_1 = b_1, a_2 = b_2, a_3 = b_3 .$$

Scalar multiplication $\quad k\mathbf{a} = (ka_1, ka_2, ka_3), \quad k \in \mathbb{R}$

Addition $\quad \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Other notation for coordinate vectors $\langle x_1, x_2, x_3 \rangle$.

Row vectors $\mathbf{a} = (a_1, a_2, a_3)$ is the other name for coordinate vectors.

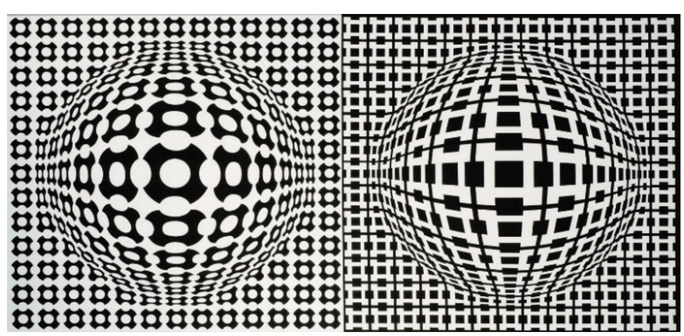
Column vectors

Column vectors are identical to coordinate vectors, the difference is only on the way they are written:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}, \qquad \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

Therefore, free vectors can be defined in the form of coordinate vectors or in the form of column vectors. It means that if a free vector $\mathbf{a}$ is given, then its coordinates $\mathbf{a} = (a_1, a_2, a_3)$ in the Cartesian coordinate system are given.

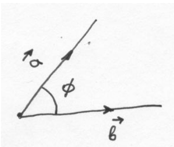
The definitions and operations with these types of vectors are summarized in the table. The generalization of the description of vector space induced by the geometrical Euclidian space is performed with the help of tensors which we will consider below.



IV.1.5. DOT PRODUCT

The geometric constructions which are used in trigonometry, analytical geometry or computer graphics can be formalized in terms of operations with geometric vectors.

Angle between two vectors



Draw the vectors $\mathbf{a}$ and $\mathbf{b}$ from the same initial point. Then draw the rays in the direction of vectors $\mathbf{a}$ and $\mathbf{b}$. These rays define two positive angles the sum of which is equal to the full angle 2π. For characterization of the angle between two vectors choose those which are between 0 and π. Use the following notation for the angle between the vectors

$$\phi = (\mathbf{a},\mathbf{b}) \quad = (\mathbf{b},\mathbf{a})$$

Dot product

The ***dot product*** (***inner product, scalar product***) of vectors $\mathbf{a}$, $\mathbf{b} \in V$ is defined as a map $\mathbf{a}\cdot\mathbf{b} : V \times V \to \mathbb{R}$ calculated according to

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos(\mathbf{a},\mathbf{b}) \qquad = \|\mathbf{a}\|\|\mathbf{b}\| \cos\phi \tag{7a}$$

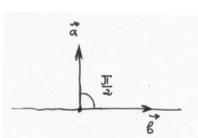
and in the form of the column vectors

$$\mathbf{a}\cdot\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{7b}$$

We will show that the second definition follows from the first one (Property 6).

The result of the dot product of two vectors is a scalar (real number). It is positive if the angle ϕ between vectors $\mathbf{a}$ and $\mathbf{b}$ is acute (less than $\pi/2$) and negative if the angle ϕ is obtuse (greater than $\pi/2$).

Orthogonal vectors



(8)

We say that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and denote it $\mathbf{a}\perp\mathbf{b}$ if the angle between them is the right angle $\phi = (\mathbf{a},\mathbf{b}) = \dfrac{\pi}{2}$. It is obvious that non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are ***orthogonal*** if and only if

$$\mathbf{a}\cdot\mathbf{b} = 0$$

The condition on the coordinates of vectors to be orthogonal is

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$$

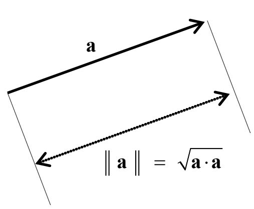
It can be shown that the dot product satisfies properties of the inner product in a vector space.

Properties of the dot product:

1) The dot product is *commutative*:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \qquad \text{(\textit{commutative law})}$$

2) The dot product of a vector with itself:

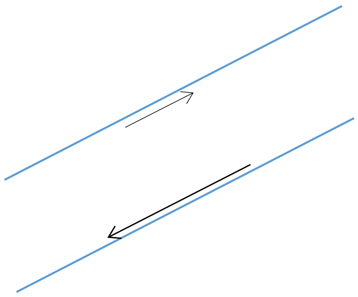


$$\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|\|\mathbf{a}\| \cos(\mathbf{a},\mathbf{a}) = \|\mathbf{a}\|^2 \cos(0) = \|\mathbf{a}\|^2 = a^2 = a_1^2 + a_2^2 + a_3^2$$

is a square of the norm (length) of a vector $\mathbf{a}$. Therefore,

$$a = \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

Also $\mathbf{a}\cdot\mathbf{a} \geq 0$, and $\mathbf{a}\cdot\mathbf{a} = 0$ only if $\mathbf{a} = \mathbf{0}$



3) If vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear (parallel) with the same direction, then

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\mathbf{a},\mathbf{b}) = \|\mathbf{a}\|\|\mathbf{b}\|\cos(0) = \|\mathbf{a}\|\|\mathbf{b}\| = ab$$

If vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear (parallel) with the opposite direction, then

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\mathbf{a},\mathbf{b}) = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\pi) = -\|\mathbf{a}\|\|\mathbf{b}\| = -ab$$

4) The dot products of the orthonormal basis vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$:

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$$

or in more compact form for basis vectors in form of $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$

Kronecker delta δ_{ij}

$$\mathbf{i}_i\cdot\mathbf{i}_j = \delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad i,j = 1,2,3 \qquad \delta_{ij} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where δ_{ij} is called the ***Kronecker delta.***

5) Distributive properties:

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$$

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$$

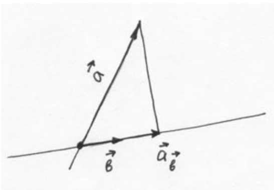
$$(\alpha\mathbf{a})\cdot(\beta\mathbf{b}) = \alpha\beta(\mathbf{a}\cdot\mathbf{b}) \qquad \alpha,\beta \in \mathbb{R}$$

6) Derivation of the equation (7b) using properties (4) and (5):

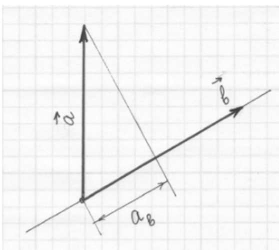
$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\
&= \left\{ a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}\cdot\left\{ b_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\
&= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i}\cdot b_1\mathbf{i} + a_1\mathbf{i}\cdot b_2\mathbf{j} + a_1\mathbf{i}\cdot b_3\mathbf{k} + a_2\mathbf{j}\cdot b_1\mathbf{i} + a_2\mathbf{j}\cdot b_2\mathbf{j} + a_2\mathbf{j}\cdot b_3\mathbf{k} \\
&\quad + a_3\mathbf{k}\cdot b_1\mathbf{i} + a_3\mathbf{k}\cdot b_2\mathbf{j} + a_3\mathbf{k}\cdot b_3\mathbf{k} \\
&= a_1b_1(\mathbf{i}\cdot\mathbf{i}) + a_1b_2(\mathbf{i}\cdot\mathbf{j}) + a_1b_3(\mathbf{i}\cdot\mathbf{k}) + a_2b_1(\mathbf{j}\cdot\mathbf{i}) + a_2b_2(\mathbf{j}\cdot\mathbf{j}) + a_2b_3(\mathbf{j}\cdot\mathbf{k}) \\
&\quad + a_3b_1(\mathbf{k}\cdot\mathbf{i}) + a_3b_2(\mathbf{k}\cdot\mathbf{j}) + a_3b_3(\mathbf{k}\cdot\mathbf{k}) \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}$$

PROJECTIONS

Projection of vector $\mathbf{a}$ on vector $\mathbf{b}$ is a vector computed in the following way



$$\begin{aligned}\mathbf{a_b} &= \|\mathbf{a}\| cos(\mathbf{a},\mathbf{b}) \frac{\mathbf{b}}{\|\mathbf{b}\|} \\ &= \|\mathbf{a}\|\|\mathbf{b}\| cos(\mathbf{a},\mathbf{b}) \frac{\mathbf{b}}{\|\mathbf{b}\|^2} \\ &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \\ &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}\right)\frac{\mathbf{b}}{\|\mathbf{b}\|} \\ &= a_b \frac{\mathbf{b}}{\|\mathbf{b}\|}\end{aligned} \tag{9}$$



where $a_b = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}$ is the length of the projection called the ***component*** of $\mathbf{a}$ on $\mathbf{b}$

Correspondingly, the projection of vector $\mathbf{b}$ on vector $\mathbf{a}$ is

$$\mathbf{b_a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|}$$

with the component $b_a = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$.

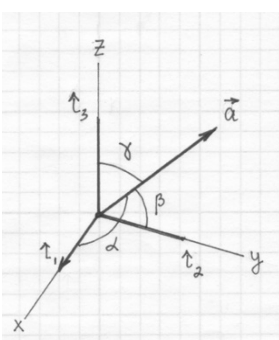
Then the dot product can be written in terms of the components in two forms:

$$a_b = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|} \quad \Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = a_b b$$

$$b_a = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} \quad \Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = b_a a$$

This means that the dot product of two vectors is equal to the product of the norm of one vector and the component of the other vector on the first one.

Projections on the basis vectors – direction angles



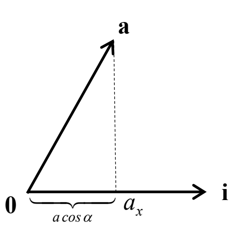
With the help of the dot product, the projections of vector $\mathbf{a}$ on the basis vectors can be determined as:

$$a_x = \frac{\mathbf{a}\cdot\mathbf{i}_1}{\|\mathbf{i}_1\|} = \|\mathbf{a}\|\|\mathbf{i}_1\| cos(\mathbf{a},\mathbf{i}_1) = a\, cos(\mathbf{a},\mathbf{i}_1) = a\, cos\,\alpha$$

$$a_y = \frac{\mathbf{a}\cdot\mathbf{i}_2}{\|\mathbf{i}_2\|} = \|\mathbf{a}\|\|\mathbf{i}_2\| cos(\mathbf{a},\mathbf{i}_2) = a\, cos(\mathbf{a},\mathbf{i}_2) = a\, cos\,\beta$$

$$a_z = \frac{\mathbf{a}\cdot\mathbf{i}_3}{\|\mathbf{i}_3\|} = \|\mathbf{a}\|\|\mathbf{i}_3\| cos(\mathbf{a},\mathbf{i}_3) = a\, cos(\mathbf{a},\mathbf{i}_3) = a\, cos\,\gamma$$

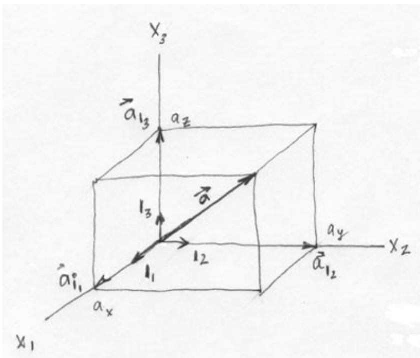
From these equations, the ***direction cosines*** of the angles between vector $\mathbf{a}$ and the coordinate axis can be defined as:



$$\begin{aligned} cos\,\alpha &= \frac{a_x}{a} = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \\ cos\,\beta &= \frac{a_y}{a} = \frac{a_y}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \\ cos\,\gamma &= \frac{a_z}{a} = \frac{a_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \end{aligned} \tag{10}$$

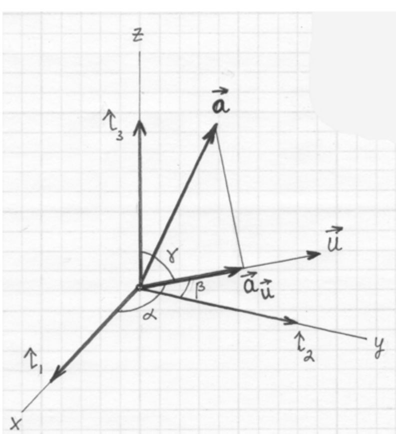
Vector's expansion in Cartesian coordinates

Consider how expansion of a vector $\mathbf{a} \in V$ in the rectangular coordinate system given previously by Equation (6) can be written with the help of the dot product:



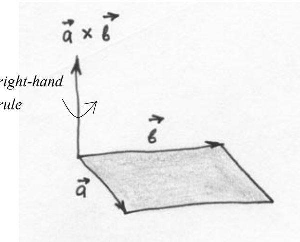
$$
\begin{aligned}
\mathbf{a} = \sum_{i=1}^{3} x_i \mathbf{i}_i \quad & = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 \\
& = \mathbf{a}_{\mathbf{i}_1} + \mathbf{a}_{\mathbf{i}_2} + \mathbf{a}_{\mathbf{i}_3} \qquad \text{(\textit{sum of projections on axis})} \\
& = \frac{\mathbf{a}\cdot\mathbf{i}_1}{\|\mathbf{i}_1\|^2}\mathbf{i}_1 + \frac{\mathbf{a}\cdot\mathbf{i}_2}{\|\mathbf{i}_2\|^2}\mathbf{i}_2 + \frac{\mathbf{a}\cdot\mathbf{i}_3}{\|\mathbf{i}_3\|^2}\mathbf{i}_3 \\
& = \|\mathbf{a}\| cos(\mathbf{a},\mathbf{i}_1)\mathbf{i}_1 + \|\mathbf{a}\| cos(\mathbf{a},\mathbf{i}_2)\mathbf{i}_2 + \|\mathbf{a}\| cos(\mathbf{a},\mathbf{i}_3)\mathbf{i}_3 \\
& = a_x\mathbf{i}_1 + a_y\mathbf{i}_2 + a_z\mathbf{i}_3 \qquad (11) \\
& = a_x\begin{bmatrix}1\\0\\0\end{bmatrix} + a_y\begin{bmatrix}0\\1\\0\end{bmatrix} + a_z\begin{bmatrix}0\\0\\1\end{bmatrix}
\end{aligned}
$$

If the coordinates of vector $\mathbf{a}$ in the coordinate system $Ox_1x_2x_3$ are known then the projection of vector $\mathbf{a}$ on the direction of *the unit vector* $\mathbf{u}$ can be determined as (derived from equation (9))



$$
\begin{aligned}
\mathbf{a}_{\mathbf{u}} = \frac{\mathbf{a}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} \quad & = (\mathbf{a}\cdot\mathbf{u})\mathbf{u} \\
& = \left[\left(a_{x_1}\mathbf{i}_1 + a_{x_2}\mathbf{i}_2 + a_{x_3}\mathbf{i}_3\right)\cdot\mathbf{u}\right]\mathbf{u} \\
& = \left(a_{x_1}\mathbf{i}_1\cdot\mathbf{u} + a_{x_2}\mathbf{i}_2\cdot\mathbf{u} + a_{x_3}\mathbf{i}_3\cdot\mathbf{u}\right)\mathbf{u} \\
& = \left[a_x\, cos(\mathbf{i}_1,\mathbf{u}) + a_y\, cos(\mathbf{i}_2,\mathbf{u}) + a_z\, cos(\mathbf{i}_3,\mathbf{u})\right]\mathbf{u} \qquad (12)
\end{aligned}
$$

IV.1.6. CROSS PRODUCT



The ***cross product*** (***outer product, vector product***) of vectors $\mathbf{a}$, $\mathbf{b} \in V$ is defined as a map $\mathbf{a}\times\mathbf{b} : V\times V \to V$. The result of the cross-product $\mathbf{a}\times\mathbf{b}$ is a vector which is orthogonal to the plane defined by the vectors $\mathbf{a}$ and $\mathbf{b}$ drawn from the same point and it is oriented according to the right-hand rule. The norm of the vector $\mathbf{a}\times\mathbf{b}$ is defined as

$$\| \mathbf{a}\times\mathbf{b} \| = \|\mathbf{a}\|\|\mathbf{b}\| sin(\mathbf{a},\mathbf{b}) \tag{20a}$$

It is equal to the area of a parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$.

Properties:

1) If vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear (parallel), then

   $\mathbf{a}\times\mathbf{b} = \mathbf{0}$

2) The cross product is *anticommutative*:

   $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$

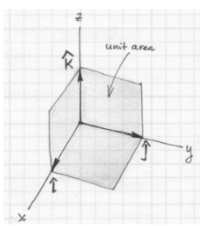
3) Distributive properties:

   $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}$

   $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$

   $(k\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(k\mathbf{b}) = k(\mathbf{a}\times\mathbf{b}), \quad k\in\mathbb{R}$

Cross products of basis vectors



It follows from the definition and the properties that

$$\begin{aligned} &\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0} \\ &\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{i} = -\mathbf{k} \\ &\mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{j} = -\mathbf{i} \\ &\mathbf{k}\times\mathbf{i} = \mathbf{j} \qquad \mathbf{i}\times\mathbf{k} = -\mathbf{j} \end{aligned} \qquad \textit{Right Orthonormal Coordinate System} \tag{21}$$

Component form of cross product

If vectors $\mathbf{a}$ and $\mathbf{b}$ are given as the column vectors:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Then using the distributive property and Equations (21) one can obtain:

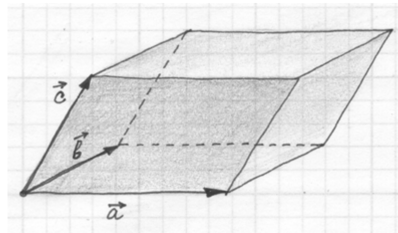
$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i}\times\mathbf{i} + a_1b_2\mathbf{i}\times\mathbf{j} + a_1b_3\mathbf{i}\times\mathbf{k} + a_2b_1\mathbf{j}\times\mathbf{i} + a_2b_2\mathbf{j}\times\mathbf{j} + a_2b_3\mathbf{j}\times\mathbf{k} \\ &\quad + a_3b_1\mathbf{k}\times\mathbf{i} + a_3b_2\mathbf{k}\times\mathbf{j} + a_3b_3\mathbf{k}\times\mathbf{k} \\ &= a_1b_2\mathbf{k} - a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} \end{aligned}$$

$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned} \tag{20b}$$

Triple scalar product

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{21}$$

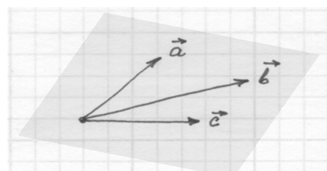
The geometric sense of the triple scalar product is the volume of a parallelepiped formed by the vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$:



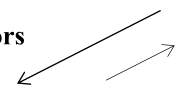
$$\textit{Volume} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

If the vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are coplanar (lie on the same plane), then:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$$



Parallel vectors



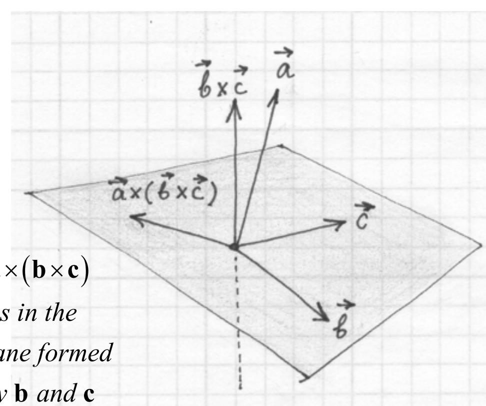
The non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if their cross product is a zero vector:

$$\mathbf{a} \parallel \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a}\times\mathbf{b} = \mathbf{0} \tag{22}$$

Triple vector product

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{23}$$

This vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to vectors $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, and therefore, it is in the plane formed by vectors $\mathbf{b}$ and $\mathbf{c}$:



$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *is in the plane formed by* $\mathbf{b}$ *and* $\mathbf{c}$

plane formed by $\mathbf{b}$ *and* $\mathbf{c}$

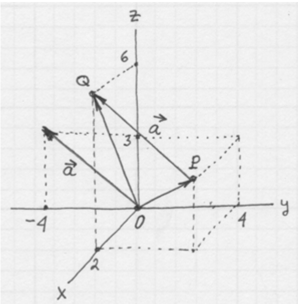
The other form of the triple vector product is given by the similar equation:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} \tag{24}$$

Lagrange Identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{25}$$

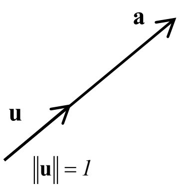
IV.1.7. EXAMPLES:



1. What is a free vector **a** defined by the initial point $P=(2,4,3)$ and the terminal point $Q=(2,0,6)$?

Points P and Q are defined by the position vectors $\vec{0P}=(2,4,3)$ and $\vec{0Q}=(2,4,3)$. Then the vector **a** can be defined as

$$\mathbf{a}=\vec{0Q}-\vec{0P}=(2,0,6)-(2,4,3)=(0,-4,3)=-4\mathbf{j}+3\mathbf{k}=\begin{bmatrix}0\\-4\\3\end{bmatrix}$$

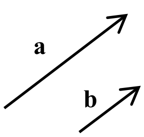


2. Find the unit vector in the direction of vector $\mathbf{a}=-4\mathbf{j}+3\mathbf{k}$.
The norm of vector **a** is

$$a=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{(-4)^2+3^2}=\sqrt{16+9}=\sqrt{25}=5$$

Then the unit vector **u** in the direction of vector **a** can be defined as

$$\mathbf{u}=\frac{\mathbf{a}}{\|\mathbf{a}\|}=\frac{\mathbf{a}}{a}=\frac{\mathbf{a}}{5}=\frac{-4\mathbf{j}+3\mathbf{k}}{5}=-\frac{4}{5}\mathbf{j}+\frac{3}{5}\mathbf{k}=\begin{bmatrix}0\\-\frac{4}{5}\\\frac{3}{5}\end{bmatrix}$$



3. Show that if the vectors $\mathbf{a}=(x_1,y_1,z_1)$ and $\mathbf{b}=(x_2,y_2,z_2)\neq 0$ are collinear then

$$\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}$$

If the vectors are collinear, then they are multiples of each other

$$\mathbf{a}=k\mathbf{b}$$
$$(x_1,y_1,z_1)=k(x_2,y_2,z_2)$$
$$(x_1,y_1,z_1)=(kx_2,ky_2,kz_2)$$

and, therefore, $x_1=kx_2 \qquad k=\frac{x_1}{x_2} \qquad x_2\neq 0$

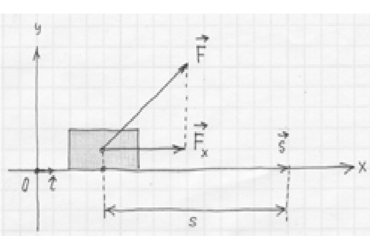
$$y_1=ky_2 \qquad k=\frac{y_1}{y_2} \qquad y_2\neq 0$$

$$z_1=kz_2 \qquad k=\frac{z_1}{z_2} \qquad z_2\neq 0$$

From which follows the required identity.

4. ***(Work by a constant force)***
Determine the work done by a constant force $\mathbf{F}=(5,4,0)$ on the object along the *x*-axis on the distance $s=11$.
The work done by the force is defined by the product of the magnitude of its projection on the direction of motion and the distance that the body moves:

$$W=\|\mathbf{F}_\mathbf{i}\|s=(\mathbf{F}\cdot\mathbf{i})s=\mathbf{F}\cdot(s\mathbf{i})$$



Let **s** be a vector of magnitude s in a direction $\mathbf{i}$: $\mathbf{s}=s\mathbf{i}=11\mathbf{i}=(11,0,0)$.
Then

$$W=\mathbf{F}\cdot\mathbf{s} \tag{26}$$

$$W=(5,4,0)\cdot(11,0,0)=55$$

5. For vectors $\mathbf{a} = (1,-2,3)$ and $\mathbf{b} = (-1,3,1)$, find:

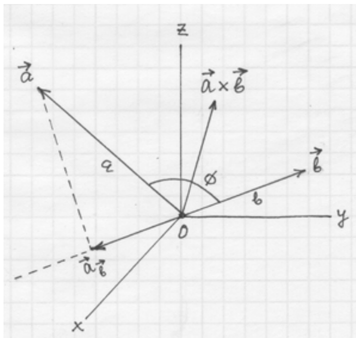
a) the **norm** of the vectors:

$$a = \|\mathbf{a}\| = \sqrt{1^2 + (-2)^2 + 3^3} = \sqrt{1+4+9} = \sqrt{14}$$

$$b = \|\mathbf{b}\| = \sqrt{(-1)^2 + 3^2 + 1^3} = \sqrt{1+9+1} = \sqrt{11}$$

b) the **sum** of the vectors:

$$\mathbf{a} + \mathbf{b} = (1-1,-2+3,3+1) = (0,1,4)$$



c) the **dot product** of the vectors:

$$\mathbf{a}\cdot\mathbf{b} = 1\cdot(-1) + (-2)\cdot 3 + 3\cdot 1 = -4$$

d) the **angle** between the vectors:

$$\phi = (\mathbf{a},\mathbf{b}) = cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right) = cos^{-1}\left(\frac{-4}{\sqrt{154}}\right) = 1.9 = 109^\circ$$

c) the **projection** of vector $\mathbf{a}$ on the direction of vector $\mathbf{b}$:

$$\mathbf{a_b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{-4}{11}(-1,3,1) = \left(\frac{4}{11},-\frac{12}{11},-\frac{4}{11}\right)$$

f) the **cross product** of vectors:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -1 & 3 & 1 \end{vmatrix} = (-2-9)\mathbf{i} - (1+3)\mathbf{j} + (3-2)\mathbf{k} = -11\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

6. *The* ***center of mass*** *of two points:*

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

The ***center of mass*** *of n points:*

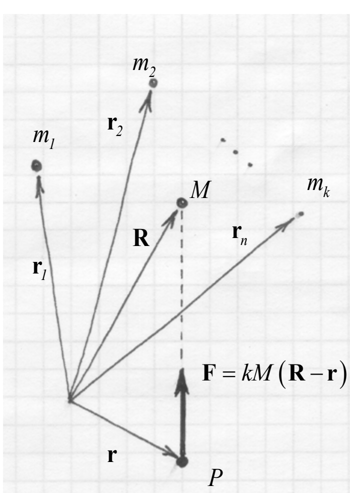
$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n}{m_1 + m_2 + \cdots m_n}$$

Let $P_1, P_2, ..., P_n$ be fixed points with masses $m_1, m_2, ..., m_n$ respectively.

Let the attraction force of the point P by the point P_j be proportional to the distance between the points and to the mass of the point P_j:

$$F_j = kr_jm_j \qquad \textit{where } k \in \mathbb{R} \textit{ is the coefficient of proportionality}$$

Determine the attraction force acting on the point P and determine the equilibrium position of the point P.



Solution: Let $\mathbf{r}_k$ be the position vector with the terminal point P_k.

Then the attraction force acting on the point P by the point P_k is

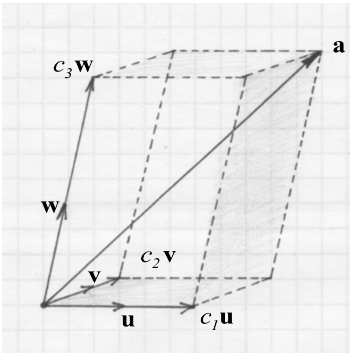
$$\mathbf{F}_k = km_k(\mathbf{r}_k - \mathbf{r})$$

Then the total force acting on the point P

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n \\ &= km_1(\mathbf{r}_1 - \mathbf{r}) + km_2(\mathbf{r}_2 - \mathbf{r}) + \cdots + km_n(\mathbf{r}_n - \mathbf{r}) \\ &= k(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n) - k(m_1 + m_2 + \cdots + m_n)\mathbf{r} \\ &= k(m_1 + m_2 + \cdots + m_n)\left[\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n}{m_1 + m_2 + \cdots + m_n} - \mathbf{r}\right] \\ &= kM(\mathbf{R} - \mathbf{r}), \quad \text{where } M = m_1 + m_2 + \cdots + m_n\end{aligned}$$

Point P is the equilibrium point if $\mathbf{F} = \mathbf{0}$, therefore, $\mathbf{R} = \mathbf{r}$. It means that the equilibrium point is located at the center of mass of the system of points.

7. (***Vector's representation by a linear combination***)



Let vectors $\left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\}$ be linearly independent. Then any vector $\mathbf{a}$ can be represented as a linear combination

$$\mathbf{a} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$$

(this representation is equivalent for writing the vector $\mathbf{a}$ in the oblique coordinate system (5)). Let us find the coefficients c_1, c_2, c_3 (coordinates of the vector $\mathbf{a}$ in the coordinate system $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Write a linear combination in the component form:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + c_3 \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

which can be written as a linear system for coefficients c_1, c_2, c_3:

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

or in the vector form:

$$\mathbf{Ac} = \mathbf{a}$$

Because the set of columns in the matrix $\mathbf{A}$ is linearly independent, the matrix $\mathbf{A}$ is invertible, and the linear system has a unique solution (Chapter x, statements 1,4,5 of the Inverse Matrix Theorem). Therefore, coefficients c_1, c_2, c_3 can be found as

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

or using Cramer's rule (Chapter x, Theorem x).

For example, find the coordinates of vector $\mathbf{a} = (-10, 5, -5)$ in the oblique coordinate system with the basis vectors

$$\left\{ \mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}$$

The determinant of the matrix of coefficients

$$det\,\mathbf{A} = \begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -5 \neq 0$$

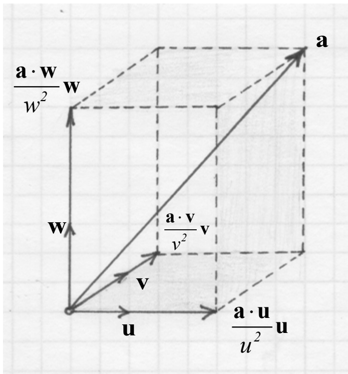
This means that the column vectors are linearly independent. Then the linear system has the solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -10 \\ 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{6}{5} \\ 1 & -1 & -1 \\ -1 & \frac{4}{5} & \frac{7}{5} \end{bmatrix} \begin{bmatrix} -10 \\ 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \\ 7 \end{bmatrix}$$

Therefore,

$$\mathbf{a} = -6\mathbf{u} - 10\mathbf{v} + 7\mathbf{w}$$

8. (***Representating a vector as linear combination of orthogonal vectors***)



Let vectors $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ be mutually orthogonal:

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{u}=0$$

Find the coefficients in the representation of $\mathbf{a}$ by a linear combination

$$\mathbf{a}=c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w}$$

Construct a dot product of the equation with the vectors $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ consequently :

$$\mathbf{a}\cdot\mathbf{u}=(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})\cdot\mathbf{u}=c_1\mathbf{u}\cdot\mathbf{u}+c_2\mathbf{v}\cdot\mathbf{u}+c_3\mathbf{w}\cdot\mathbf{u}=c_1\mathbf{u}\cdot\mathbf{u}=c_1\|\mathbf{u}\|^2$$

$$\mathbf{a}\cdot\mathbf{v}=(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})\cdot\mathbf{v}=c_1\mathbf{u}\cdot\mathbf{v}+c_2\mathbf{v}\cdot\mathbf{v}+c_3\mathbf{w}\cdot\mathbf{v}=c_2\mathbf{v}\cdot\mathbf{v}=c_2\|\mathbf{v}\|^2$$

$$\mathbf{a}\cdot\mathbf{w}=(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})\cdot\mathbf{w}=c_1\mathbf{u}\cdot\mathbf{w}+c_2\mathbf{v}\cdot\mathbf{w}+c_3\mathbf{w}\cdot\mathbf{w}=c_3\mathbf{w}\cdot\mathbf{w}=c_3\|\mathbf{w}\|^2$$

Then the coefficients can be determined as:

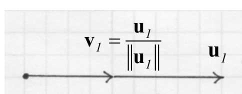
$$c_1=\frac{\mathbf{a}\cdot\mathbf{u}}{\|\mathbf{u}\|^2},\quad c_2=\frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|^2},\quad c_3=\frac{\mathbf{a}\cdot\mathbf{w}}{\|\mathbf{w}\|^2} \tag{27}$$

If in addition, vectors $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ are normalized, $\|\mathbf{u}\|=\|\mathbf{v}\|=\|\mathbf{w}\|=1$, then

$$c_1=\mathbf{a}\cdot\mathbf{u},\quad c_2=\mathbf{a}\cdot\mathbf{v},\quad c_3=\mathbf{a}\cdot\mathbf{w}$$

For an orthogonal basis, solving the linear system is not necessary – each coefficient can be determined individually. This is a key advantage of using an orthogonal basis. In the following example, we will demonstrate how a linearly independent set can be used to construct an orthonormal basis .

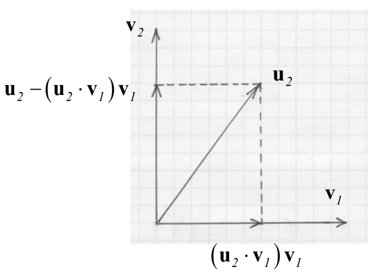
9. (***Gram-Schmidt orthogonalization process***)



Let the set of vectors $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ be linearly independent. Then the set of ***orthonormal vectors*** $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ can be constructed with the help of the so called Gram-Schmidt process which consists of the following steps:

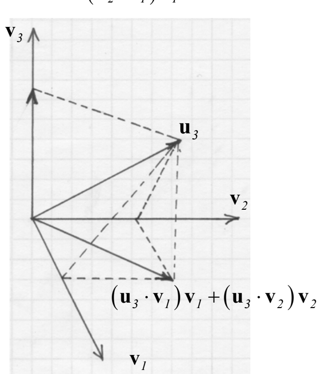
1) Normalize the first vector $\mathbf{u}_1$ and call it $\mathbf{v}_1$:

$$\mathbf{v}_1=\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$



2) Find the component of vector $\mathbf{u}_2$ orthogonal to vector $\mathbf{v}_1$, normalize it and call it $\mathbf{v}_2$:

$$\mathbf{v}_2=\frac{\mathbf{u}_2-(\mathbf{u}_2\cdot\mathbf{v}_1)\mathbf{v}_1}{\|\mathbf{u}_2-(\mathbf{u}_2\cdot\mathbf{v}_1)\mathbf{v}_1\|}$$



3) Projection of the vector $\mathbf{u}_3$ on the plane defined by the vectors $\mathbf{v}_1,\mathbf{v}_2$ can be found as the sum of projections on the directions of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$(\mathbf{u}_3)_{\mathbf{v}_1\mathbf{v}_2}=(\mathbf{u}_3\cdot\mathbf{v}_1)\mathbf{v}_1+(\mathbf{u}_3\cdot\mathbf{v}_2)\mathbf{v}_2$$

Then the vector $\mathbf{u}_3-(\mathbf{u}_3)_{\mathbf{v}_1\mathbf{v}_2}=\mathbf{u}_3-(\mathbf{u}_3\cdot\mathbf{v}_1)\mathbf{v}_1-(\mathbf{u}_3\cdot\mathbf{v}_2)\mathbf{v}_2$ is orthogonal to the plane $\mathbf{v}_1,\mathbf{v}_2$. Normalize it and call it $\mathbf{v}_3$:

$$\mathbf{v}_3=\frac{\mathbf{u}_3-(\mathbf{u}_3\cdot\mathbf{v}_1)\mathbf{v}_1-(\mathbf{u}_3\cdot\mathbf{v}_2)\mathbf{v}_2}{\|\mathbf{u}_3-(\mathbf{u}_3\cdot\mathbf{v}_1)\mathbf{v}_1-(\mathbf{u}_3\cdot\mathbf{v}_2)\mathbf{v}_2\|}$$

Gram-Schmidt orthogonalization:

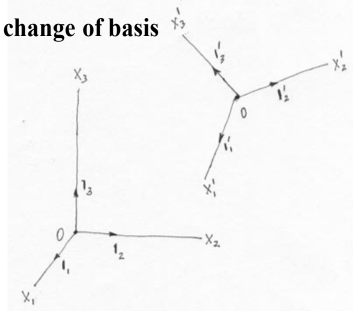
$$\mathbf{v}_1=\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$

$$\mathbf{v}_2=\frac{\mathbf{u}_2-(\mathbf{u}_2\cdot\mathbf{v}_1)\mathbf{v}_1}{\|\mathbf{u}_2-(\mathbf{u}_2\cdot\mathbf{v}_1)\mathbf{v}_1\|} \tag{28}$$

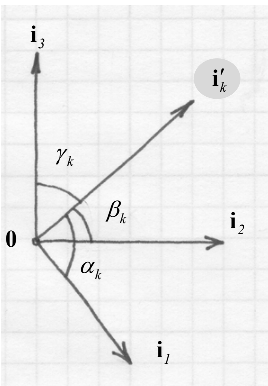
$$\mathbf{v}_3=\frac{\mathbf{u}_3-(\mathbf{u}_3\cdot\mathbf{v}_1)\mathbf{v}_1-(\mathbf{u}_3\cdot\mathbf{v}_2)\mathbf{v}_2}{\|\mathbf{u}_3-(\mathbf{u}_3\cdot\mathbf{v}_1)\mathbf{v}_1-(\mathbf{u}_3\cdot\mathbf{v}_2)\mathbf{v}_2\|}$$

10. Transformation of Coordinates

change of basis



$\left(\mathbf{i}_i, \mathbf{i}'_k\right) =$ *angle between* $\mathbf{i}_i$ *and* $\mathbf{i}'_k$



directional cosines:

$\alpha_k = cos\left(\mathbf{i}_1, \mathbf{i}'_k\right)$

$\beta_k = cos\left(\mathbf{i}_2, \mathbf{i}'_k\right)$

$\gamma_k = cos\left(\mathbf{i}_3, \mathbf{i}'_k\right)$

Consider two orthogonal coordinte systems $Oxyz$ and $Ox'y'z'$ defined by the orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ and $\{\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3\}$ having the same point O.
Consider first how one basis can be written in terms of another basis.
Using expansion (11), write vectors $\{\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3\}$ in terms of $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$:

$$\begin{aligned}
\mathbf{i}'_1 &= cos\left(\mathbf{i}'_1, \mathbf{i}_1\right)\mathbf{i}_1 + cos\left(\mathbf{i}'_1, \mathbf{i}_2\right)\mathbf{i}_2 + cos\left(\mathbf{i}'_1, \mathbf{i}_3\right)\mathbf{i}_3 &&= \alpha_1\mathbf{i}_1 + \beta_1\mathbf{i}_2 + \gamma_1\mathbf{i}_3 \\
\mathbf{i}'_2 &= cos\left(\mathbf{i}'_2, \mathbf{i}_1\right)\mathbf{i}_1 + cos\left(\mathbf{i}'_2, \mathbf{i}_2\right)\mathbf{i}_2 + cos\left(\mathbf{i}'_2, \mathbf{i}_3\right)\mathbf{i}_3 &&= \alpha_2\mathbf{i}_1 + \beta_2\mathbf{i}_2 + \gamma_2\mathbf{i}_3 \qquad (13) \\
\mathbf{i}'_3 &= cos\left(\mathbf{i}'_3, \mathbf{i}_1\right)\mathbf{i}_1 + cos\left(\mathbf{i}'_3, \mathbf{i}_2\right)\mathbf{i}_2 + cos\left(\mathbf{i}'_3, \mathbf{i}_3\right)\mathbf{i}_3 &&= \alpha_3\mathbf{i}_1 + \beta_3\mathbf{i}_2 + \gamma_3\mathbf{i}_3
\end{aligned}$$

and write vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ in terms of $\{\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3\}$:

$$\begin{aligned}
\mathbf{i}_1 &= cos\left(\mathbf{i}_1, \mathbf{i}'_1\right)\mathbf{i}'_1 + cos\left(\mathbf{i}_1, \mathbf{i}'_2\right)\mathbf{i}'_2 + cos\left(\mathbf{i}_1, \mathbf{i}'_3\right)\mathbf{i}'_3 &&= \alpha_1\mathbf{i}'_1 + \alpha_2\mathbf{i}'_2 + \alpha_3\mathbf{i}'_3 \\
\mathbf{i}_2 &= cos\left(\mathbf{i}_2, \mathbf{i}'_1\right)\mathbf{i}'_1 + cos\left(\mathbf{i}_2, \mathbf{i}'_2\right)\mathbf{i}'_2 + cos\left(\mathbf{i}_2, \mathbf{i}'_3\right)\mathbf{i}'_3 &&= \beta_1\mathbf{i}'_1 + \beta_2\mathbf{i}'_2 + \beta_3\mathbf{i}'_3 \qquad (14) \\
\mathbf{i}_3 &= cos\left(\mathbf{i}_3, \mathbf{i}'_1\right)\mathbf{i}'_1 + cos\left(\mathbf{i}_3, \mathbf{i}'_2\right)\mathbf{i}'_2 + cos\left(\mathbf{i}_3, \mathbf{i}'_3\right)\mathbf{i}'_3 &&= \gamma_1\mathbf{i}'_1 + \gamma_2\mathbf{i}'_2 + \gamma_3\mathbf{i}'_3
\end{aligned}$$

Here we use the following notation for cosines of the angles between $\mathbf{i}_i$ *and* $\mathbf{i}'_k$:

$$\begin{aligned}
\alpha_k &= cos\left(\mathbf{i}_1, \mathbf{i}'_k\right) &&= cos\left(\mathbf{i}'_k, \mathbf{i}_1\right) = \mathbf{i}_1 \cdot \mathbf{i}'_k = \mathbf{i}'_k \cdot \mathbf{i}_1 \quad k = 1,2,3 \\
\beta_k &= cos\left(\mathbf{i}_2, \mathbf{i}'_k\right) &&= cos\left(\mathbf{i}'_k, \mathbf{i}_2\right) = \mathbf{i}_2 \cdot \mathbf{i}'_k = \mathbf{i}'_k \cdot \mathbf{i}_2 \quad k = 1,2,3 \qquad (15) \\
\gamma_k &= cos\left(\mathbf{i}_3, \mathbf{i}'_k\right) &&= cos\left(\mathbf{i}'_k, \mathbf{i}_3\right) = \mathbf{i}_3 \cdot \mathbf{i}'_k = \mathbf{i}'_k \cdot \mathbf{i}_3 \quad k = 1,2,3
\end{aligned}$$

These equations include nine coefficients which are cosines of angles between the axes of the different basis. We can find the relationships for these coefficients. Multiply correspondingly each of the equations (13) and (14) by a vector which is in the left hand side of the equation and use definition of direction cosines (15):

$$\begin{aligned}
\mathbf{i}'_1 \cdot \mathbf{i}'_1 &= \alpha_1\mathbf{i}_1 \cdot \mathbf{i}'_1 + \beta_1\mathbf{i}_2 \cdot \mathbf{i}'_1 + \gamma_1\mathbf{i}_3 \cdot \mathbf{i}'_1 &&= \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1 \\
\mathbf{i}'_2 \cdot \mathbf{i}'_2 &= \alpha_2\mathbf{i}_1 \cdot \mathbf{i}'_2 + \beta_2\mathbf{i}_2 \cdot \mathbf{i}'_2 + \gamma_2\mathbf{i}_3 \cdot \mathbf{i}'_2 &&= \alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1 \\
\mathbf{i}'_3 \cdot \mathbf{i}'_3 &= \alpha_3\mathbf{i}_1 \cdot \mathbf{i}'_3 + \beta_3\mathbf{i}_2 \cdot \mathbf{i}'_3 + \gamma_3\mathbf{i}_3 \cdot \mathbf{i}'_3 &&= \alpha_3^2 + \beta_3^2 + \gamma_3^2 = 1
\end{aligned}$$

$$\begin{aligned}
\mathbf{i}_1 \cdot \mathbf{i}_1 &= \alpha_1\mathbf{i}'_1 \cdot \mathbf{i}_1 + \alpha_2\mathbf{i}'_2 \cdot \mathbf{i}_1 + \alpha_3\mathbf{i}'_3 \cdot \mathbf{i}_1 &&= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \\
\mathbf{i}_2 \cdot \mathbf{i}_2 &= \beta_1\mathbf{i}'_1 \cdot \mathbf{i}_2 + \beta_2\mathbf{i}'_2 \cdot \mathbf{i}_2 + \beta_3\mathbf{i}'_3 \cdot \mathbf{i}_2 &&= \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \\
\mathbf{i}_3 \cdot \mathbf{i}_3 &= \gamma_1\mathbf{i}'_1 \cdot \mathbf{i}_3 + \gamma_2\mathbf{i}'_2 \cdot \mathbf{i}_3 + \gamma_3\mathbf{i}'_3 \cdot \mathbf{i}_3 &&= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1
\end{aligned}$$

This procedure yields six equations for coefficients, but the first three equations are equivalent to the last three equations. Now form other products with the vectors from the same basis and use the condition of orthogonality

$$\begin{aligned}
\mathbf{i}'_1 \cdot \mathbf{i}'_2 &= \alpha_1\mathbf{i}_1 \cdot \mathbf{i}'_2 + \beta_1\mathbf{i}_2 \cdot \mathbf{i}'_2 + \gamma_1\mathbf{i}_3 \cdot \mathbf{i}'_2 &&= \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0 \\
\mathbf{i}'_1 \cdot \mathbf{i}'_3 &= \alpha_1\mathbf{i}_1 \cdot \mathbf{i}'_3 + \beta_1\mathbf{i}_2 \cdot \mathbf{i}'_3 + \gamma_1\mathbf{i}_3 \cdot \mathbf{i}'_3 &&= \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 = 0 \\
\mathbf{i}'_2 \cdot \mathbf{i}'_3 &= \alpha_2\mathbf{i}_1 \cdot \mathbf{i}'_3 + \beta_2\mathbf{i}_2 \cdot \mathbf{i}'_3 + \gamma_2\mathbf{i}_3 \cdot \mathbf{i}'_3 &&= \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 0
\end{aligned}$$

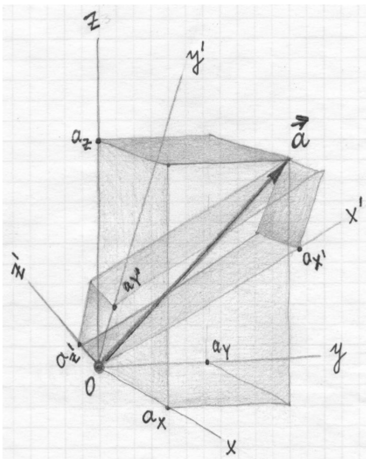
$$\begin{aligned}
\mathbf{i}_1 \cdot \mathbf{i}_2 &= \alpha_1\mathbf{i}'_1 \cdot \mathbf{i}_2 + \alpha_2\mathbf{i}'_2 \cdot \mathbf{i}_2 + \alpha_3\mathbf{i}'_3 \cdot \mathbf{i}_2 &&= \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0 \\
\mathbf{i}_1 \cdot \mathbf{i}_3 &= \alpha_1\mathbf{i}'_1 \cdot \mathbf{i}_3 + \alpha_2\mathbf{i}'_2 \cdot \mathbf{i}_3 + \alpha_3\mathbf{i}'_3 \cdot \mathbf{i}_3 &&= \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 = 0 \\
\mathbf{i}_3 \cdot \mathbf{i}_2 &= \gamma_1\mathbf{i}'_1 \cdot \mathbf{i}_2 + \gamma_2\mathbf{i}'_2 \cdot \mathbf{i}_2 + \gamma_3\mathbf{i}'_3 \cdot \mathbf{i}_2 &&= \gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3 = 0
\end{aligned}$$

Again, only three of these equations for coefficients are independent. Therefore, the nine coefficients in the basis expansions are connected only by six equations. Three coefficients remain free – it provides three conditions for the rotation of the rectangular coordinates system which can be defined by three parameters (for example, by three Euler angles).

Coordinates of vector *a*

Consider now, what will happen to the coordinates of some vector $\mathbf{a}$ under the change of the coordinate system from $Oxyz$ to $Ox'y'z'$. Write the expansion of vector $\mathbf{a}$ in $Oxyz$ (Equation 11):

$$\begin{aligned}\mathbf{a}=\sum_{i=1}^{3}a_{x_i}\mathbf{i}_i &=\|\mathbf{a}\|cos(\mathbf{a},\mathbf{i}_1)\mathbf{i}_1+\|\mathbf{a}\|cos(\mathbf{a},\mathbf{i}_2)\mathbf{i}_2+\|\mathbf{a}\|cos(\mathbf{a},\mathbf{i}_3)\mathbf{i}_3\\ &=(\mathbf{a}\cdot\mathbf{i}_1)\mathbf{i}_1+(\mathbf{a}\cdot\mathbf{i}_2)\mathbf{i}_2+(\mathbf{a}\cdot\mathbf{i}_3)\mathbf{i}_3\\ &=a_x\mathbf{i}_1+a_y\mathbf{i}_2+a_z\mathbf{i}_3\end{aligned}$$



Use equation (12): $\mathbf{a}_\mathbf{u}=\left[a_x\,cos(\mathbf{i}_1,\mathbf{u})+a_y\,cos(\mathbf{i}_2,\mathbf{u})+a_z\,cos(\mathbf{i}_3,\mathbf{u})\right]\mathbf{u}$,

then projections of vector $\mathbf{a}$ on the vectors $\{\mathbf{i}'_1,\mathbf{i}'_2,\mathbf{i}'_3\}$ are

$$\mathbf{a}_{\mathbf{i}'_1}=\left[a_x\,cos(\mathbf{i}_1,\mathbf{i}'_1)+a_y\,cos(\mathbf{i}_2,\mathbf{i}'_1)+a_z\,cos(\mathbf{i}_3,\mathbf{i}'_1)\right]\mathbf{i}'_1=\left(a_x\alpha_1+a_y\beta_1+a_z\gamma_1\right)\mathbf{i}'_1$$

$$\mathbf{a}_{\mathbf{i}'_2}=\left[a_x\,cos(\mathbf{i}_1,\mathbf{i}'_2)+a_y\,cos(\mathbf{i}_2,\mathbf{i}'_2)+a_z\,cos(\mathbf{i}_3,\mathbf{i}'_2)\right]\mathbf{i}'_2=\left(a_x\alpha_2+a_y\beta_2+a_z\gamma_2\right)\mathbf{i}'_2$$

$$\mathbf{a}_{\mathbf{i}'_3}=\left[a_x\,cos(\mathbf{i}_1,\mathbf{i}'_3)+a_y\,cos(\mathbf{i}_2,\mathbf{i}'_3)+a_z\,cos(\mathbf{i}_3,\mathbf{i}'_3)\right]\mathbf{i}'_3=\left(a_x\alpha_3+a_y\beta_3+a_z\gamma_3\right)\mathbf{i}'_3$$

It means that coordinates of vector $\mathbf{a}$ in the new coordinate system are:

$$\begin{aligned}a_{x'}&=a_x\alpha_1+a_y\beta_1+a_z\gamma_1\\ a_{y'}&=a_x\alpha_2+a_y\beta_2+a_z\gamma_2\\ a_{z'}&=a_x\alpha_3+a_y\beta_3+a_z\gamma_3\end{aligned}\tag{16}$$

They provide the direct transformation of the vector's coordinates under the change of coordinate system from $Oxyz$ to $Ox'y'z'$.

Similarly, it can be shown that under the change of coordinate system from $Oxyz$ to $Ox'y'z'$ the coordinates of vector $\mathbf{a}$ are transformed according to

$$\begin{aligned}a_x&=a_{x'}\alpha_1+a_{y'}\alpha_2+a_{z'}\alpha_3\\ a_y&=a_{x'}\beta_1+a_{y'}\beta_2+a_{z'}\beta_3\\ a_y&=a_{x'}\gamma_1+a_{y'}\gamma_2+a_{z'}\gamma_3\end{aligned}\tag{17}$$

In particular, if we consider the transformation of coordinates of the point (x,y,z) to (x',y',z') under the change of coordinate system from $Oxyz$ to $Ox'y'z'$ with the same origin, we have: the direct transformation of coordinates

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=\begin{pmatrix}\alpha_1&\beta_1&\gamma_1\\\alpha_2&\beta_2&\gamma_3\\\alpha_3&\beta_3&\gamma_3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{aligned}x'&=\alpha_1x+\beta_1y+\gamma_1z\\ y'&=\alpha_2x+\beta_2y+\gamma_2z\\ z'&=\alpha_3x+\beta_3y+\gamma_3z\end{aligned}\tag{18}$$

and the inverse transformation

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\\\beta_1&\beta_2&\beta_3\\\gamma_1&\gamma_2&\gamma_3\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$$

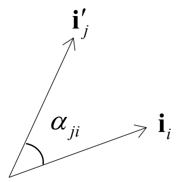
$$\begin{aligned}x&=\alpha_1x'+\alpha_2y'+\alpha_3z'\\ y&=\beta_1x'+\beta_2y'+\beta_3z'\\ z&=\gamma_1x'+\gamma_2y'+\gamma_3z'\end{aligned}\tag{19}$$

11. Alternative matrix representation of transformation of coordinates (transiotional stage to tensor notations)

Consider two orthogonal coordinate systems $Oxyz$ and $Ox'y'z'$ defined by the orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ and $\{\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3\}$.

introduce the new notation for directional cosines α_{ij}

Define the cosines of the angles between coordinate vectors $\mathbf{i}_i$ *and* $\mathbf{i}'_j$



$\alpha_{ji} = cos\left(\mathbf{i}'_j, \mathbf{i}_i\right) = \mathbf{i}'_j \cdot \mathbf{i}_i$

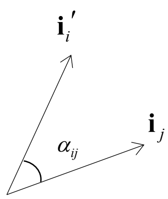
$$cos\left(\mathbf{i}'_j, \mathbf{i}_1\right) = \mathbf{i}'_j \cdot \mathbf{i}_1 = \alpha_{j1}$$
$$cos\left(\mathbf{i}'_j, \mathbf{i}_2\right) = \mathbf{i}'_j \cdot \mathbf{i}_2 = \alpha_{j2}$$
$$cos\left(\mathbf{i}'_j, \mathbf{i}_3\right) = \mathbf{i}'_j \cdot \mathbf{i}_3 = \alpha_{j3}$$

α_{ji}

Directional cosines can be written in a matix form

$$\alpha_{ji} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_2 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_1 \\ \mathbf{i}'_1 \cdot \mathbf{i}_2 & \mathbf{i}'_2 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_2 \\ \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_3 \end{pmatrix}$$

α_{ji} $\mathbf{i}'_j \cdot \mathbf{i}_i$



$\alpha_{ij} = cos\left(\mathbf{i}'_i, \mathbf{i}_j\right) = \mathbf{i}'_i \cdot \mathbf{i}_j$

$$\alpha_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{i}_1 \cdot \mathbf{i}'_1 & \mathbf{i}_2 \cdot \mathbf{i}'_1 & \mathbf{i}_3 \cdot \mathbf{i}'_1 \\ \mathbf{i}_1 \cdot \mathbf{i}'_2 & \mathbf{i}_2 \cdot \mathbf{i}'_2 & \mathbf{i}_3 \cdot \mathbf{i}'_2 \\ \mathbf{i}_1 \cdot \mathbf{i}'_3 & \mathbf{i}_2 \cdot \mathbf{i}'_3 & \mathbf{i}_3 \cdot \mathbf{i}'_3 \end{pmatrix}$$

α_{ij} $\mathbf{i}_j \cdot \mathbf{i}'_i$

Representation of coordinate vectors $\mathbf{i}'_i$ in coordinate system $Oxyz$

$$\mathbf{i}'_1 = \left(\mathbf{i}'_1 \cdot \mathbf{i}_1\right)\mathbf{i}_1 + \left(\mathbf{i}'_1 \cdot \mathbf{i}_2\right)\mathbf{i}_2 + \left(\mathbf{i}'_1 \cdot \mathbf{i}_3\right)\mathbf{i}_3 = \alpha_{11}\mathbf{i}_1 + \alpha_{12}\mathbf{i}_2 + \alpha_{13}\mathbf{i}_3$$
$$\mathbf{i}'_2 = \left(\mathbf{i}'_2 \cdot \mathbf{i}_1\right)\mathbf{i}_1 + \left(\mathbf{i}'_2 \cdot \mathbf{i}_2\right)\mathbf{i}_2 + \left(\mathbf{i}'_2 \cdot \mathbf{i}_3\right)\mathbf{i}_3 = \alpha_{21}\mathbf{i}_1 + \alpha_{22}\mathbf{i}_2 + \alpha_{23}\mathbf{i}_3$$
$$\mathbf{i}'_3 = \left(\mathbf{i}'_3 \cdot \mathbf{i}_1\right)\mathbf{i}_1 + \left(\mathbf{i}'_3 \cdot \mathbf{i}_2\right)\mathbf{i}_2 + \left(\mathbf{i}'_3 \cdot \mathbf{i}_3\right)\mathbf{i}_3 = \alpha_{31}\mathbf{i}_1 + \alpha_{32}\mathbf{i}_2 + \alpha_{33}\mathbf{i}_3$$

$$\begin{pmatrix} \mathbf{i}'_1 \\ \mathbf{i}'_2 \\ \mathbf{i}'_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix}$$

$\mathbf{i}'_j = \alpha_{ji}\mathbf{i}_i$

Representation of coordinate vectors $\mathbf{i}_i$ in coordinate system $Ox'y'z'$

$$\mathbf{i}_1 = \left(\mathbf{i}_1 \cdot \mathbf{i}'_1\right)\mathbf{i}'_1 + \left(\mathbf{i}_1 \cdot \mathbf{i}'_2\right)\mathbf{i}'_2 + \left(\mathbf{i}_1 \cdot \mathbf{i}'_3\right)\mathbf{i}'_3 = \alpha_{11}\mathbf{i}'_1 + \alpha_{21}\mathbf{i}'_2 + \alpha_{31}\mathbf{i}'_3$$
$$\mathbf{i}_2 = \left(\mathbf{i}_2 \cdot \mathbf{i}'_1\right)\mathbf{i}'_1 + \left(\mathbf{i}_2 \cdot \mathbf{i}'_2\right)\mathbf{i}'_2 + \left(\mathbf{i}_2 \cdot \mathbf{i}'_3\right)\mathbf{i}'_3 = \alpha_{12}\mathbf{i}'_1 + \alpha_{22}\mathbf{i}'_2 + \alpha_{32}\mathbf{i}'_3$$
$$\mathbf{i}_3 = \left(\mathbf{i}_3 \cdot \mathbf{i}'_1\right)\mathbf{i}'_1 + \left(\mathbf{i}_3 \cdot \mathbf{i}'_2\right)\mathbf{i}'_2 + \left(\mathbf{i}_3 \cdot \mathbf{i}'_3\right)\mathbf{i}'_3 = \alpha_{13}\mathbf{i}'_1 + \alpha_{23}\mathbf{i}'_2 + \alpha_{33}\mathbf{i}'_3$$

$$\begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \mathbf{i}'_1 \\ \mathbf{i}'_2 \\ \mathbf{i}'_3 \end{pmatrix}$$

$\mathbf{i}_i = \alpha_{ji}\mathbf{i}'_j$

Representation of vector **a** in $Ox'y'z'$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{i}'_1)\mathbf{i}'_1 + (\mathbf{a}\cdot\mathbf{i}'_2)\mathbf{i}'_2 + (\mathbf{a}\cdot\mathbf{i}'_3)\mathbf{i}'_3 = x'_1\mathbf{i}'_1 + x'_2\mathbf{i}'_2 + x'_3\mathbf{i}'_3$$

Use transformation of basis vectors

$$x'_1 = \mathbf{a}\cdot\mathbf{i}'_1 = \mathbf{a}\cdot\left[(\mathbf{i}'_1\cdot\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{i}'_1\cdot\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{i}'_1\cdot\mathbf{i}_3)\mathbf{i}_3\right] = \left[(\mathbf{i}'_1\cdot\mathbf{i}_1)\mathbf{a}\cdot\mathbf{i}_1 + (\mathbf{i}'_1\cdot\mathbf{i}_2)\mathbf{a}\cdot\mathbf{i}_2 + (\mathbf{i}'_1\cdot\mathbf{i}_3)\mathbf{a}\cdot\mathbf{i}_3\right] = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3$$

$$x'_2 = \mathbf{a}\cdot\mathbf{i}'_2 = \mathbf{a}\cdot\left[(\mathbf{i}'_2\cdot\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{i}'_2\cdot\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{i}'_2\cdot\mathbf{i}_3)\mathbf{i}_3\right] = \left[(\mathbf{i}'_2\cdot\mathbf{i}_1)\mathbf{a}\cdot\mathbf{i}_1 + (\mathbf{i}'_2\cdot\mathbf{i}_2)\mathbf{a}\cdot\mathbf{i}_2 + (\mathbf{i}'_2\cdot\mathbf{i}_3)\mathbf{a}\cdot\mathbf{i}_3\right] = \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3$$

$$x'_3 = \mathbf{a}\cdot\mathbf{i}'_3 = \mathbf{a}\cdot\left[(\mathbf{i}'_3\cdot\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{i}'_3\cdot\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{i}'_3\cdot\mathbf{i}_3)\mathbf{i}_3\right] = \left[(\mathbf{i}'_3\cdot\mathbf{i}_1)\mathbf{a}\cdot\mathbf{i}_1 + (\mathbf{i}'_3\cdot\mathbf{i}_2)\mathbf{a}\cdot\mathbf{i}_2 + (\mathbf{i}'_3\cdot\mathbf{i}_3)\mathbf{a}\cdot\mathbf{i}_3\right] = \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3$$

$$x'_j = \alpha_{ji}x_i \qquad \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{18b}$$

Representation of vector **a** in $Oxyz$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{a}\cdot\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{a}\cdot\mathbf{i}_3)\mathbf{i}_3 = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$$

Use transformation of basis vectors

$$x_1 = \mathbf{a}\cdot\mathbf{i}_1 = \mathbf{a}\cdot\left[(\mathbf{i}_1\cdot\mathbf{i}'_1)\mathbf{i}'_1 + (\mathbf{i}_1\cdot\mathbf{i}'_2)\mathbf{i}'_2 + (\mathbf{i}_1\cdot\mathbf{i}'_3)\mathbf{i}'_3\right] = \left[(\mathbf{i}_1\cdot\mathbf{i}'_1)\mathbf{a}\cdot\mathbf{i}'_1 + (\mathbf{i}_1\cdot\mathbf{i}'_2)\mathbf{a}\cdot\mathbf{i}'_2 + (\mathbf{i}_1\cdot\mathbf{i}'_3)\mathbf{a}\cdot\mathbf{i}'_3\right] = \alpha_{11}x'_1 + \alpha_{21}x'_2 + \alpha_{31}x'_3$$

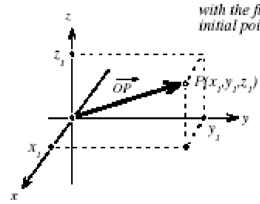
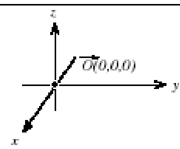
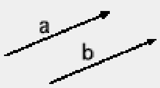
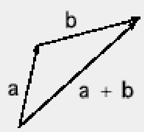
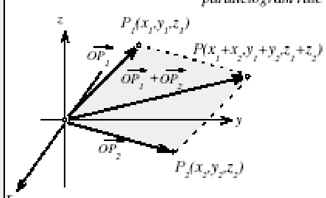
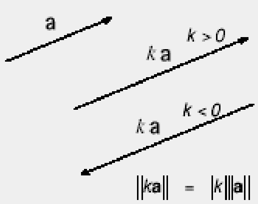
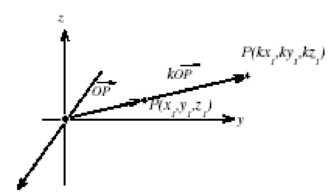
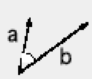
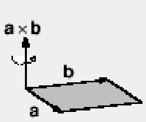
$$x_2 = \mathbf{a}\cdot\mathbf{i}_2 = \mathbf{a}\cdot\left[(\mathbf{i}_2\cdot\mathbf{i}'_1)\mathbf{i}'_1 + (\mathbf{i}_2\cdot\mathbf{i}'_2)\mathbf{i}'_2 + (\mathbf{i}_2\cdot\mathbf{i}'_3)\mathbf{i}'_3\right] = \left[(\mathbf{i}_2\cdot\mathbf{i}'_1)\mathbf{a}\cdot\mathbf{i}'_1 + (\mathbf{i}_2\cdot\mathbf{i}'_2)\mathbf{a}\cdot\mathbf{i}'_2 + (\mathbf{i}_2\cdot\mathbf{i}'_3)\mathbf{a}\cdot\mathbf{i}'_3\right] = \alpha_{12}x'_1 + \alpha_{22}x'_2 + \alpha_{32}x'_3$$

$$x_3 = \mathbf{a}\cdot\mathbf{i}_3 = \mathbf{a}\cdot\left[(\mathbf{i}_3\cdot\mathbf{i}'_1)\mathbf{i}'_1 + (\mathbf{i}_3\cdot\mathbf{i}'_2)\mathbf{i}'_2 + (\mathbf{i}_3\cdot\mathbf{i}'_3)\mathbf{i}'_3\right] = \left[(\mathbf{i}_3\cdot\mathbf{i}'_1)\mathbf{a}\cdot\mathbf{i}'_1 + (\mathbf{i}_3\cdot\mathbf{i}'_2)\mathbf{a}\cdot\mathbf{i}'_2 + (\mathbf{i}_3\cdot\mathbf{i}'_3)\mathbf{a}\cdot\mathbf{i}'_3\right] = \alpha_{13}x'_1 + \alpha_{23}x'_2 + \alpha_{33}x'_3$$

$$x_i = \alpha_{ji}x'_j \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \tag{19b}$$

The coordinates of a vector are transformed in the same way as the basis vectors.

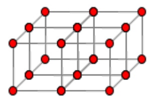
Vectors in Euclidian Space

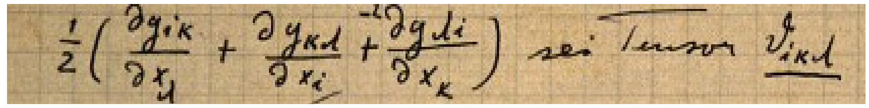
free vector	position vector	coordinate vector	1st order tensor
 *directed segment*	*directed segment with the fixed initial point* 	*triple of real numbers* $\mathbf{a} = \langle a_1,a_2,a_3\rangle$ or $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$	a_i with index convention: $i = 1,2,3$ and summation convention: $a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3$ $\delta_{ij} a_j = a_1 + a_2 + a_3$
zero vector · *any point*		$\mathbf{0} = \langle 0,0,0\rangle$	0
norm $\lVert \mathbf{a} \rVert$ = *length of segment*	$\lVert \vec{OP} \rVert = \sqrt{x_1^2 + y_1^2 + z_1^2}$	$\lVert \mathbf{a} \rVert = \sqrt{a_1^2 + a_2^2 + a_3^2}$	$\sqrt{\delta_{ij} x_i x_j}$
equality $\mathbf{a} = \mathbf{b}$  *two vectors are equal if they have the same norm and direction*	$\vec{OP}_1 = \vec{OP}_2 \Leftrightarrow$ $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$	$\mathbf{a} = \mathbf{b} \Leftrightarrow$ $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$	$a_i = b_i$
summation *triangle rule* 	*parallelogram rule* 	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2+b_2, a_3+b_3\rangle$	$a_i + b_i$
multiplication by a scalar 		$k\,\mathbf{a} = \langle ka_1, ka_2, ka_3\rangle$	ka_i
dot product  $\mathbf{a}\cdot\mathbf{b} = \lVert \mathbf{a} \rVert \lVert \mathbf{b} \rVert \cos(\mathbf{a},\mathbf{b})$	$\vec{OP}_1 \cdot \vec{OP}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$	$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$	$a_i b_i$
cross product 	$\vec{OP}_1 \times \vec{OP}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$	$\mathbf{a}\times\mathbf{b}$ $= \{a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1\}$	$\varepsilon_{ijk} a_j b_k$ $(\mathbf{a}\times\mathbf{b})_i = a_j b_k - a_k b_j$ i,j,k is cyclic permutation of 1,2,3

The relationship (homomorphism) between these vector spaces can be established, making them equivalent in this sense. As a result, we can flexibly choose the most suitable representation for a given situation. Geometric vectors are ideal for visualizing physical models, coordinate vectors are more convenient for calculations, and tensors simplify and streamline the derivation of equations.

IV.1.8. TENSORS Index notation and the summation convention

A_{ijk}





A ***tensor*** is an organized multidimensional *array* of numerical values (numbers) which are called the components of a tensor.

Each tensor comes equipped with a ***transformation law*** that details how the components of the tensor respond to a change of basis.

Order of a tensor is a number of dimensions needed for its representations (number of indices needed to label the components).

The following ***convention*** is universally accepted in modern mathematical and physical literature:

Einstein convention

1. Any index can appear in lower or upper position:

$$a_k\ ,\quad x^i\ ,\quad c_{ij}^k$$

2. Any index which appears **once** in the expression can take values $1\,,2\,,3$

a_k	denotes 3 quantities:	a_1, a_2, a_3
$a_i b_j$	denotes 9 quantities:	$a_1b_1, a_1b_2, ..., a_3b_3$
A_{ijk}	denotes 27 quantities:	$A_{111}, A_{112}, ..., A_{333}$

3. Any index which appears exactly **twice** in any terms of an expression denotes **summation** with respect to this index from 1 to 3

$$a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

$$a_k b_k = \sum_{k=1}^{3} a_k b_k = a_1b_1 + a_2b_2 + a_3b_3$$

$$a_{ij} b_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} b_{ij} = a_{11}b_{11} + a_{12}b_{12} + \ldots + a_{33}b_{33}$$

With this convention the summation sign can be dropped and expressions are simplified. Note that index of summation is a "dummy" variable, that means that any other index in the same position produces the same result:

$$A_{ii} = A_{kk} = A_{11} + A_{22} + A_{33}$$

4. The coordinates of a point are usually denoted:
in the *oblique coordinate system* by x^i (*upper index*)
in the *rectangular coordinate system* by x_i (*lower index*)

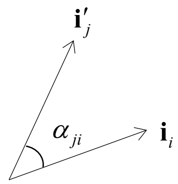
5. The change of coordinate system is denoted by a prime.
The coordinates of the same point are denoted
in the *rectangular* coordinate system $Oxyz$ by x_i
in the *rectangular* coordinate system $O'x'y'z'$ by x'_i

We will consider tensors in the rectangular coordinate systems.
They are called **Cartesian tensors** (or **affine orthogonal tensors**).

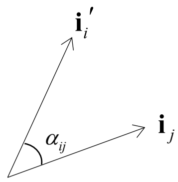
Transformation of coordinates

Consider how the equations for transformation of coordinates can be rewritten using the index convention and produce some additional results.

Directional cosines
$$\alpha_{ji} = cos\left(\mathbf{i}'_j, \mathbf{i}_i\right) = \mathbf{i}'_j \cdot \mathbf{i}_i$$
$$\alpha_{ij} = cos\left(\mathbf{i}'_i, \mathbf{i}_j\right) = \mathbf{i}'_i \cdot \mathbf{i}_j \qquad (21)$$



$$\alpha_{ji} = cos\left(\mathbf{i}'_j, \mathbf{i}_i\right) = \mathbf{i}'_j \cdot \mathbf{i}_i$$



$$\alpha_{ij} = cos\left(\mathbf{i}'_i, \mathbf{i}_j\right) = \mathbf{i}'_i \cdot \mathbf{i}_j$$

Directional cosines can be written in a matix form

$$\alpha_{ji} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_2 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_1 \\ \mathbf{i}'_1 \cdot \mathbf{i}_2 & \mathbf{i}'_2 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_2 \\ \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_3 \end{pmatrix}$$

α_{ji} $\qquad$ $\mathbf{i}'_j \cdot \mathbf{i}_i$

$$\alpha_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{i}_1 \cdot \mathbf{i}'_1 & \mathbf{i}_2 \cdot \mathbf{i}'_1 & \mathbf{i}_3 \cdot \mathbf{i}'_1 \\ \mathbf{i}_1 \cdot \mathbf{i}'_2 & \mathbf{i}_2 \cdot \mathbf{i}'_2 & \mathbf{i}_3 \cdot \mathbf{i}'_2 \\ \mathbf{i}_1 \cdot \mathbf{i}'_3 & \mathbf{i}_2 \cdot \mathbf{i}'_3 & \mathbf{i}_3 \cdot \mathbf{i}'_3 \end{pmatrix}$$

α_{ij} $\qquad$ $\mathbf{i}_j \cdot \mathbf{i}'_i$

Transformation of the basis vectors, see equations (13) and (14):

$$\mathbf{i}'_j = \alpha_{ji}\mathbf{i}_i \qquad (22')$$

$$\mathbf{i}_i = \alpha_{ji}\mathbf{i}'_j \qquad (23')$$

Then transformation of coordinates (18b) with the Einstein convention becomes

$$x'_j = \alpha_{ji} x_i \qquad (22)$$

and the inverse transformation (19b) is given by

$$x_i = \alpha_{ji} x'_j \qquad (23)$$

Some useful identities for coefficients can be derived with tensor notations:

Kronecker delta

$$\delta_{ij} \equiv \mathbf{i}_i \cdot \mathbf{i}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{i}_i = \alpha_{ki}\mathbf{i}'_k, \quad \mathbf{i}_j = \alpha_{kj}\mathbf{i}'_k \quad \Rightarrow \quad \mathbf{i}_i \cdot \mathbf{i}_j = \alpha_{ki}\mathbf{i}'_k \cdot \alpha_{kj}\mathbf{i}'_k = \alpha_{ki}\alpha_{kj}\, \mathbf{i}'_k \cdot \mathbf{i}'_k = \alpha_{ki}\alpha_{kj} = \delta_{ij} \qquad (26)$$

$$\mathbf{i}'_j = \alpha_{jk}\mathbf{i}_k, \quad \mathbf{i}'_i = \alpha_{ik}\mathbf{i}_k \quad \Rightarrow \quad \mathbf{i}'_i \cdot \mathbf{i}'_j = \alpha_{ik}\mathbf{i}_k \cdot \alpha_{jk}\mathbf{i}_k = \alpha_{ik}\alpha_{jk}\, \mathbf{i}_k \cdot \mathbf{i}_k = \alpha_{ik}\alpha_{jk} = \delta_{ij} \qquad (27)$$

The zero order tensors (scalars) a

Definition of the 0th order tensor

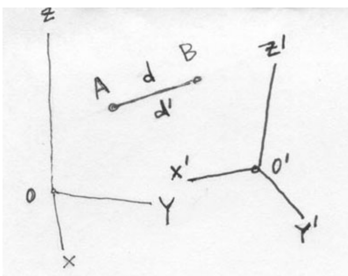
The *zero-order tensors* are the elements of the field of real numbers $\mathbb{R}$, which are uniquely specified in any coordinate system by a single number $a \in \mathbb{R}$, and are invariant under the change of coordinate system:

$$a' = a$$

Example:

The ***distance between two points*** is the same in any coordinate system and is represented by the zero order tensor (scalar).

Indeed, consider two points:



Point A with coordinates a_i in $Oxyz$ and a'_i in $O'x'y'z'$
Point B with coordinates b_i in $Oxyz$ and b'_i in $O'x'y'z'$

Let the coordinates of the origin O' in the system $Oxyz$ be $x_i^{o'}$
the coordinates of the origin O in the system $O'x'y'z'$ be x_i^{o}

Then

$$a'_i = \alpha_{ij} a_j + x_i^{o'} \qquad b'_i = \alpha_{ij} b_j + x_i^{o'} \qquad \Rightarrow \quad a'_i - b'_i = \alpha_{ij}\left(a_j - b_j\right)$$
$$a_i = \alpha_{ij} a'_j + x_i^{o} \qquad b_i = \alpha_{ij} b'_j + x_i^{o} \qquad \Rightarrow \quad a_i - b_i = \alpha_{ij}\left(a'_j - b'_j\right) \qquad (29)$$

By the Pythagorean Theorem:

$$\left(d'\right)^2 = \sum_{i=1}^{3}\left(a'_i - b'_i\right)^2$$

$$= \sum_{i=1}^{3}\alpha_{ij}\left(a_j - b_j\right)\alpha_{ik}\left(a_k - b_k\right)$$ use Einstein convention

$$= \alpha_{ij}\alpha_{ik}\left(a_j - b_j\right)\left(a_k - b_k\right)$$

$$= \delta_{jk}\left(a_j - b_j\right)\left(a_k - b_k\right)$$ from (26) $\alpha_{ji}\alpha_{ki} = \delta_{jk}$

$$= \delta_{jk}\left(a_j - b_j\right)^2$$ rewrite using sigma

$$= \sum_{j=1}^{3}\left(a_j - b_j\right)^2$$

$$= d^2$$

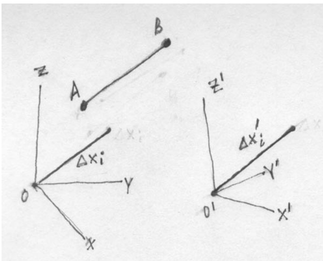
Therefore, by taking the square roots, we formally establish this geometrical fact that

$$d = d'$$

i.e. distance between points does not depend on the choice of coordinate system and it is invariant under linear transformation of coordinates.

The first-order tensors (vectors) a_i

Consider the vector space $\mathbb{R}_3$. Let $Oxyz$ and $O'x'y'z'$ be two rectangular coordinate systems in this space. Consider two points



Point A with coordinates a_i in $Oxyz$ and a'_i in $O'x'y'z'$
Point B with coordinates b_i in $Oxyz$ and b'_i in $O'x'y'z'$

Let the coordinates of the origin O' in the system $Oxyz$ be $x_i^{o'}$
the coordinates of the origin 0 in the system $O'x'y'z'$ be x_i^{o}

Then

$$a'_i = \alpha_{ji} a_j + x_i^{o'} \qquad b'_i = \alpha_{ji} b_j + x_i^{o'}$$
$$a_i = \alpha_{ij} a'_j + x_i^{o} \qquad b_i = \alpha_{ij} b'_j + x_i^{o} \qquad (30)$$

The increments of coordinates in two systems are connected through the relation

$$\begin{aligned} \Delta x_i &= a_i - b_i \\ &= \left(\alpha_{ij} a'_j + x_i^o\right) - \left(\alpha_{ij} b'_j + x_i^o\right) \\ &= \alpha_{ij}\left(a'_j - b'_j\right) \\ &= \alpha_{ij} \Delta x'_j \end{aligned}$$

This equation determines the transformation of the difference between the coordinates of two points under the change of coordinate system from $Oxyz$ to $O'x'y'z'$. This transformation is also equivalent to the transformation of the coordinates of the point under the change of coordinate system from $Oxyz$ to $Ox'y'z'$ when the origin of the coordinate system is fixed (just rigid rotation):

$$x_i = \alpha_{ji} x'_j$$

This consideration is a foundation for the following definition:

Definition of the 1st order tensor

The ***first-order tensor*** (affine vector or just a vector) is given in any coordinate system $Oxyz$ by a triple x_i which is transformed under the change of coordinate system to $Ox'y'z'$ according to the law:

$$x_i = \alpha_{ji} x'_j \qquad (31a)$$

$$x'_j = \alpha_{ji} x_i \qquad (31b)$$

.

Note that a zero vector is a zero vector in all coordinate systems. The 1st order tensors are equivalent to coordinate vectors; the comparison of them in the Table of Vectors in Euclidean Space (p.230) shows only some simplification in the notations. But the advantage is in the possibility of generalizing them to arbitrary order tensors.

Example

Suppose that the function $x_i(t)$ determines the position (trajectory) of a particle of mass *m* in space with the coordinate system $Oxyz$. Show that the force acting on this particle is a vector.

For the time interval from t to $t+\Delta t$, the displacement of the particle in coordinate system $Oxyz$ is given by

$$\Delta x_i = x_i(t+\Delta t) - x_i(t)$$

which in the other coordinate system $Ox'y'z'$ is written as:

$$\Delta x_i' = x_i'(t+\Delta t) - x_i'(t)$$

If we assume that time does not depend on coordinate system, $t = t'$, then according to (31)

$$x_i'(t+\Delta t) - x_j'(t) = \alpha_{ji}\left[x_i(t+\Delta t) - x_i(t)\right]$$

Therefore, displacement $\Delta x_i'$ is a vector, and $\dfrac{\Delta x_i'}{\Delta t}$ is also a vector. Moreover, provided that the limit exists,

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta x_i'}{\Delta t}$$

is also a vector, which defines instantaneous velocity of the particle at the moment of time t.

By similar arguments, the acceleration $\mathbf{a}$ is also a vector with components a_i. Then, according to Newton's Second Law

$$F_i = ma_i$$

holds in any coordinate system, and the force is a vector

$$\mathbf{F} = m\mathbf{a}$$

i.e. force is defined both by amplitude and direction and cannot become a scalar by a choice of coordinate system.

In general, vector cannot become a scalar by a choice of coordinate system.

Dot product

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i \quad \text{tensor notation}$$

Norm

$$a = \|\mathbf{a}\| = = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_i a_i} \quad \text{tensor notation}$$

The second-order tensors (matrices) A_{ij}

Consider the ordered triple of vectors which in the coordinate system $Oxyz$ are written as

$$a_{i1}, a_{i2}, a_{i3}$$

They are described by the 9 components

$$a_{i1} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, a_{i2} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, a_{i3} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

This elements can be organized into one unit as

$$A_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{32}$$

According to (31), under the change of coordinates from $Oxyz$ to $Ox'y'z'$, vectors a_{i1}, a_{i2}, a_{i3} are transformed to

$$a'_{i1} = \alpha_{ik} a_{k1}$$
$$a'_{i2} = \alpha_{ik} a_{k2}$$
$$a'_{i3} = \alpha_{ik} a_{k3}$$

The simultaneous transformation of the components of all three vectors under the change of coordinate system can be performed in the following way

$$A'_{ij} = \alpha_{ik} \alpha_{jm} A_{km}$$

Definition of the 2nd order tensor

The quantity defined by nine components $A_{ij} \in \mathbb{R}$ which are transformed under the change of coordinate system according to the law

$$A'_{ij} = \alpha_{ik} \alpha_{jm} A_{km} \tag{33}$$

is called a ***second-order tensor***.

A second-order tensor can be written as A_{ij} or in the matrix form (32).

A second order tensor defined in one coordinate system can be determined in any other coordinate system according to transformation (33).

If all components of a tensor are equal to zero, then the tensor is called a **zero tensor**. It is obvious that a zero tensor is a zero tensor in all coordinate systems.

Dyadic

A 2nd order tensor can be obtained as a listing of all cross products of the components of two vectors a_i and b_j

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \qquad \textbf{dyadic (outer product)} \qquad (34)$$

The transformation of the vectors is given by

$$a_i' = \alpha_{ik} a_k$$

$$b_j' = \alpha_{jm} b_m$$

Then transformation of (34) is defined by

$$a_i' b_j' = \alpha_{ik} \alpha_{jm} a_k b_m$$

Therefore (34) is a second-order tensor. It is called ***dyadic*** and is denoted by

$$a_i b_j$$

Vector notation for dyadic is $\mathbf{a}\,\mathbf{b}$. Note that $\mathbf{ab} \neq \mathbf{ba}$.

Example 1

(*stress tensor)*

Consider a point *M* in space.
A force acting on some element of area dS containing point *M* is

$$\mathbf{f} = \mathbf{p} dS$$

where $\mathbf{p}$ is a stress. Vector $\mathbf{p}$ can be expanded into

$$\mathbf{p} dS = \mathbf{p}_1 dS_1 + \mathbf{p}_2 dS_2 + \mathbf{p}_3 dS_3$$

where vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are the components of the stress tensor.

Example 2

(*the deformation tensor in the linear theory of elasticity)*

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

Example 3

(*the rate of the deformation tensor)*

$$v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)$$

IV.1.9. TENSOR ALGEBRA

Consider some operations with tensors of different order.

Transpose $A^T_{ij} = A_{ji}$

Addition Let A_{ij}, B_{ij} be the 2nd order tensors, then define

$$C_{ij} = A_{ij} + B_{ij}$$

Check if the result is a 2nd order tensor:

$$A'_{ij} = \alpha_{ik} a_{jm} A_{km}, \quad B'_{ij} = \alpha_{ik} a_{jm} B_{km}$$

$$C'_{ij} = \alpha_{ik} a_{jm} A_{km} + \alpha_{ik} a_{jm} B_{km} = \alpha_{ik} a_{jm} \left(A_{km} + B_{km} \right) = \alpha_{ik} a_{jm} C_{km}$$

Therefore, the defined sum is a tensor and transforms the same way.

Multiplication by a scalar ca, cx_i, cA_{ij}

Outer product of tensors $C_{ijkm} = A_{ij} B_{km}$ fourth-order tensor (outer product)

Contraction of tensors A_{ii} zero order tensor

$A_{ik} b_k$ multiplication of a matrix by a vector, the result is a vector (1st order tensor)

Matrix multiplication $A_{ik} B_{kj}$ matrix multiplication (2nd order tensor)

Dot product $a_i b_i$ inner product, dot or scalar product (0th order tensor)

Note that contraction reduces the order of a tensor.

Symmetry $A_{ij} = A_{ji}$ ***symmetric tensor***

$A_{ij} = -A_{ji}$ ***antisymmetric tensor***

Symmetry properties of tensors are not changed under the change coordinates: a tensor which is symmetric (antisymmetric) in one coordinate system is symmetric (antisymmetric) in any other coordinate system. General forms

symmetric *antisymmetric*

$$A_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \qquad A_{ij} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

Any *2nd* order tensor T_{ij} can be represented as a sum of a symmetric and antisymmetric tensors. In fact, such an expansion can be written as

$T_{ij} = S_{ij} + A_{ij}$, where $S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji} \right)$ *symmetric*

$A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji} \right)$ *antisymmetric*

Kronecker delta δ_{ij} The Kronecker delta is a symmetric tensor (**unit tensor**) defined as

Getulio Alviani

$$\delta_{ij} = \mathbf{i}_i \cdot \mathbf{i}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{35}$$

Operations with δ_{ij}

symmetry $\delta_{ij} = \delta_{ji}$

change of index $\delta_{ik} a_i = a_k$ This expression according to tensor convention yields:

$k = 1 \quad \delta_{i1} a_i = \delta_{11} a_1 + \delta_{21} a_2 + \delta_{31} a_3 = a_1$

$k = 2 \quad \delta_{i2} a_i = \delta_{12} a_1 + \delta_{22} a_2 + \delta_{32} a_3 = a_2$

$k = 3 \quad \delta_{i3} a_i = \delta_{13} a_1 + \delta_{23} a_2 + \delta_{33} a_3 = a_3$

change of index $\delta_{ik} A_{km} = A_{im}$

$A_{ij} \delta_{jk} = A_{ij} \delta_{kj} = A_{ik}$

tensor inverse $A_{ik} A_{kj}^{-1} = \delta_{ij}$

contractions $\delta_{ii} = 3$

$\delta_{ik} \delta_{kj} = \delta_{ij}$

$\delta_{im} \delta_{mk} \delta_{kj} = \delta_{ij}$

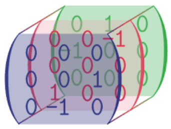
factoring $A_{ik} b_k - c b_i = A_{ik} b_k - c \delta_{ik} b_k = \left(A_{ik} - c \delta_{ik} \right) b_k$

Leopold Kronecker
$(1823-1891)$

Tullio Levi-Civita
$(1873-1941)$

Alternating unit tensor ε_{ijk} **(Levi-Civita tensor)** is a special *3^rd^* order tensor defined as

$$\varepsilon_{ijk} = \left(\mathbf{i}_i \times \mathbf{i}_j\right) \cdot \mathbf{i}_k$$ *consists of 27 entries*

$\epsilon_{ijk} =$ 

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if} \quad ijk = 123,\ 231,\ \text{or } 312 \\ 0 & \text{if} \quad \text{any two indices are alike} \\ -1 & \text{if} \quad ijk = 321,\ 213,\ \text{or } 132 \end{cases} \tag{36}$$

Rules

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$
$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$
$$\varepsilon_{ijk} = -\varepsilon_{kji}$$

sign is changed under interchange of any pair of subscripts (totally antisymmetric tensor)

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$ *repeated interchange of subscripts*

Relations between Levi-Civita tensor and Kronecker delta

permutation tensor $(6^{th}\,order)$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{37a}$$ *6^{th} order tensor*

contractions

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{37b}$$ *4^{th} order tensor*

contractions

$$\varepsilon_{ijk}\varepsilon_{mjk} = 2\delta_{im} \tag{37c}$$ *2^{nd} order tensor*

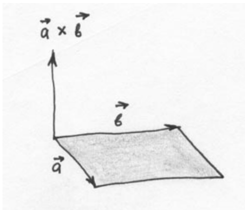
"6^th^ order tensor"

Many Hands for Thee
Fidalis Buehler, BYU

Cross-product

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i}_1 - (a_1b_3 - a_3b_1)\mathbf{i}_2 + (a_1b_2 - a_2b_1)\mathbf{i}_3$$



$$(\mathbf{a}\times\mathbf{b})_i = \varepsilon_{ijk} a_j b_k \qquad (38)$$

$$(\mathbf{a}\times\mathbf{b})_1 = \varepsilon_{1jk} a_j b_k + \varepsilon_{1jk} a_j b_k = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2b_3 - a_3b_2$$

$$(\mathbf{a}\times\mathbf{b})_2 = \varepsilon_{2jk} a_j b_k + \varepsilon_{2jk} a_j b_k = \varepsilon_{231} a_3 b_1 + \varepsilon_{213} a_1 b_3 = a_3b_1 - a_1b_3$$

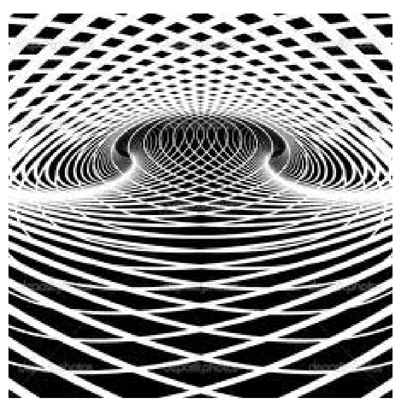
$$(\mathbf{a}\times\mathbf{b})_3 = \varepsilon_{3jk} a_j b_k + \varepsilon_{3jk} a_j b_k = \varepsilon_{312} a_1 b_2 + \varepsilon_{321} a_2 b_1 = a_1b_2 - a_2b_1$$

Exercise

Transformation of δ_{ij} obeys the tensor rule (33).

Transformation of ε_{ijk} obeys the tensor rule (44).

Therefore, indeed, they are the 2nd oder and the 3rd order tensors.



Examples of the application of tensors δ_{ij} and ε_{ijk} to prove vector identities

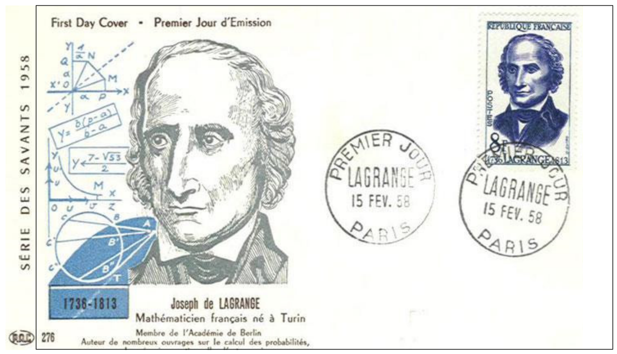
Example 1

Prove that the cross product is *anticommutative*:

$$\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$$

Proof: apply Eqn.(38), and since $\varepsilon_{ijk} = -\varepsilon_{ikj}$

$$\left(\mathbf{b}\times\mathbf{a}\right)_i = \varepsilon_{ijk} b_j a_k = \varepsilon_{ijk} a_k b_j = -\varepsilon_{ikj} a_k b_j = -\left(\mathbf{a}\times\mathbf{b}\right)_i$$ ■



Rue Lagrange, Paris

Example 2

Prove ***Lagrange's identity***, Eqn.(25):

$$\left(\mathbf{a}\times\mathbf{b}\right)\cdot\left(\mathbf{c}\times\mathbf{d}\right) = \left(\mathbf{a}\cdot\mathbf{c}\right)\left(\mathbf{b}\cdot\mathbf{d}\right) - \left(\mathbf{a}\cdot\mathbf{d}\right)\left(\mathbf{b}\cdot\mathbf{c}\right)$$

Proof: apply Eqn. (38) for tensor representation of cross-product

$$\left(\mathbf{a}\times\mathbf{b}\right)\cdot\left(\mathbf{c}\times\mathbf{d}\right) = \left(\mathbf{a}\times\mathbf{b}\right)_i\cdot\left(\mathbf{c}\times\mathbf{d}\right)_i$$

$$= \varepsilon_{ijk} a_j b_k \varepsilon_{ilm} c_l d_m$$

$$= \varepsilon_{ijk}\varepsilon_{ilm} a_j b_k c_l d_m$$ *contraction (37a)*

$$= \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) a_j b_k c_l d_m$$

$$= \delta_{jl}\delta_{km} a_j b_k c_l d_m - \delta_{jm}\delta_{kl} a_j b_k c_l d_m$$

$$= \left(\delta_{jl} c_l\right)\left(\delta_{km} d_m\right) a_j b_k - \left(\delta_{jm} d_m\right)\left(\delta_{kl} c_l\right) a_j b_k$$ *change of index*

$$= c_j d_k a_j b_k - d_j c_k a_j b_k$$

$$= a_j c_j b_k d_k - a_m d_m b_l c_l$$

$$= \left(\mathbf{a}\cdot\mathbf{c}\right)\left(\mathbf{b}\cdot\mathbf{d}\right) - \left(\mathbf{a}\cdot\mathbf{d}\right)\left(\mathbf{b}\cdot\mathbf{c}\right)$$ ■

Reduction to principal axes

Contraction operation

$$a_i = A_{ik} b_k$$

results in a vector. It can be treated as a rotation of a vector and changing of its length (linear transformation).

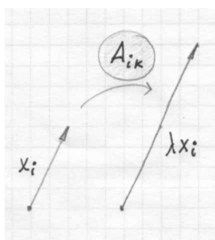
For a given 2nd order tensor A_{ik} it is important to determine if there are some vectors which are not rotated after transformation.

This question is formulated in the familiar form of an eigenvalue problem:

Eigenvalue problem

Find values of parameter λ for which equation

$$A_{ik} x_k = \lambda x_i \qquad (39)$$



has a non-trivial solution x_i .

They are called: λ *eigenvalue*
x_i *eigenvector*

Eigenvectors if they exist determine the principle axes (coordinate system) of the tensor A_{ij} . The problem is to find this coordinate system and to transform a tensor to it. Rewrite equation (35) in the form:

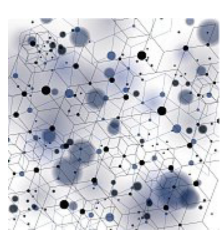
$$\left(A_{ik} - \lambda \delta_{ik} \right) x_k = 0$$

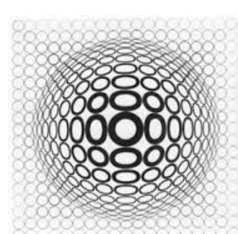
The necessary condition for this equation to have a non-trivial solution is:

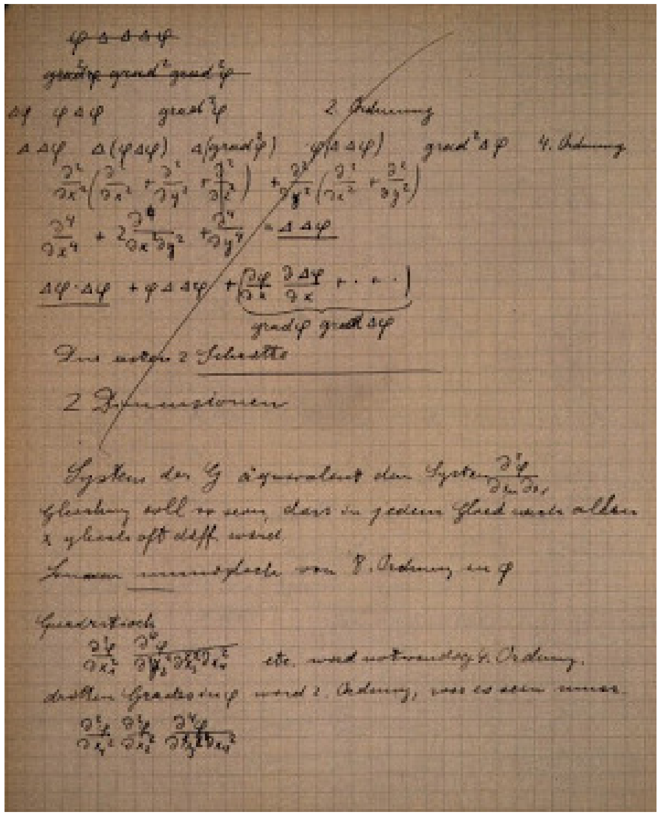
$$\left| A_{ik} - \lambda \delta_{ik} \right| = 0 \quad \textit{characteristic equation} \qquad (40)$$

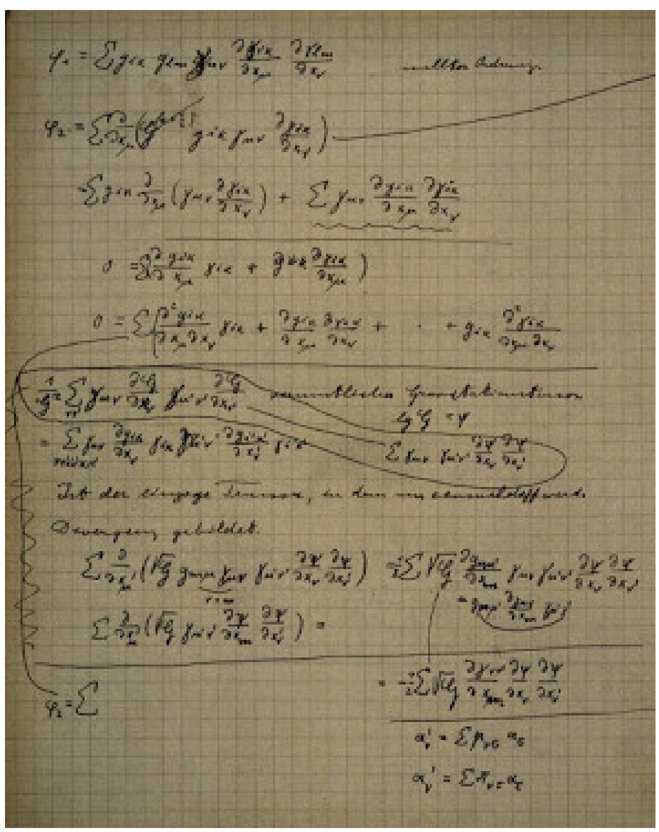
The tensor written in the principle coordinate system has the simplest form.

Formulate the eigenvalue problem in the traditional vector-matrix form.









Einstein's manuscripts (from the former Museum of Letters and Manuscripts, Paris)

IV.1.10. SUMMARY OF TENSORS

The ***Cartesian tensors*** are defined in the rectangular coordinate system as the quantities which under the change of the coordinate system obey the following laws of transformation of its components:

$a' = a$	*Zero-order tensors (scalars)*	(41)
$x'_i = \alpha_{ij} x_j$	*First-order tensors (vectors)*	(42)
$A'_{ij} = \alpha_{ik}\alpha_{jm} A_{km}$	*Second-order tensors (matrices)*	(43)
$\dots$		
$A'_{i_1 i_2 \cdots i_n} = \alpha_{i_1 k_1}\alpha_{i_2 k_2} \cdots \alpha_{i_n k_n} A_{k_1 k_2 \cdots k_n}$	*n^{th}-order tensors*	(44)

The Einstein convention on notation and summation is used in these definitions.

The coefficients

$$\alpha_{ji} = \mathbf{i}_i \cdot \mathbf{i}'_j = cos\left(\mathbf{i}_i, \mathbf{i}'_j\right)$$

are the cosines of the angles between the basis vectors of coordinate systems (21).

The tensors with the higher order *n* consist of 3^n real numbers.

If the components of a tensor in one coordinate system $0xyz$ are known, then using equations (41-44), we can determine the components of a tensor in the rotated coordinate system $0x'y'z'$.

Vector space of tensors V_n

The set of all tensors of order *n* together with operations multiplication by a scalar and addition, form a *vector space* V_n

$$A_{i_1 i_2 \cdots i_n} \in V_n \quad (45)$$

Tensors can also be defined in the generalized curvilinear coordinate systems. Definitions in the *m*-dimensional geometrical space yield n^{th} order tensors which have m^n components.

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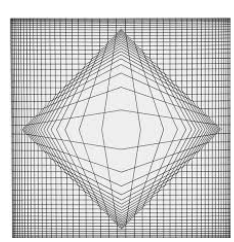
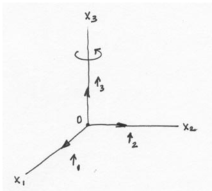


Orthonormal right coordinate system

$\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$, $\mathbf{i}_i \in \mathbb{R}_3$



$$\delta_{ij} = \mathbf{i}_i \cdot \mathbf{i}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{i}_1 \times \mathbf{i}_2 = \mathbf{i}_3 \quad \mathbf{i}_2 \times \mathbf{i}_1 = -\mathbf{i}_3$

$\mathbf{i}_2 \times \mathbf{i}_3 = \mathbf{i}_1 \quad \mathbf{i}_3 \times \mathbf{i}_2 = -\mathbf{i}_1$

$\mathbf{i}_3 \times \mathbf{i}_1 = \mathbf{i}_2 \quad \mathbf{i}_1 \times \mathbf{i}_3 = -\mathbf{i}_2$

$\mathbf{i}_1 \times \mathbf{i}_1 = \mathbf{i}_2 \times \mathbf{i}_2 = \mathbf{i}_3 \times \mathbf{i}_3 = \mathbf{0}$

$$\varepsilon_{ijk} = (\mathbf{i}_i \times \mathbf{i}_j) \cdot \mathbf{i}_k = \begin{cases} 1 & \text{if} \quad ijk = 123,\ 231,\ \text{or}\ 312 \\ 0 & \text{if} \quad \text{any two indices are alike} \\ -1 & \text{if} \quad ijk = 321,\ 213,\ \text{or}\ 132 \end{cases}$$

Kronecker delta

$$\delta_{ij} = \mathbf{i}_i \cdot \mathbf{i}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\delta_{ik} a_i = a_k$ *change of index*

$\delta_{ik} A_{km} = \delta_{ki} A_{km} = A_{im}$

$A_{ik} b_k - c b_i = (A_{ik} - c\delta_{ik}) b_k$ *factoring*

$A_{ik} A_{kj}^{-1} = \delta_{ij}$ *tensor inverse*

Levi-Civita tensor

$\varepsilon_{ijk} = (\mathbf{i}_i \times \mathbf{i}_j) \cdot \mathbf{i}_k$

$\varepsilon_{ijk} = -\varepsilon_{jik}$

$\varepsilon_{ijk} = -\varepsilon_{ikj}$

$\varepsilon_{ijk} = -\varepsilon_{kji}$

$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$

$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ *contruction*

Transformation of coordinates

$\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ and $\{\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3\}$ $\quad 0xyz \rightarrow 0x'y'z'$ (rotation)

$\alpha_{ij} = \mathbf{i}'_i \cdot \mathbf{i}_j$

$\alpha_{ji} = \mathbf{i}'_j \cdot \mathbf{i}_i$

$\mathbf{i}_i = \alpha_{ji} \mathbf{i}'_j \qquad \mathbf{i}_i \cdot \mathbf{i}_j = \alpha_{ki} \mathbf{i}'_k \cdot \alpha_{kj} \mathbf{i}'_k = \alpha_{ki}\alpha_{kj} \mathbf{i}'_k \cdot \mathbf{i}'_k = \alpha_{ki}\alpha_{kj} = \delta_{ij}$

$\mathbf{i}'_i = \alpha_{ij} \mathbf{i}_j \qquad \mathbf{i}'_i \cdot \mathbf{i}'_j = \alpha_{ik} \mathbf{i}_k \cdot \alpha_{jk} \mathbf{i}_k = \alpha_{ik}\alpha_{jk} \mathbf{i}_k \cdot \mathbf{i}_k = \alpha_{ik}\alpha_{jk} = \delta_{ij}$

$\alpha_{ki}\alpha_{kj} = \delta_{ij}$

$\alpha_{ik}\alpha_{jk} = \delta_{ij}$

Cartesian tensors

$a' = a$ *zero-order tensors (scalars)*

$x'_i = \alpha_{ij} x_j$ *first-order tensors (vectors)*

$A'_{ij} = \alpha_{ik}\alpha_{jm} A_{km}$ *second-order tensors (matrices)*

Vector operations	$\mathbf{a}$		a_i
	$k\mathbf{a}$		ka_i
	$\mathbf{a}+\mathbf{b}$		a_i+b_i
	$\mathbf{a}\cdot\mathbf{b}$		a_ib_i
	$\mathbf{a}\times\mathbf{b}$		$\varepsilon_{ijk}a_jb_k$
Properties	$\mathbf{a}\cdot\mathbf{b}$	$=\mathbf{b}\cdot\mathbf{a}$	
	$(k\mathbf{a})\cdot\mathbf{b}$	$=\mathbf{a}\cdot(k\mathbf{b})=k(\mathbf{a}\cdot\mathbf{b})$	
	$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$	$=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$	
	$\mathbf{a}\times\mathbf{b}$	$=-\mathbf{b}\times\mathbf{a}$	
	$\mathbf{a}\times(\mathbf{b}+\mathbf{c})$	$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$	
	$\mathbf{a}\times\mathbf{a}$	$=\mathbf{0}$	
Triple Scalar product	$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$		$\varepsilon_{ijk}a_ib_jc_k$
Triple Vector product	$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$		$\varepsilon_{ijk}\varepsilon_{klm}a_jb_lc_m$
Identities	$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$	$=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$	
	$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$	$=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$	
	$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$	$=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$	*Lagrange identitity*
	$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$	$=((\mathbf{a}\times\mathbf{b})\cdot\mathbf{d})\mathbf{c}-((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\mathbf{d}$ $=((\mathbf{a}\times\mathbf{c})\cdot\mathbf{d})\mathbf{b}-((\mathbf{b}\times\mathbf{c})\cdot\mathbf{d})\mathbf{a}$ $=((\mathbf{c}\times\mathbf{d})\cdot\mathbf{a})\mathbf{b}-((\mathbf{c}\times\mathbf{d})\cdot\mathbf{b})\mathbf{a}$	
	$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{0}$		
	$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})=0$		
	$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{b})$	$=\ ?$	

May, 2019

May, 2024