VIII.3 Method of Separation of Variables – Transient Initial-Boundary Value Problems

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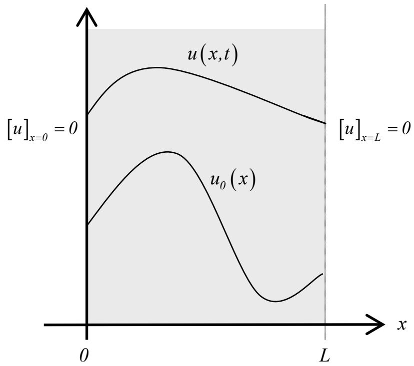
VIII.3.1 HEAT EQUATION IN PLANE WALL – 1-D Heat Equation

VIII.3.1.1 BASIC CASE: Homogeneous equation, Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad x\in(0,L),\ t>0$$

Initial condition: $u(x,0)=u_0(x)$

Boundary conditions: $[u]_{x=0}=0,\ t>0$ (*I, II or III kind*)

$[u]_{x=L}=0,\ t>0$ (*I, II or III kind*)

1) Separation of variables: $u(x,t)=X(x)T(t)$

Boundary conditions:

$$[u]_{x=0}=[X]_{x=0}T(t)=0 \quad\Rightarrow\quad [X]_{x=0}=0$$
$$[u]_{x=L}=[X]_{x=L}T(t)=0 \quad\Rightarrow\quad [X]_{x=L}=0$$

$$\frac{X''}{X}=\frac{1}{\alpha}\frac{T'}{T}=\mu$$

2) Sturm-Liouville Problem:

$$X''-\mu X=0$$
$$[X]_{x=0}=0 \quad\Rightarrow\quad \mu=-\lambda_n^2 \qquad n=1,2,\ldots$$
$$[X]_{x=L}=0 \qquad\qquad X_n(x)$$

3) Equation for T : $T'-\alpha\mu T=0$

$$T'+\alpha\lambda_n^2 T=0 \quad\Rightarrow\quad T_n(t)=e^{-\alpha\lambda_n^2 t}$$

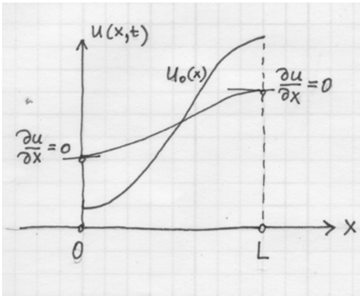
4) Solution:

$$u(x,t)=\sum_{n=1}^{\infty}a_n X_n T_n \qquad =\sum_{n=1}^{\infty}a_n\ X_n\ e^{-\alpha\lambda_n^2 t}$$

Initial condition:

$$u(x,0)=u_0(x)=\sum_{n=1}^{\infty}a_n X_n \quad\Rightarrow\quad a_n=\frac{\int_0^L u_0(x)X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

Example 1 **Neumann-Neumann Problem**



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in (0,L), \quad t>0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions:
$$\left[\frac{\partial u}{\partial x}\right]_{x=0} = 0 \qquad t>0 \qquad (Neumann)$$
$$\left[\frac{\partial u}{\partial x}\right]_{x=L} = 0 \qquad t>0 \qquad (Neumann)$$

(both boundaries are insulated)

Separation of variables: $u(x,t) = X(x)T(t)$

Boundary conditions:
$$x=0 \qquad \frac{\partial u(0,t)}{\partial x} = X'(0)T(t) = 0 \quad \Rightarrow \quad X'(0) = 0$$
$$x=L \qquad \frac{\partial u(L,t)}{\partial x} = X'(L)T(t) = 0 \quad \Rightarrow \quad X'(L) = 0$$

Solution of SLP:
$$X'' - \mu X = 0 \qquad \mu_n = -\lambda_n^2$$
$$\lambda_0 = 0 \qquad X_0 = 1$$
$$\lambda_n = \frac{n\pi}{L} \qquad X_n = cos\left(\frac{n\pi}{L}x\right) \quad n = 1,2,...$$

Solution for T:
$$T' + \alpha\lambda_n^2 T = 0 \qquad T_n(t) = e^{-\alpha\lambda_n^2 t}$$
$$T' + \alpha \cdot 0 \cdot T = 0 \qquad T_0(t) = 1$$

Solution:
$$u(x,t) = a_0 X_0 T_0 + \sum_{n=1}^{\infty} a_n X_n T_n \quad = a_0 + \sum_{n=1}^{\infty} a_n X_n e^{-\alpha\lambda_n^2 t}$$

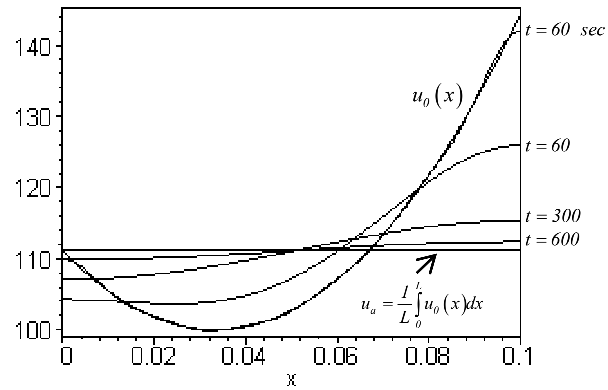
$$a_0 = \frac{1}{L}\int_0^L u_0(x)dx$$

$$a_n = \frac{2}{L}\int_0^L u_0(x)cos\left(\frac{n\pi}{L}x\right)dx$$

Solution of IBVP:

$$u(x,t) = \frac{\int_0^L u_0(x)\,dx}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L u_0(x)cos\left(\frac{n\pi}{L}x\right)dx\right]cos\left(\frac{n\pi}{L}x\right)e^{-\alpha\frac{n^2\pi^2}{L^2}t}$$

Particular case: $u_0(x) = 100 + 10000\left(x - \frac{L}{3}\right)^2 \left[{}^oC\right], \ \frac{1}{\alpha} = a^2 = 500^2\left[\frac{s}{m^2}\right] (steel), \quad L=0.1m$



Comments:

1) The solution is in the form of an infinite series.
If the initial temperature distribution given by the function $u_0(x)$ is integrable, then the Fourier series is absolutely convergent and the function $u(x,t)$ satisfies the Heat Equation and initial and boundary conditions.

Therefore, it is an *analytical solution* of the given IBVP.

2) With the increase of time, the solution approaches the steady state (the averaged temperature in the slab). Boundaries are insulated, and there are no heat sources. As a result, no heat escapes into the surroundings. The driving force – temperature gradient – is directed toward the areas with lower temperature. There exists a process of redistribution of heat energy that produces the uniform temperature in the slab.

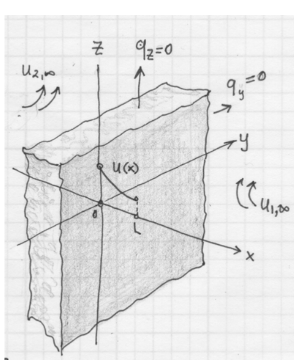
3) Basic functions consist of the product

$$u_n(x,t) = cos\left(\frac{n\pi}{L}x\right)e^{-\alpha\frac{n^2\pi^2}{L^2}t} = cos\left(\frac{n\pi}{L}x\right)e^{-(n\pi)^2\overbrace{\left(\frac{\alpha t}{L^2}\right)}^{\substack{\text{non-dimensional time}\\ Fo}}}$$

where the cosine function provides the spatial shape of the temperature profile; and the exponential function is responsible for decay of the temperature profile in time.

4) The rate of change of temperature depends on the thermal diffusivity α .

5) Very often, a *1-D* Heat Equation is treated as a model for heat transfer in a long very thin rod of constant cross-section whose surface, except for the ends, is insulated against the flow of heat Although, it is formally a correct model, the practical application of it is very limited. But there is another interpretation of a *1-D* model, which is more reliable.



Consider a *3-D* wall with finite dimension in the x-direction (within $x = 0$ and $x = L$) and elongated dimensions (may be infinite) in y- and z-directions. If the conditions at the walls $x = 0$ and $x = L$ are uniform, and the initial condition is independent of variables *y* and *z*, then the variation of temperature in the *y*- and *z*-directions is negligible (no heat flux in these directions)

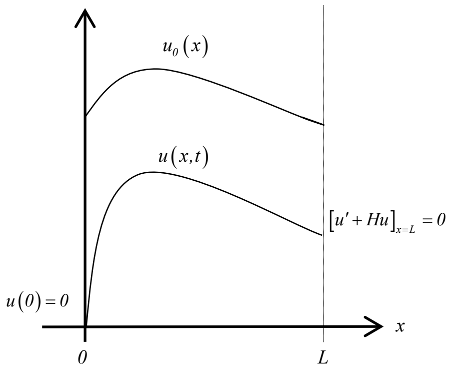
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$$

and the heat equation becomes *1-D*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

It defines the variation of temperature along any line perpendicular to the wall.

Example 2 — *Dirichlet-Robin Problem*



$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t):\quad x\in(0,L)\,,\quad t>0$$

Initial condition: $u(x,0)=u_0(x)$

Boundary conditions: $[u]_{x=0}=0$ (*Dirichlet*)

$$\left[\frac{\partial u}{dx}+Hu\right]_{x=L}=0 \qquad (Robin) \qquad H=\frac{h}{k}$$

Separation of variables: $u(x,t)=X(x)T(t)$

$$X''-\mu X=0 \qquad T'-\alpha\mu T=0$$

Boundary conditions :

$\underline{x=0}$ $\quad X(0)T(t)=0 \quad \Rightarrow \quad X(0)=0$

$\underline{x=L}$ $\quad X'(L)T(t)+HX(L)T(t)=0 \quad \Rightarrow \quad X'(L)+HX(L)=0$

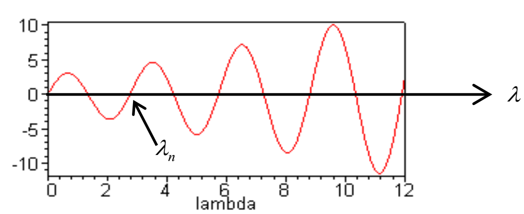
Solution of Sturm-Liouville problem:

$\mu_n=-\lambda_n^2$

$X_n=sin(\lambda_n x) \qquad n=1,2,...$

where eigenvalues λ_n are positive roots of the characteristic equation:

$\lambda\, cos\, \lambda L+H\, sin\, \lambda x=0$



Solution for *T(t):*

With determined eigenvalues, the solution for T becomes:
$T_n(t)=e^{-\alpha\lambda_n^2 t}$

Solution:

$$u(x,t)=\sum_{n=1}^{\infty}a_n\, sin(\lambda_n x)e^{-\alpha\lambda_n^2 t}$$

This solution satisfies the heat equation and boundary conditions. We want to define coefficients a_n in a such a way that the obtained solution satisfies also the initial condition at $t=0$:

$$u(x,0)=\sum_{n=1}^{\infty}a_n\, sin(\lambda_n x)=u_0(x)$$

In our problem, functions $\{X_n(x)=sin(\lambda_n x)\}$ are obtained as eigenfunctions of the Sturm-Liouville problem for the equation $X''+\lambda^2 X=0$; therefore, the set of all eigenfunctions is a complete system of functions orthogonal with respect to the weight function $p=1$. Then, the last equation is an expansion of the function $u_0(x)$ in a generalized Fourier series over the interval $(0,L)$ with coefficients defined by

$$a_n=\frac{\int_0^L u_0(x)\,sin(\lambda_n x)\,dx}{\int_0^L sin^2(\lambda_n x)\,dx}$$

Then, the solution of the initial-boundary value problem is given by

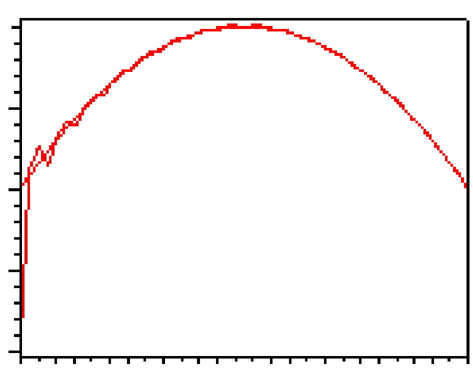
$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{\int_0^L u_0(x)\,sin(\lambda_n x)\,dx}{\int_0^L sin^2(\lambda_n x)\,dx}\right]sin(\lambda_n x)\,e^{-\alpha\lambda_n^2 t}$$

where the squared norm of eigenfunctions may be evaluated after integration as

$$\|X_n\|^2=\int_0^L sin^2(\lambda_n x)\,dx=\frac{L}{2}-\frac{sin(2\lambda_n L)}{4\lambda_n}$$

Finally, the solution is:

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{\int_0^L u_0(x)\,sin(\lambda_n x)\,dx}{\frac{L}{2}-\frac{sin(2\lambda_n L)}{4\lambda_n}}\right]sin(\lambda_n x)\,e^{-\alpha\lambda_n^2 t}$$



MAPLE: Let $L=2$, $H=3$, $u_0(x)=x(2-x)$, $\alpha=0.0625$

```
> restart;
> with(plots):
> L:=2;H:=3;A:=0.0625;
```

$$L := 2$$

$$H := 3$$

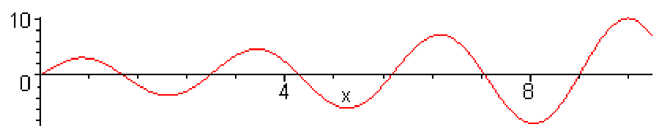
$$A := 0.0625$$

Characteristic equation:

```
> w(x):=x*cos(x*L)+H*sin(x*L);
```

$$\mathrm{w}(x) := x\cos(2\,x)+3\sin(2\,x)$$

```
> plot(w(x),x=0..10);
```



Eigenvalues:

```
> n:=1: for m from 1 to 500 do z:=fsolve(w(x)=0,x=m/10..(m+1)/10):
if type(z,float) then lambda[n]:=z: n:=n+1 fi od:
> for i to 5 do lambda[i] od;
```

$$1.358229874$$

$$2.768911636$$

$$4.235147453$$

$$5.738636645$$

$$7.264403196$$

```
> N:=n-1;
```

$$N := 32$$

```
> n:='n':i:='i':
```

Eigenfunctions:

```
> X[n]:=sin(lambda[n]*x);
```

$$X_n := \sin(\lambda_n\,x)$$

Squared-norm:

```
> NX[n]:=int(X[n]^2,x=0..L);
```

$$NX_n := \frac{1}{2}\,\frac{-\cos(2\,\lambda_n)\sin(2\,\lambda_n)+2\,\lambda_n}{\lambda_n}$$

Initial condition:

```
> u0(x):=x*(L-x)+1;
```

$$\mathrm{u0}(x) := x\,(2-x)+1$$

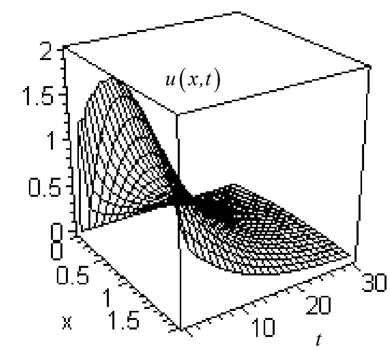
Fourier coefficients:

```
> a[n]:=simplify(int(u0(x)*X[n],x=0..L)/NX[n]);
```

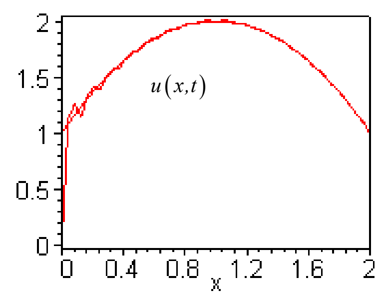
$$a_n := -\frac{2\,(2\,\lambda_n\sin(2\,\lambda_n)+\lambda_n^{\,2}\cos(2\,\lambda_n)+2\cos(2\,\lambda_n)-2-\lambda_n^{\,2})}{\lambda_n^{\,2}\,(-\cos(2\,\lambda_n)\sin(2\,\lambda_n)+2\,\lambda_n)}$$

Solution - Generalized Fourier series:

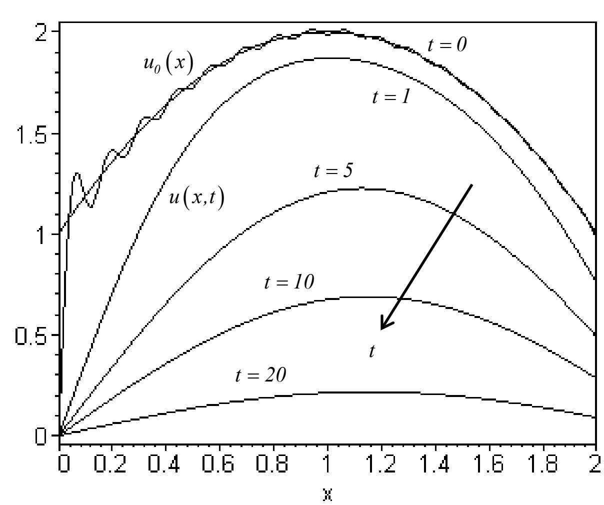
```
> u(x,t):=sum(a[n]*X[n]*exp(-lambda[n]^2*t/A^2),n=1..N):
> plot3d(u(x,t),x=0..L,t=0..30,axes=boxed,style=wireframe);
```



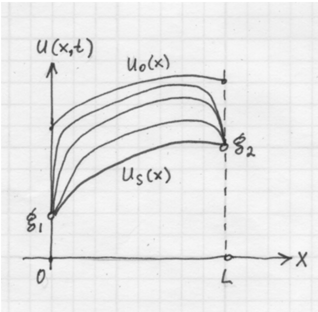

```
> animate({u0(x),u(x,t)},x=0..L,t=0..50,frames=200,axes=boxed);
```




```
> u(x,0):=subs(t=0,u(x,t)):
> u(x,1):=subs(t=1,u(x,t)):
> u(x,5):=subs(t=5,u(x,t)):
> u(x,10):=subs(t=10,u(x,t)):
> u(x,20):=subs(t=20,u(x,t)):
> plot({u0(x),u(x,0),u(x,1),u(x,5),u(x,10),u(x,20)},x=0..L);
```



VIII.3.1.2 GENERAL CASE Non-Homogeneous Equation, Non-Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t),\quad x\in(0,L),\qquad t>0$$

Initial condition: $u(x,0)=u_0(x)$

Boundary conditions: $[u]_{x=0}=g_1,\ t>0$ (*I, II or IIIrd kind*)

$[u]_{x=L}=g_2,\ t>0$ (*I, II or IIIrd kind*)

I Steady State Solution

> *Definition* A time-independent function which satisfies the heat equation and boundary conditions obtained as
> $$u_s(x)=\lim_{t\to\infty}u(x,t)$$
> is called a ***steady state solution***

Substitution of a time-independent function into the heat equation leads to the following ordinary differential equation:

$$\frac{\partial^2 u_s}{\partial x^2}+F(x)=0 \qquad u_s(x),\quad x\in(0,L)$$

subject to the boundary conditions of the same kind as for PDE:

$[u_s]_{x=0}=g_1,\ t>0$ (*I, II or IIIrd kind*)

$[u_s]_{x=L}=g_2,\ t>0$ (*I, II or IIIrd kind*)

General solution of ODE:

$$u_s(x)=-\int\left[\int F(x)\,dx\right]dx+c_1x+c_2$$

Solutions of BVPs for plane wall with uniform heat generation are provided by the Table.

II Transient Solution:

Define the transient solution by equation:

$$U(x,t)=u(x,t)-u_s(x)$$

then solution of the original problem is a sum of transient solution and steady state solution:

$$u(x,t)=U(x,t)+u_s(x)$$

Substitute it into the Heat Equation:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 u_s}{\partial x^2}+F(x)=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

Since $\frac{\partial^2 u_s}{\partial x^2}+F(x)=0$, it yields

$$\frac{\partial^2 U}{\partial x^2}=a^2\frac{\partial U}{\partial t}$$

We obtained the equation for the new unknown function $U(x,t)$ which has homogeneous boundary conditions:

$$x=0 \qquad [U]_{x=0}=[u]_{x=0}-[u_s]_{x=0}=g_1-g_1=0$$

$$x=L \qquad [U]_{x=L}=[u]_{x=L}-[u_s]_{x=L}=g_2-g_2=0$$

As a result, we reduced the non-homogeneous problem to a homogeneous equation for $U(x,t)$ with homogeneous boundary conditions. Initial condition for function $U(x,t)$:

$$U(x,0)=u(x,0)-u_s(x)=u_0(x)-u_s(x)$$

Solution for *U(x,t)*

We consider the following ***basic*** initial boundary value problem:

$$\frac{\partial^2 U}{\partial x^2}=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U(x,t), \quad x\in(0,L), \; t>0$$

initial condition: $U(x,0)=u_0(x)-u_s(x)$

boundary conditions: $[U]_{x=0}=0, \quad t>0$
$[U]_{x=L}=0 \quad t>0$

We already know a solution of this basic problem obtained by separation of variables:

$$U(x,t) \; = \sum_{n=1}^{\infty} a_n X_n e^{-\alpha\lambda_n^2 t}$$

where coefficients a_n are the Fourier coefficients determined with the corresponding initial condition for the function $U(x,t)$:

$$a_n = \frac{\int_0^L \left[u_0(x)-u_s(x)\right] X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

<u>III Solution of IBVP</u>:

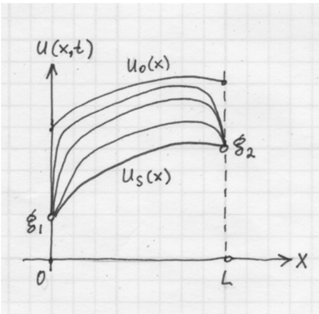
Solution of the original IBVP is a sum of steady state solution and transient solution:

$$u(x,t) \; = u_s(x)+U(x,t)$$

$$= u_s(x)+\sum_{n=1}^{\infty} a_n X_n e^{-\alpha\lambda_n^2 t}$$

$$a_n \; = \frac{\int_0^L \left[u_0(x)-u_s(x)\right] X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

Example 3 **Dirichlet-Dirichlet problem with a uniform heat generation:**



$$\frac{\partial^2 u}{\partial x^2} + F = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t):\ x \in (0,L), \quad t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $u(0,t) = g_1$ $\quad t > 0$ $\quad$ (*Dirichlet*)

$u(L,t) = g_2$ $\quad t > 0$ $\quad$ (*Dirichlet*)

1) Steady State Solution:

Let $F = const$, then integrating the equation twice, we come up with the following solution:

$$\frac{\partial u_s}{\partial x} = -F\,x + c_1$$

$$u_s = -\frac{F}{2}x^2 + c_1 x + c_2$$

Apply boundary conditions to determine the constants of integration:

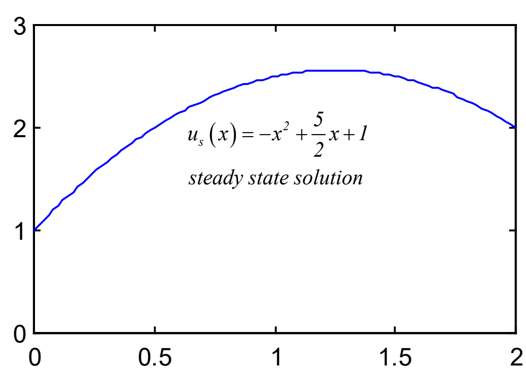
$\underline{x = 0}$ $\Rightarrow$ $c_2 = g_1$

$\underline{x = L}$ $\Rightarrow$ $-\frac{F}{2}L^2 + c_1 L + g_1 = g_2$

$\Rightarrow$ $c_1 = \frac{g_2 - g_1}{L} + \frac{FL}{2}$

$$u_s(x) = -\frac{F}{2}x^2 + \left(\frac{g_2 - g_1}{L} + \frac{FL}{2}\right)x + g_1$$

Example: $F = 2$, $g_1 = 1$, $g_2 = 2$, $L = 2$



2) Transient Problem:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U(x,t):\ x \in (0,L),\ t > 0$$

initial condition: $U(x,0) = u_0(x) - u_s(x)$

boundary conditions: $U(0,t) = 0$ $\quad$ (*Dirichlet*)

$U(L,t) = 0$ $\quad$ (*Dirichlet*)

Solution of this basic problem (Dirichlet-Dirichlet) obtained by separation of variables:

$$\lambda_n = \frac{n\pi}{L}, \qquad X_n(x) = sin\left(\frac{n\pi}{L}x\right)$$

$$U(x,t) = \sum_{n=1}^{\infty} a_n X_n e^{-\alpha\lambda_n^2 t} = \sum_{n=1}^{\infty} a_n\, sin\left(\frac{n\pi}{L}x\right) e^{-\frac{\alpha n^2\pi^2}{L^2}t}$$

where coefficients a_n are the Fourier coefficients determined by the corresponding initial condition for the function $U(x,t)$:

$$a_n = \frac{\int_0^L \left[u_0(x) - u_s(x)\right] X_n(x)dx}{\int_0^L X_n^2(x)dx} = \frac{2}{L}\int_0^L \left[u_0(x) - u_s(x)\right] sin\left(\frac{n\pi}{L}x\right)dx$$

3) Solution of IBVP:
Return to the original function $u(x,t)$:

$$u(x,t) = U(x,t) + u_s(x) = u_s(x) + \sum_{n=1}^{\infty} a_n\, sin\left(\frac{n\pi}{L}x\right) e^{-\frac{\alpha n^2\pi^2}{L^2}t}$$

Then the solution of the non-homogeneous heat equation with non-homogeneous Dirichlet boundary conditions becomes:

$$u(x,t) = \left[-\frac{F}{2}x^2 + \left(\frac{g_2 - g_1}{L} + \frac{FL}{2}\right)x + g_1\right] + \frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L \left[u_0(x) - u_s(x)\right] sin\left(\frac{n\pi}{L}x\right)dx\right\} sin\left(\frac{n\pi}{L}x\right) e^{-\frac{\alpha n^2\pi^2}{L^2}t}$$

Remark: In practice, instead of the exact solution defined by the infinite series, the truncated series is used for calculation of the approximate solution. How many terms are needed in the truncated series for the accurate approximation? Comparison of the exact solution (which is also a truncated series but with a very large number of terms, which we assume, provides an accurate result) with the calculation with a small number of terms in a truncated series shows that the accuracy depends on time: the further we proceed in time, the more accurate becomes an approximate solution (why?). For uniform characterization of physical processes, the non-dimensional parameters are used in engineering. In heat transfer, non-dimensional time is defined by the Fourier number:

$$Fo = \frac{\alpha t}{L^2}$$

where α is the *thermal diffusivity*.

In engineering heat transfer analysis, a 4 term approximation is considered as an accurate approximation for all values of the Fourier number. For simplicity, very often even a 1 term approximation is used, which is considered to be accurate for $Fo > 0.2$ (error in most cases does not exceed *1%,* and this is a convention in engineering heat transfer).

Consider comparison of the exact solution (100 terms) with 1 and 4 terms approximations.

Results are calculated for:

$Fo = 0.0$

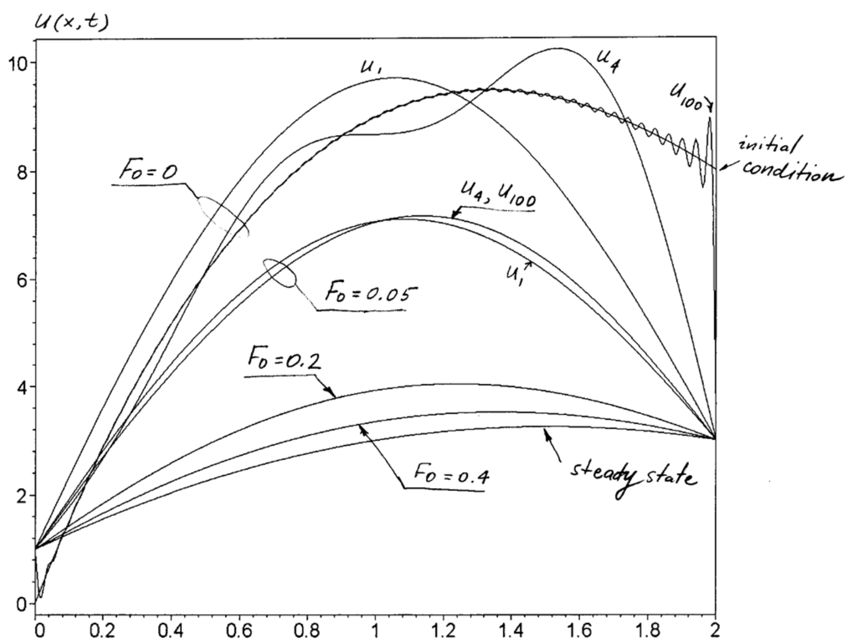
$Fo = 0.05$

$Fo = 0.2$

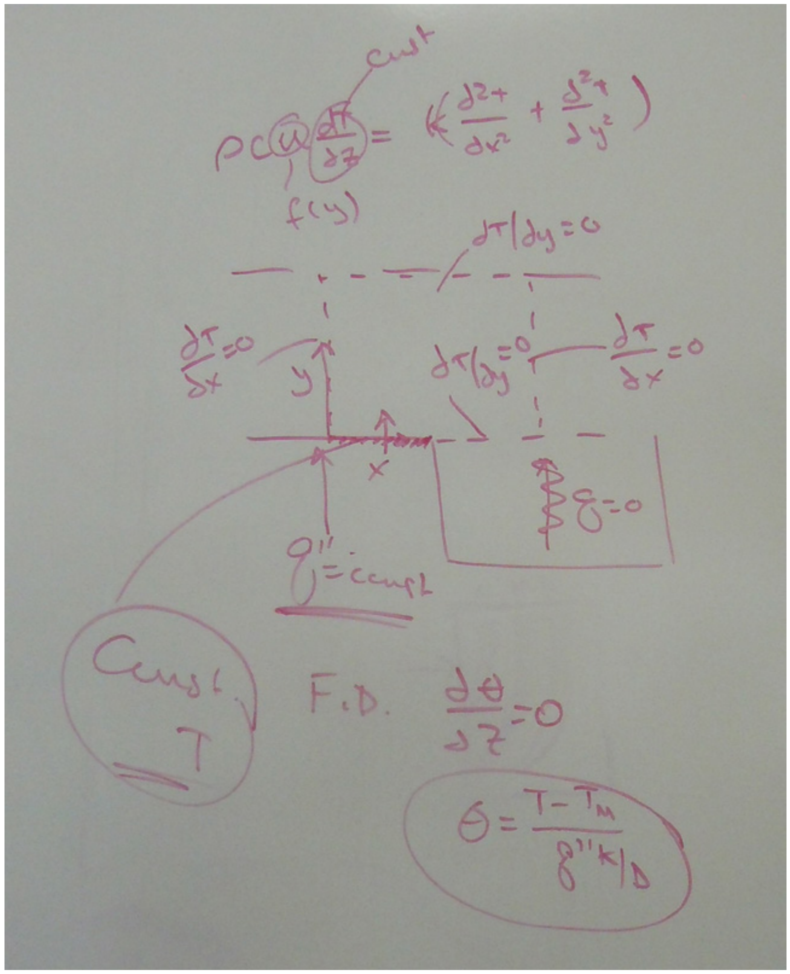
$Fo = 0.4$

The lowest curve is a steady state solution.

As can be seen from the figure, for $Fo > 0.2$, all results coincide.

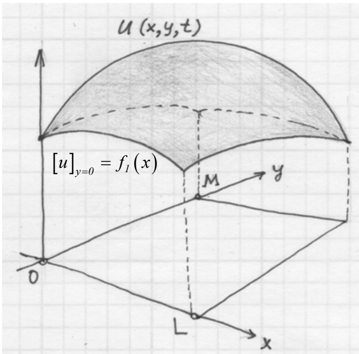


$Fo > 0.2$	is generally adopted [see Incroper and De Witt] as a condition for application one term approximation: one-term solution becomes accurate for $Fo > 0.2$.


const
f(y)
dT/dy=0
dT/dx=0
y
x
q=0
q''=const
Const. T
F.D.
T−T_M

VIII.3.2.1 HEAT EQUATION in CARTESIAN COORDINATES 2-D

General Problem:



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+F(x,y)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,y,t): \qquad (x,y)\in(0,L)\times(0,M)$$

Initial Condition: $u(x,y,0)=u_0(x,y) \qquad (x,y)\in[0,L]\times[0,M]$

Boundary Conditions:

$x=0 \quad [u]_{x=0}=f_3(y) \quad y\in(0,M) \quad t>0$

$x=L \quad [u]_{x=L}=f_4(y) \quad y\in(0,M) \quad t>0$

$y=0 \quad [u]_{y=0}=f_1(x) \quad x\in(0,L) \quad t>0$

$y=M \quad [u]_{y=M}=f_2(x) \quad x\in(0,L) \quad t>0$

<u>1. Steady State Solution</u>

Find time-independent solution $u_s(x,y)$. We are looking for a steady state solution which satisfies the differential equation:

$$\frac{\partial^2 u_s}{\partial x^2}+\frac{\partial^2 u_s}{\partial y^2}+F(x,y)=0$$

and the boundary conditions of the same type as in the general problem

$x=0 \quad [u_s]_{x=0}=f_3(y) \quad y\in(0,M) \quad t>0$

$x=L \quad [u_s]_{x=L}=f_4(y) \quad y\in(0,M) \quad t>0$

$y=0 \quad [u_s]_{y=0}=f_1(x) \quad x\in(0,L) \quad t>0$

$y=M \quad [u_s]_{y=M}=f_2(x) \quad x\in(0,L) \quad t>0$

This is the BVP for Poisson's Equation for which, in general, all boundary conditions are non-homogeneous. The superposition principle should be used to reduce the problem to the set of supplemental basic problems (see VIII.3.4, p.597).

<u>2. Transient Solution (Basic Case)</u>

Introduce the transient function as

$$U(x,y,t)=u(x,y,t)-u_s(x,y)$$

It can be verified that function U satisfies *homogeneous* Heat Equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

with four *homogeneous* boundary conditions (of the same type):

$x=0 \quad [U]_{x=0}=0 \quad y\in(0,M) \quad t>0$

$x=L \quad [U]_{x=L}=0 \quad y\in(0,M) \quad t>0$

$y=0 \quad [U]_{y=0}=0 \quad x\in(0,L) \quad t>0$

$y=M \quad [U]_{y=M}=0 \quad x\in(0,L) \quad t>0$

and the initial condition:

$$U(x,y,0)=u_0(x,y)-u_s(x,y) \quad \equiv U_0(x,y)$$

Separation of variables – 1st stage:

We assume that the function $U(x,y,t)$ can be written as a product of two functions

$$U(x,y,t)=\Phi(x,y)T(t)$$

where $\Phi(x,y)$ is the function of space variables. Substitute it into the Heat Equation

$$\frac{\partial^2\Phi}{\partial x^2}T+\frac{\partial^2\Phi}{\partial y^2}T=\frac{1}{\alpha}\Phi T'$$

Divide equation by ΦT :

$$\frac{\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2}}{\Phi}=\frac{1}{\alpha}\frac{T'}{T}$$

or using Laplacian operator

$$\frac{\nabla^2\Phi}{\Phi}=\frac{1}{\alpha}\frac{T'}{T}$$

Left hand side is a function of space variables only and the right hand side is a function of the time variable, therefore, they have to be equal to a constant (separation constant):

$$\frac{\nabla^2\Phi}{\Phi}=\frac{1}{\alpha}\frac{T'}{T}=\beta$$

Boundary conditions for separated functions are:

$$[U]_{x=0}=[\Phi]_{x=0}T(t)=0 \quad y\in(0,M) \quad t>0 \quad \Rightarrow \quad [\Phi]_{x=0}=0$$
$$[U]_{x=L}=[\Phi]_{x=L}T(t)=0 \quad y\in(0,M) \quad t>0 \quad \Rightarrow \quad [\Phi]_{x=L}=0$$
$$[U]_{y=0}=[\Phi]_{y=0}T(t)=0 \quad x\in(0,L) \quad t>0 \quad \Rightarrow \quad [\Phi]_{y=0}=0$$
$$[U]_{y=M}=[\Phi]_{y=M}T(t)=0 \quad x\in(0,L) \quad t>0 \quad \Rightarrow \quad [\Phi]_{y=M}=0$$

There are four homogeneous boundary conditions for the function Φ .

From the separated equations, consider the equation

Helmholtz Equation

$$\nabla^2\Phi=\beta\Phi$$

which has a structure of equation of the *eigenvalue problem* for differential operator ∇^2 . It is called the ***Helmholtz Equation***.

The solution of the Helmholtz Equation subject to boundary conditions can be easily obtained by the eigenfunction expansion method.

Separation of variables – 2nd stage:

Assume $\Phi(x,y)=X(x)Y(y)$

Substitute into the Helmholtz Equation

$$\nabla^2(XY)\equiv X''Y+XY''=\zeta XY$$

Divide by XY

$$\frac{X''}{X}+\frac{Y''}{Y}=\beta$$

Separation of variables in the boundary conditions yield:

$$y\in(0,M) \quad [\Phi]_{x=0}=[X(0)]Y(y)=0 \quad \Rightarrow \quad [X]_{x=0}=0$$
$$y\in(0,M) \quad [\Phi]_{x=L}=[X(L)]Y(y)=0 \quad \Rightarrow \quad [X]_{x=L}=0$$
$$x\in(0,L) \quad [\Phi]_{y=0}=X(x)[Y(0)]=0 \quad \Rightarrow \quad [Y]_{y=0}=0$$
$$x\in(0,L) \quad [\Phi]_{y=M}=X(x)[Y(M)]=0 \quad \Rightarrow \quad [Y]_{y=M}=0$$

Note, that we have complete pairs of homogeneous boundary conditions both for X and Y .

Now, solve consequently the Sturm-Liouville problems for X and Y:

$$\frac{X''}{X} = -\frac{Y''}{Y} + \beta = \mu$$

Equation is separated. It yields first SLP:

$$\begin{array}{lcll} X'' - \mu X = 0 & & & \\ \left[X\right]_{x=0} = 0 & \overset{SLP}{\Rightarrow} & \mu = -\lambda_n^2 & n = 1,2,... \\ \left[X\right]_{x=L} = 0 & & X_n\left(x\right) & \end{array}$$

Then the second equation becomes:

$$-\frac{Y''}{Y} + \beta = -\lambda_n^2$$

which in its turn is a separated equation:

$$\frac{Y''}{Y} = \beta + \lambda_n^2 = \eta$$

It yields the second Sturm-Liouville Problem:

$$\begin{array}{lcll} Y'' - \eta Y = 0 & & & \\ \left[Y\right]_{y=0} = 0 & \overset{SLP}{\Rightarrow} & \eta = -\nu_m^2 & m = 1,2,... \\ \left[Y\right]_{y=M} = 0 & & Y_m\left(y\right) & \end{array}$$

Equation for separation constants yields:

$$\beta + \lambda_n^2 = -\nu_m^2 \quad \Rightarrow \quad \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Then equation for T becomes

$$\frac{1}{\alpha}\frac{T'}{T} = \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Which is the 1st order ordinary differential equation:

$$T' + \alpha\left(\lambda_n^2 + \nu_m^2\right)T = 0$$

with the solutions:

$$T_{nm}\left(t\right) = e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Solution of the Transient Problem:

Construct the solution in the form of double infinite series (eigenfunction expansion):

$$U\left(x,y,t\right) = \sum_n \sum_m A_{nm} X_n Y_m e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Where the coefficients A_{nm} can be found from the initial condition

$$U\left(x,y,0\right) = U_0\left(x,y\right) = \sum_n \sum_m A_{nm} X_n Y_m$$

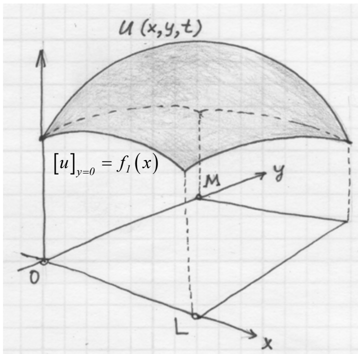
as the Fourier coefficients of the double Generalized Fourier series:

$$A_{nm} = \frac{\int_0^M \int_0^L U_0\left(x,y\right) X_n\left(x\right) Y_m\left(y\right) dxdy}{\left\|X_n\right\|^2 \left\|Y_m\right\|^2}$$

Example: DDNN

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$

$u(x,y,t):\ (x,y)\in(0,L)\times(0,M)\ ,t>0$



Initial Condition: $u(x,y,0)=u_0(x,y)$

Boundary Conditions:

$x=0$	$[u]_{x=0}=0$	$y\in(0,M)$	$t>0$	(*Dirichlet*)
$x=L$	$[u]_{x=L}=f_4(y)$	$y\in(0,M)$	$t>0$	(*Dirichlet*)
$y=0$	$\left[\frac{\partial u}{\partial y}\right]_{y=0}=0$	$x\in(0,L)$	$t>0$	(*Neumann*)
$y=M$	$\left[\frac{\partial u}{\partial y}\right]_{y=M}=0$	$x\in(0,L)$	$t>0$	(*Neumann*)

---

1. Steady State Solution

Find time-independent solution $u_s(x,y)$:

$$\frac{\partial^2 u_s}{\partial x^2}+\frac{\partial^2 u_s}{\partial y^2}=0$$

subject to the boundary conditions:

$x=0$	$[u_s]_{x=0}=0$	$y\in(0,M)$	$t>0$
$x=L$	$[u_s]_{x=L}=f_4(y)$	$y\in(0,M)$	$t>0$
$y=0$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=0}=0$	$x\in(0,L)$	$t>0$
$y=M$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=M}=0$	$x\in(0,L)$	$t>0$

This is the basic problem for Laplace's Equation when, three boundary conditions are non-homogeneous.

Separation of variables: $u_s(x,y)=XY$

$x=0$	$[u_s]_{x=0}=0$	$\Rightarrow\ X(0)=0$
$x=L$	$[u_s]_{x=L}=f_4(y)$	
$y=0$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=0}=0$	$\Rightarrow\ Y'(0)=0$
$y=M$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=M}=0$	$\Rightarrow\ Y'(M)=0$

Separated equation:

$$\frac{Y''}{Y}=-\frac{X''}{X}=\mu$$

First, consider equation for Y (two conditions):

$Y''-\mu Y=0$ $\qquad \mu=-\lambda_n^2$

$Y'(0)=0$ $\qquad \overset{SLP}{\Rightarrow}\ \lambda_0=0 \qquad Y_0=1$

$Y'(M)=0$ $\qquad \lambda_n=\frac{n\pi}{M} \qquad Y_n(y)=cos(\lambda_n y)=cos\left(\frac{n\pi}{M}y\right)$

Then equations for X:

$$X_0'' = 0 \qquad \Rightarrow \quad X_0(x) = c_1 + c_2 x$$

$$X_n'' - \lambda_n^2 X = 0 \qquad \Rightarrow \quad X_n(x) = c_1 \cosh(\lambda_n x) + c_2 \sinh(\lambda_n x)$$

Boundary condition at $x = 0$ yields

$$X_0(0) = 0 = c_1 + c_2 \cdot 0 = c_1 \qquad \Rightarrow \quad c_1 = 0$$

$$X_n(0) = 0 = c_1 \cdot 1 + c_2 \cdot 0 = c_1 \qquad \Rightarrow \quad c_1 = 0$$

Then

$$X_0(x) = x$$

$$X_n(x) = \sinh\left(\frac{n\pi}{M}x\right)$$

Construct the steady state solution as

$$u_s(x,y) = a_0 X_0 Y_0 + \sum_{n=1}^{\infty} a_n X_n Y_n \quad = a_0 x + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{M}x\right)\cos\left(\frac{n\pi}{M}y\right)$$

This solution should satisfy the boundary condition at $x = L$:

$$u_s(L,y) = f_3(y) = a_0 L + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{M}L\right)\cos\left(\frac{n\pi}{M}y\right)$$

Which is a cosine Fourier series expansion of $f_3(y)$ with

$$a_0 = \frac{1}{LM}\int_0^M f_3(y)dy$$

$$a_n = \frac{2}{M \sinh\left(\frac{n\pi}{M}L\right)}\int_0^M f_3(y)\cos\left(\frac{n\pi}{M}y\right)dy$$

Then the steady state solution becomes:

$$u_s(x,y) \quad = \left[\frac{1}{LM}\int_0^M f_3(y)dy\right]x + \sum_{n=1}^{\infty}\left[\frac{2}{M \sinh\left(\frac{n\pi}{M}L\right)}\int_0^M f_3(y)\cos\left(\frac{n\pi}{M}y\right)dy\right]\sinh\left(\frac{n\pi}{M}x\right)\cos\left(\frac{n\pi}{M}y\right)$$

2. Transient Solution

Introduce the transient function as

$$U(x,y,t) = u(x,y,t) - u_s(x,y)$$

Function U satisfies *homogeneous* Heat Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

with four *homogeneous* boundary conditions:

$$x = 0 \quad [U]_{x=0} = 0 \qquad y \in (0,M) \quad t > 0$$

$$x = L \quad [U]_{x=L} = 0 \qquad y \in (0,M) \quad t > 0$$

$$y = 0 \quad \left[\frac{\partial U}{\partial y}\right]_{y=0} = 0 \qquad x \in (0,L) \quad t > 0$$

$$y = M \quad \left[\frac{\partial U}{\partial y}\right]_{y=M} = 0 \qquad x \in (0,L) \quad t > 0$$

and the initial condition:

$$U(x,y,0) = u_0(x,y) - u_s(x,y) \quad \equiv U_0(x,y)$$

Separation of variables $U = XYT$ yields a separated equation

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

with homogeneous boundary conditions:

$$x = 0 \quad \left[U\right]_{x=0} = 0 \quad \Rightarrow \quad X(0) = 0$$

$$x = L \quad \left[U\right]_{x=L} = 0 \quad \Rightarrow \quad X(L) = 0$$

$$y = 0 \quad \left[\frac{\partial U}{\partial y}\right]_{y=0} = 0 \quad \Rightarrow \quad Y'(0) = 0$$

$$y = M \quad \left[\frac{\partial U}{\partial y}\right]_{y=M} = 0 \quad \Rightarrow \quad Y'(M) = 0$$

Solve consequently the Sturm-Liouville problems for X and Y:

$$\frac{X''}{X} = -\frac{Y''}{Y} + \beta = \mu$$

$$\begin{array}{lcl} X'' - \mu X = 0 & & \\ \left[X\right]_{x=0} = 0 & \overset{DD}{\Rightarrow} & \mu = -\lambda_n^2, \quad \lambda_n = \dfrac{n\pi}{L}, \quad n = 1,2,\ldots \\ \left[X\right]_{x=L} = 0 & & X_n(x) = \sin(\lambda_n x) = \sin\left(\dfrac{n\pi}{L}x\right) \end{array}$$

Then the second equation becomes:

$$-\frac{Y''}{Y} + \beta = \mu = -\lambda_n^2$$

which in its turn is a separated equation:

$$\frac{Y''}{Y} = \beta + \lambda_n^2 = \eta$$

It yields the second Sturm-Liouville Problem:

$$\begin{array}{lclc} Y'' - \eta Y = 0 & & \eta = -\nu_m^2 & m = 0,1,2,\ldots \\ \left[Y'\right]_{y=0} = 0 & \overset{NN}{\Rightarrow} & \nu_0 = 0 & Y_0(y) = 1 \\ \left[Y'\right]_{y=M} = 0 & & \nu_m = \dfrac{m\pi}{M} & Y_m(y) = \cos\left(\dfrac{m\pi}{M}y\right) \end{array}$$

Equation for separation constants yields:

$$\beta + \lambda_n^2 = \eta = -\nu_m^2 \quad \Rightarrow \quad \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Then equation for T becomes

$$\frac{1}{\alpha}\frac{T'}{T} = \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Which is the 1st order ordinary differential equation:

$$T' + \alpha\left(\lambda_n^2 + \nu_m^2\right)T = 0$$

with the solutions:

$$T_{nm}(t) = e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Solution of the Transient Problem:

Construct the solution in the form of double infinite series (eigenfunction expansion):

$$U(x,y,t) = \sum_{n=1}^{\infty} A_{n0} X_n Y_0 e^{-\alpha\lambda_n^2 t} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} X_n Y_m e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Where the coefficients A_{nm} can be found from the initial condition:

$$U(x,y,0) = U_0(x,y) = \sum_{n=1}^{\infty} A_{n0} X_n Y_0 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} X_n Y_m$$

$$U_0(x,y) = \left[\sum_{n=1}^{\infty} A_{n0} X_n\right] Y_0 + \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} A_{nm} X_n\right] Y_m$$

where

$$\left[\sum_{n=1}^{\infty} A_{n0} X_n\right] = \frac{1}{M}\int_0^M U_0(x,y)dy$$

$$\left[\sum_{n=1}^{\infty} A_{nm} X_n\right] = \frac{2}{M}\int_0^M U_0(x,y) Y_m(y)\,dy$$

Then

$$A_{n0} = \frac{2}{LM}\int_0^L\int_0^M U_0(x,y) X_n(x)dydx$$

$$A_{nm} = \frac{4}{LM}\int_0^M\int_0^L U_0(x,y) X_n(x) Y_m(y)\,dxdy$$

3. Solution of IBVP

$$u(x,y,t) \quad = U(x,y,t) + u_s(x,y)$$

$$u(x,y,t) \quad = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{M}y\right) e^{-\left(\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{M^2}\right)\frac{1}{a^2}t}$$

$$+a_0 x + \sum_{m=1}^{\infty} a_m \sinh\left(\frac{m\pi}{M}x\right)\cos\left(\frac{m\pi}{M}y\right)$$

where coefficients are

$$A_{0n} \quad = \frac{2}{LM}\int_0^L\int_0^M \left[g(x,y) - u_s(x,y)\right]\sin\left(\frac{n\pi}{L}x\right)dydx$$

$$A_{mn} \quad = \frac{4}{LM}\int_0^L\int_0^M \left[g(x,y) - u_s(x,y)\right]\cos\left(\frac{m\pi}{M}y\right)\sin\left(\frac{n\pi}{L}y\right)dydx$$

$$a_0 \quad = \frac{1}{LM}\int_0^M f(y)dy$$

$$a_m \quad = \frac{2}{M\sinh\left(\frac{m\pi}{M}L\right)}\int_0^M f(y)\cos\left(\frac{m\pi}{M}y\right)dy$$

4. Maple Example: *heat5dn-2.mws* $L = 2,\quad M = 4,\ \alpha = 0.5,\ f(y) = 1,\ g(x,y) = x(x-L) + y(y-M)$

2-D Heat Equation Example DD-NN

```
> restart;
> with(plots):
> L:=2;M:=4;alpha:=0.5;
```

$$L := 2$$

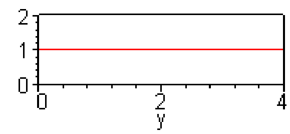
$$M := 4$$

$$\alpha := 0.5$$

```
> f(y):=1;
```

$$f(y) := 1$$

```
> plot(f(y),y=0..M,axes=boxed);
```



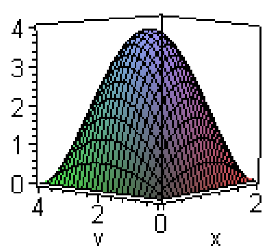
function in non-homogeneous boundary condition

$[u]_{x=L} = f_4(y)$

```
> u0(x,y):=x*(x-L)*y*(y-M);
```

$$u0(x, y) := x\,(x-2)\,y\,(y-4)$$

```
> plot3d(u0(x,y),x=0..L,y=0..M,axes=boxed);
```



initial temperature distribution

$u_0(x,y) = x(x-L) + y(y-M)$

Steady State Solution:

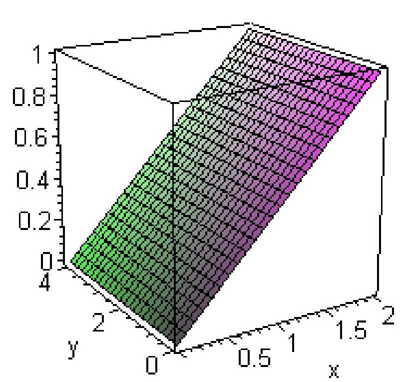
```
> a[0]:=int(f(y),y=0..M)/L/M;
```

$$a_0 := \frac{1}{2}$$

```
> a[m]:=2/M*int(f(y)*cos(m*Pi*y/M),y=0..M)/sinh(m*Pi*L/M);
```

$$a_m := \frac{2\sin(m\,\pi)}{m\,\pi\,\sinh\left(\frac{m\,\pi}{2}\right)}$$

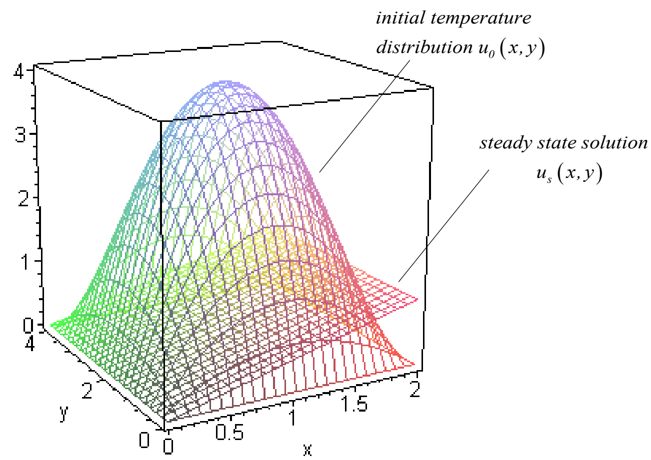
```
> us[m](x,y):=a[m]*sinh(m*Pi*x/M)*cos(m*Pi*y/M):
> us(x,y):=a[0]*x+sum(us[m](x,y),m=1..2):
> plot3d(us(x,y),x=0..L,y=0..M,axes=boxed,projection=0.92);
```



steady state solution

$u_s(x,y)$

```
> plot3d({us(x,y),u0(x,y)},x=0..L,y=0..M,axes=boxed,style=wireframe);
```



Transient Solution:

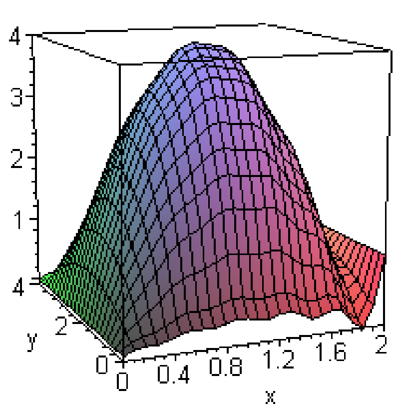
```
> U0(x,y):=u0(x,y)-us(x,y);
```

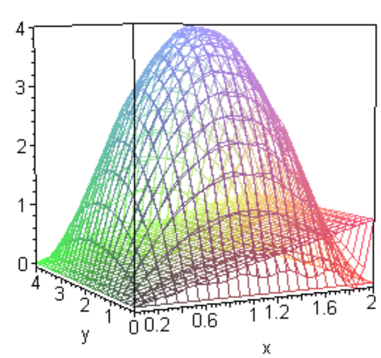
$$U0(x,y) := x\,(x-2)\,y\,(y-4) - \frac{x}{2}$$

```
> A[n,0]:=2*int(int(U0(x,y),y=0..M)*sin(n*Pi*x/L),x=0..L)/L/M:

> A[n,m]:=4*int(int(U0(x,y)*cos(m*Pi*y/M),y=0..M)*sin(n*Pi*x/L),x=0..L)/L/M:

> U[n,0](x,y,t):=A[n,0]*sin(n*Pi*x/L)*exp(-n^2/L^2*Pi^2*t*alpha):

> U[n,m](x,y,t):=A[n,m]*sin(n*Pi*x/L)*cos(m*Pi*y/M)*exp(-
  (m^2/M^2+n^2/L^2)*Pi^2*t*alpha):

>
U(x,y,t):=sum(U[n,0](x,y,t),n=1..10)+sum(sum(U[n,m](x,y,t),m=1..10),n=1..10):

> U(x,y,0):=subs(t=0,U(x,y,t)):
```

Solution of IBVP:

```
> u(x,y,t):=us(x,y)+U(x,y,t):

> u(x,y,0):=subs(t=0,u(x,y,t)):

> animate3d(u(x,y,t),x=0..L,y=0..M,t=0..3,frames=100,axes=boxed);
```



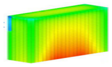


Professor Gabriel Węcel

(Institute of Thermal Technology, Silesian University of Technology, Gliwice, Poland)

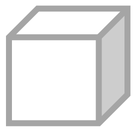
has visited our class on February 1, 2013

Silesian University of Technology



THE HEAT EQUATION 3-D Cartesian Coordinates

$u(x,y,z,t)$



$[u]_S = f$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+F(x,y,z) = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad (x,y,z)\in(0,L)\times(0,M)\times(0,K)\subset\mathbb{R}^3,\ t>0$$

$$[u]_S = f \qquad t>0$$

$$[u]_{t=0} = u_0(x,y,z) \qquad t=0 \qquad (x,y,z)\in[0,L]\times[0,M]\times[0,K]\subset\mathbb{R}^3$$

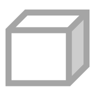
$$u(x,y,z,t) = u_{ss}(x,y,z) + U(x,y,z,t)$$

$u_{ss}(x,y,z)$ $\qquad$ $U(x,y,z,t)$

STEADY STATE PROBLEM - PELE

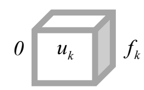
$$\frac{\partial^2 u_{ss}}{\partial x^2}+\frac{\partial^2 u_{ss}}{\partial y^2}+\frac{\partial^2 u_{ss}}{\partial z^2} + F(x,y,z) = 0$$

Laplace Eqn $\nabla^2 u = 0$

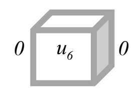


$[u_{ss}]_S = f$

six basic problems

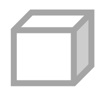


Poisson Eqn $\nabla^2 u + F = 0$



TRANSIENT PROBLEM (basic)

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$



$[U]_S = 0 \qquad [U]_{t=t_0} = u_0 - u_{ss}$

supplemental eigenvalue problems

$X'' = \mu X$ $[X]_{x=0} = 0$ $[X]_{x=L} = 0$	$\overset{SLP}{\Rightarrow}$	$X_n'' = -\lambda_n^2 X_n$ $\mu_n = -\lambda_n^2$ $X_n(x)$
$Y'' = \eta Y$ $[Y]_{y=0} = 0$ $[Y]_{y=M} = 0$	$\overset{SLP}{\Rightarrow}$	$Y_m'' = -\nu_m^2 Y_m$ $\eta_m = -\nu_m^2$ $Y_m(y)$
$Z'' = \gamma Z$ $[Z]_{z=0} = 0$ $[Z]_{z=K} = 0$	$\overset{SLP}{\Rightarrow}$	$Z_k'' = -\omega_k^2 Z_k$ $\gamma_k = -\omega_k^2$ $Z_k(z)$

SEPARATION OF VARIABLES

$$U(x,y,z,t) = \Phi(x,y,z)\,T(t) \qquad \frac{\nabla^2\Phi}{\Phi} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

HELMHOLTZ EQUATION

$$\nabla^2\Phi = \beta\Phi$$

$$\Phi(x,y,z) = X(x)Y(y)Z(z)$$

$\lambda_n,\ \nu_m,\ \omega_k$

$X_n,\ Y_m,\ Z_k$

$$\beta_{nmk} = -\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)$$

$$\frac{1}{\alpha}\frac{T'}{T} = \beta$$

$$T = e^{-\alpha\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)t}$$

STEADY STATE SOLUTION (PELE)

$$u_{ss}(x,y,z) = u_1+u_2+\ldots+u_6 + \sum_n\sum_m\sum_k A_{nmk}\,X_nY_mZ_k$$

$$\text{where} \quad A_{mnk} = \frac{\int_0^K\int_0^M\int_0^L F(x,y,z)\,X_n\,Y_m\,Z_k\,dx\,dy\,dz}{\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)\|X_n\|^2\,\|Y_m\|^2\,\|Z_k\|^2}$$

$u_{ss}(x,y,z)$ = *solution of six basic problems for Laplace's Equation plus solution of Poisson's equation with zero b.c.'s*

TRANSIENT SOLUTION

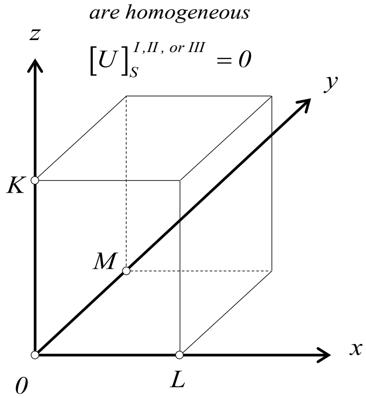
$$U(x,y,z,t) = \sum_n\sum_m\sum_k B_{nmk}\,X_nY_mZ_k\,e^{-\alpha\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)t}$$

$$\text{where} \quad B_{nmk} = \frac{\int_0^K\int_0^M\int_0^L (u_0-u_s)\,X_nY_mZ_k\,dx\,dy\,dz}{\|X_n\|^2\,\|Y_m\|^2\,\|Z_k\|^2}$$

THE SOLUTION OF THE IBVP is a superposition of steady-state and transient solutions

$$u(x,y,z,t) = u_{ss}(x,y,z) + U(x,y,z,t)$$

VIII.3.2.2 3-D TRANSIENT PROBLEM. HELMHOLTZ EQUATION.



Consider transient problem from the solution of the 3-D Heat Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad (x,y,z) \in (0,L)\times(0,M)\times(0,K)\ ,\ t>0$$

Separation of variables:

$$U(x,y,z,t) = \Phi(x,y,z)T(t)$$

Separated equation:

$$\frac{\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2} + \dfrac{\partial^2 \Phi}{\partial z^2}}{\Phi} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

Separated equation yields the Helmholtz Equation:

Helmholtz Equation

$$\nabla^2 \Phi = \beta \Phi$$

which constitutes the *eigenvalue problem* for differential operator ∇^2.

The solution of the Helmholtz Equation subject to boundary conditions can be easily obtained by the eigenfunction expansion method.

Assume

$$\Phi(x,y,z) = X(x)Y(y)Z(z)$$

Substitute into the Helmholtz Equation

$$\nabla^2(XYZ) \equiv X''YZ + XY''Z + XYZ'' = \beta XYZ$$

Divide by XYZ

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \beta$$

Separation of variables in the boundary conditions yields:

$$\begin{array}{lll} x=0 & [\Phi]_{x=0} = [X(0)]Y(y)Z(z) = 0 \quad \Rightarrow & [X]_{x=0} = 0 \\ x=L & [\Phi]_{x=L} = [X(L)]Y(y)Z(z) = 0 \quad \Rightarrow & [X]_{x=L} = 0 \\ y=0 & [\Phi]_{y=0} = X(x)[Y(0)]Z(z) = 0 \quad \Rightarrow & [Y]_{y=0} = 0 \\ y=M & [\Phi]_{y=M} = X(x)[Y(M)]Z(z) = 0 \quad \Rightarrow & [Y]_{y=M} = 0 \\ z=0 & [\Phi]_{z=0} = X(x)Y(y)[Z(0)] = 0 \quad \Rightarrow & [Z]_{z=0} = 0 \\ z=K & [\Phi]_{z=K} = X(x)Y(y)[Z(K)] = 0 \quad \Rightarrow & [Z]_{z=K} = 0 \end{array}$$

Note, that we have complete pairs of homogeneous boundary conditions for X, Y and Z.

Solve consequently the Sturm-Liouville problems for X, Y, and Z:

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} + \beta = \mu$$

Supplemental Eigenvalue problems

The first Sturm-Liouville Problem:

$$X'' - \mu X = 0$$
$$[X]_{x=0} = 0 \quad \overset{SLP}{\Rightarrow} \quad \mu = -\lambda_n^2 \quad n = (0), 1, 2, \ldots$$
$$[X]_{x=L} = 0 \quad \quad X_n(x)$$

Then the equation becomes:

$$-\frac{Y''}{Y} - \frac{Z''}{Z} + \beta = \mu = -\lambda_n^2$$

which in its turn is a separated equation:

$$\frac{Y''}{Y} = -\frac{Z''}{Z} + \beta + \lambda_n^2 = \eta$$

It yields the second Sturm-Liouville Problem:

$$Y'' - \eta Y = 0$$
$$[Y]_{y=0} = 0 \quad \overset{SLP}{\Rightarrow} \quad \eta = -\nu_m^2 \quad m = (0), 1, 2, \ldots$$
$$[Y]_{y=M} = 0 \quad \quad Y_m(y)$$

Then one more step produces equation

$$-\frac{Z''}{Z} + \beta + \lambda_n^2 = -\nu_m^2$$

which also can be separated

$$\frac{Z''}{Z} = \beta + \lambda_n^2 + \nu_m^2 = \gamma$$

It yields the third Sturm-Liouville Problem:

$$Z'' - \gamma Z = 0$$
$$[Z]_{z=0} = 0 \quad \overset{SLP}{\Rightarrow} \quad \gamma = -\omega_k^2 \quad k = (0), 1, 2, \ldots$$
$$[Z]_{z=K} = 0 \quad \quad Z_k(z)$$

Then the second part of the last equation becomes

$$\beta + \lambda_n^2 + \nu_m^2 = -\omega_k^2$$

and the constant of separation is

$$\beta_{nmk} = -\left(\lambda_n^2 + \nu_m^2 + \omega_k^2\right)$$

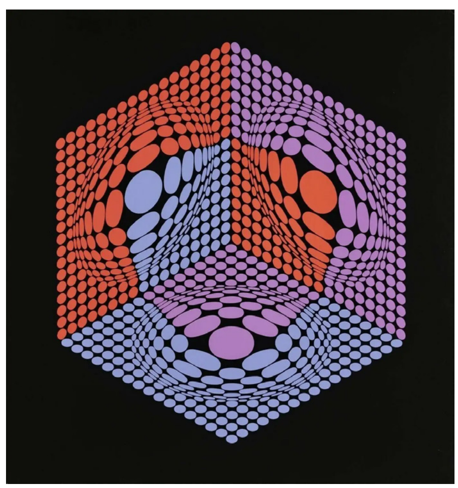
Then the solution of the Basic IBVP for the Heat Equation is:

Solution of Basic IBVP:

$$U(x,y,z,t) = \sum_n \sum_m \sum_k B_{nmk} X_n(x) Y_m(y) Z_k(z) e^{-\alpha\left(\lambda_n^2 + \nu_m^2 + \omega_k^2\right)t}$$

where the coefficients B_{nmk} can be found from the initial condition as the Fourier coefficients of the triple Generalized Fourier Series:

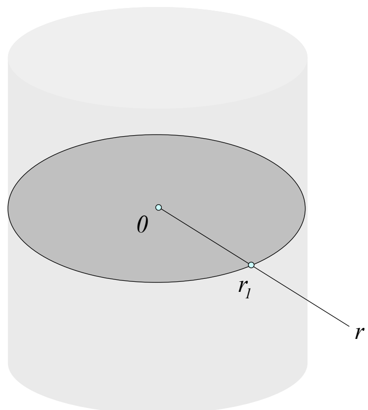
$$B_{nmk} = \frac{\int_0^K \int_0^M \int_0^L U_0(x,y,z) X_n(x) Y_m(y) Z_k(z)\, dxdydz}{\|X_n\|^2 \|Y_m\|^2 \|Z_k\|^2}$$



VIII.3.3. HEAT EQUATION IN CYLINDRICAL COORDINATES

VIII.3.3.1 LONG SOLID CYLINDER long solid cylinder with angular symmetry: $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \theta} = 0$

BASIC CASE Homogeneous Equation and Boundary Conditions



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(r,t): \quad r \in [0, r_1), \quad t > 0$$

Initial condition: $u(r,0) = u_0(r)$

Boundary conditions: $u(0,t) < \infty \quad t > 0 \quad \text{bounded}$

$[u]_{r=r_1} = 0 \quad t > 0 \quad (I, II \text{ or } III^{rd} \text{ kind})$

1) Separation of variables: $u(r,t) = R(r)T(t)$ $\quad u(r,t)$ *bounded* $\Rightarrow R(0) < \infty$

$[R(r_1)]T(t) = 0 \quad \Rightarrow \quad [R]_{r=r_1} = 0$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{\alpha}\frac{T'}{T} = \mu$$ separate variables in the equation

2) Sturm-Liouville Problem: $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \mu$

See VII.12, p.509	$r^2 R'' + rR + [\lambda^2 r^2 - 0^2]R = 0$	*Bessel Equation of 0th order*
	$\mu_n = -\lambda_n^2 \quad n = 1,2,\ldots$	are positive roots of
	$J_0(\lambda_n r_1) = 0$	characteristic equation
	$R_n(r) = J_0(\lambda_n r)$	Eigenfunctions
	$\lVert R_n(r) \rVert_p^2 = \int_0^{r_1} J_0^2(\lambda_n r)\, r\, dr = (r_1^2/2) J_1^2(\lambda_n r_1)$	norm
	$p(r) = r$	weight function:

3) Equation for T : $T' - \alpha\mu T = 0$

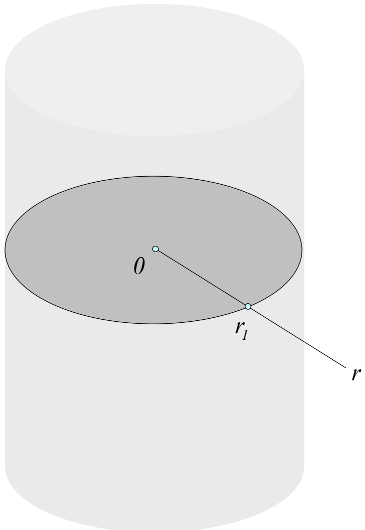
$T' + \alpha\lambda_n^2 T = 0 \quad \Rightarrow \quad T_n(t) = e^{-\alpha\lambda_n^2 t}$

4) Solution:

$$u(r,t) = \sum_{n=1}^{\infty} a_n R_n T_n = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) e^{-\alpha\lambda_n^2 t}, \quad \text{where}$$

Initial condition $u(r,0) = u_0(r) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) \quad \Rightarrow \quad a_n = \dfrac{\int_0^{r_1} u_0(r) J_0(\lambda_n r) r dr}{\int_0^{r_1} J_0^2(\lambda_n r) r dr}$

GENERAL CASE: Non-Homogeneous Equation, Non-Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(r)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad r\in(0,r_1),\ t>0$$

Initial condition: $u(r,0)=u_0(r)$

Boundary conditions: $u(r,t)<\infty \quad t>0 \quad$ *bounded*

$[u]_{r=r_1}=f_1 \quad t>0 \quad$ (*I, II or IIIrd kind*)

<u>I Steady State Solution</u>

Time-independent solution $u_s(r)$

Substitution of a time-independent function into the heat equation leads to the following ordinary differential equation:

$$\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u_s}{\partial r}+F(r)=0 \qquad u_s(r),\ \ r\in(0,r_1)$$

subject to the boundary conditions of the same kind as for PDE:

$[u_s]_{x=0}$, $\quad t>0 \quad$ bounded

$[u_s]_{r=r_1}=f_1$, $\quad t>0 \quad$ (*I, II or IIIrd kind*)

General solution of ODE:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_s}{\partial r}\right)+F(r)=0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_s}{\partial r}\right)=-rF(r)$$

$$\frac{\partial u_s}{\partial r}=\frac{1}{r}\int\left[-rF(r)\right]dr+\frac{c_1}{r}$$

$$u_s(r)=\int\left\{\frac{1}{r}\int\left[-r\,F(r)\right]dr\right\}dr+c_1\ln r+c_2$$

For bounded solution, it is necessarily $c_1=0$, therefore the general steady state solution in circular domain is

$$u_s(r)=\int\left\{\frac{1}{r}\int\left[-rF(r)\right]dr\right\}dr+c_2$$

Solutions of BVPs for circular domain with uniform heat generation are provided by the Table.

<u>II Transient Solution:</u>

Define the transient solution by equation:

$$U(r,t)=u(r,t)-u_s(r)$$

then solution of the original problem is a sum of transient solution and steady state solution:

$$u(r,t)=U(r,t)+u_s(r)$$

Substitute it into the Heat Equation:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u_s}{\partial r}+F(r)=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

Since $\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(x)=0$, it yields

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

We obtained the equation for the new unknown function $U(r,t)$ which has homogeneous boundary condition:

$r=r_1 \quad [U]_{r=r_1}=[u]_{r=r_1}-[u_s]_{r=r_1}=f_1-f_1=0$

As a result, we reduced the non-homogeneous problem to a homogeneous equation for $U(r,t)$ with homogeneous boundary conditions. Initial condition for function $U(r,t)$:

$U(r,0)=u(r,0)-u_s(r)=u_0(r)-u_s(r)$

Solution for *U(r,t)*

We consider the following BASIC initial boundary value problem:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U(r,t),\ \ r\in(0,r_1),\ t>0$$

initial condition: $U(r,0)=u_0(r)-u_s(r)$

boundary conditions: $U(0,t)<\infty \qquad t>0$

$[U]_{r=r_1}=0 \qquad t>0$

We already know a solution of this basic problem obtained by separation of variables:

$$U(x,t)=\sum_{n=1}^{\infty}a_n R_n T_n=\sum_{n=1}^{\infty}a_n J_0(\lambda_n r)e^{-\alpha\lambda_n^2 t}$$

where coefficients a_n are the Fourier coefficients determined by the corresponding initial condition for the function $U(x,t)$:

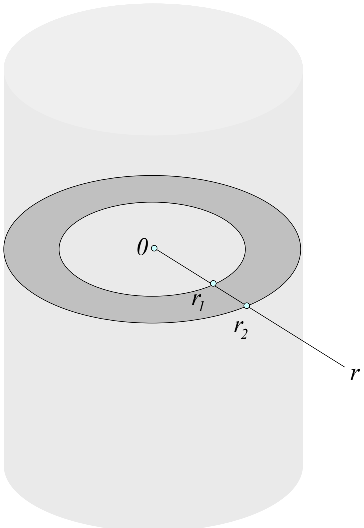
$$a_n=\frac{\int_0^{r_1}\left[u_0(r)-u_s(r)\right]R_n(r)rdr}{\int_0^{r_1}R_n^2(r)rdr}=\frac{\int_0^{r_1}\left[u_0(r)-u_s(r)\right]J_0(\lambda_n r)rdr}{\int_0^{r_1}J_0^2(\lambda_n r)rdr}$$

<u>III Solution of IBVP:</u>

Solution of the original IBVP is a sum of steady state solution and transient solution:

$$\boxed{u(r,t)=u_s(r)+U(r,t)}$$

VIII.3.3.2 HOLLOW CYLINDER BASIC CASE: Homogeneous Equation and Boundary Conditions



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad r \in (0, r_1), \ t > 0$$

Initial condition: $u(r,0) = u_0(r)$

Boundary conditions: $[u]_{r=r_1} = 0 \quad t > 0 \quad$ (*I, II or IIIrd kind*)

$[u]_{r=r_2} = 0 \quad t > 0 \quad$ (*I, II or IIIrd kind*)

1) Separation of variables: $u(r,t) = R(x)T(t)$

$$[u]_{r=r_1} = [R]_{r=r_1} T(t) = 0 \quad \Rightarrow \quad [R]_{r=r_1} = 0$$
$$[u]_{r=r_2} = [R]_{r=r_2} T(t) = 0 \quad \Rightarrow \quad [R]_{r=r_2} = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{\alpha}\frac{T'}{T} = \mu$$

2) Sturm-Liouville Problem: $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \mu \qquad$ ($\mu = -\lambda^2$ SLP)

$r^2 R'' + rR + \left[\lambda^2 r^2 - 0^2\right] R = 0 \qquad$ *Bessel Equation of 0 order*

Eigenvalues: $\mu_n = -\lambda_n^2 \qquad n = 1, 2, \ldots$

λ_n are roots of characteristic eqn

Eigenfunctions: $R_n(r) = c_{1,n} J_0(\lambda_n r) + c_{2,n} Y_0(\lambda_n r)$

3) Equation for T : $T' - \alpha\mu T = 0$

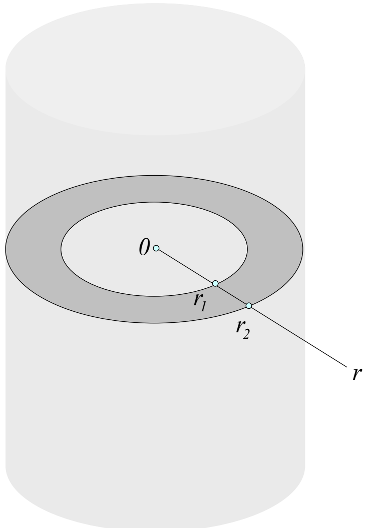
$$T' + \alpha\lambda_n^2 T = 0 \quad \Rightarrow \quad T_n(t) = e^{-\alpha\lambda_n^2 t}$$

4) Solution:

$$u(r,t) = \sum_{n=1}^{\infty} a_n R_n T_n \qquad = \sum_{n=1}^{\infty} a_n R_n(r) e^{-\alpha\lambda_n^2 t}$$

Initial condition: $u(r,0) = u_0(r) = \sum_{n=1}^{\infty} a_n R_n \quad \Rightarrow \quad a_n = \dfrac{\int_0^{r_1} u_0(r) R_n(r) r\,dr}{\int_0^{r_1} R_n^2(r) r\,dr}$

GENERAL CASE: Non-Homogeneous Equation, Non-Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(r)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad r\in(0,r_1),\ t>0$$

Initial condition: $u(r,0)=u_0(r)$

Boundary conditions: $[u]_{r=r_1}=f_1 \quad t>0 \qquad$ (*I, II or IIIrd kind*)

$[u]_{r=r_2}=f_2 \quad t>0 \qquad$ (*I, II or IIIrd kind*)

I Steady State Solution

Time-independent solution $u_s(r)$

Substitution of a time-independent function into the heat equation leads to the following ordinary differential equation:

$$\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u_s}{\partial r}+F(r)=0 \qquad u_s(r),\ \ r\in(0,r_1)$$

subject to the boundary conditions of the same kind as for PDE:

$[u_s]_{r=r_1}=f_1, \qquad t>0 \qquad$ (*I, II or IIIrd kind*)

$[u_s]_{r=r_2}=f_2, \qquad t>0 \qquad$ (*I, II or IIIrd kind*)

General solution of ODE:

$$u_s(r)=\int\left\{\frac{1}{r}\int\left[-rF(r)\right]dr\right\}dr+c_1\ln r+c_2$$

Coefficients c_1, c_2 have to be determined from boundary conditions.

II Transient Solution:

Define the transient solution by equation:

$$U(r,t)=u(r,t)-u_s(r), \qquad u(r,t)=U(r,t)+u_s(r)$$

Substitution into the Heat Equation yields an equation for transient solution:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

for the new unknown function $U(r,t)$ which has two homogeneous boundary conditions:

$$r=r_1 \qquad [U]_{r=r_1}=[u]_{r=r_1}-[u_s]_{r=r_1}=f_1-f_1=0$$

$$r=r_2 \qquad [U]_{r=r_2}=[u]_{r=r_2}-[u_s]_{r=r_2}=f_2-f_2=0$$

and initial condition:

$$U(r,0)=u(r,0)-u_s(r)=u_0(r)-u_s(r)$$

Solution for *U(r,t)*

We consider the following BASIC initial boundary value problem:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U(r,t),\ \ r\in(0,r_1),\ t>0$$

initial condition: $U(r,0)=u_0(r)-u_s(r)$

boundary conditions: $[U]_{r=r_1}=0 \qquad t>0$

$[U]_{r=r_2}=0 \qquad t>0$

We already know a solution of this basic problem obtained by separation of variables:

$$U(x,t)=\sum_{n=1}^{\infty}a_n R_n T_n=\sum_{n=1}^{\infty}a_n R_n(r)e^{-\alpha\lambda_n^2 t}$$

where coefficients a_n are the Fourier coefficients determined by the corresponding initial condition for the function $U(x,t)$:

$$a_n=\frac{\int_0^{r_1}\left[u_0(r)-u_s(r)\right]R_n(r)rdr}{\int_0^{r_1}R_n^2(r)rdr}$$

<u>III Solution of IBVP:</u>

Solution of the original IBVP is a sum of steady state solution and transient solution:

$$u(r,t)=u_s(r)+U(r,t)$$

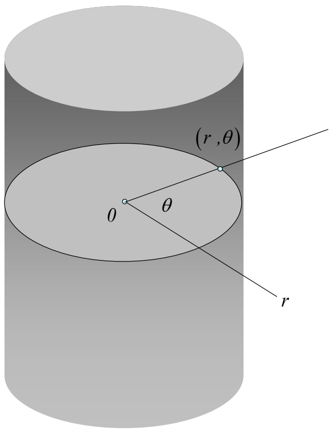
VIII.3.3.4 HEAT EQUATION in Cylindrical Coordinates:

$$u(r,\theta,z,t)$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$

1) Long cylinder $\left(\frac{\partial u}{\partial z}=0\right)$:

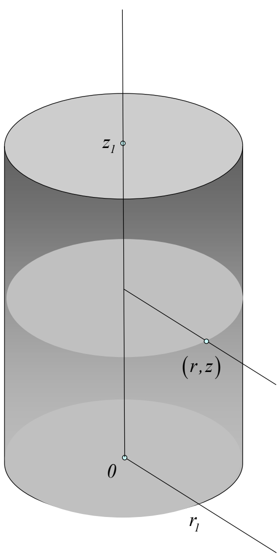
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$



$$u(r,\theta,t)$$

2) Short cylinder with angular symmetry $\left(\frac{\partial u}{\partial \theta}=0\right)$:

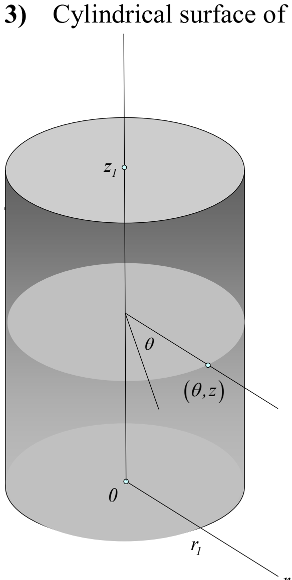
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$



$$u(r,z,t)$$

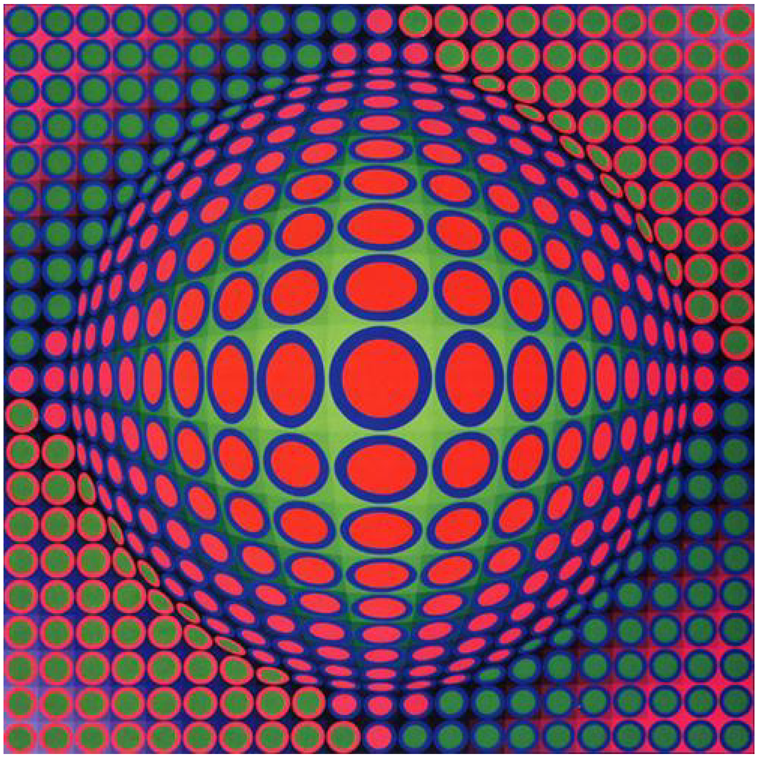
3) Cylindrical surface of fixed radius r_1 $\left(\frac{\partial u}{\partial r}=0\right)$:

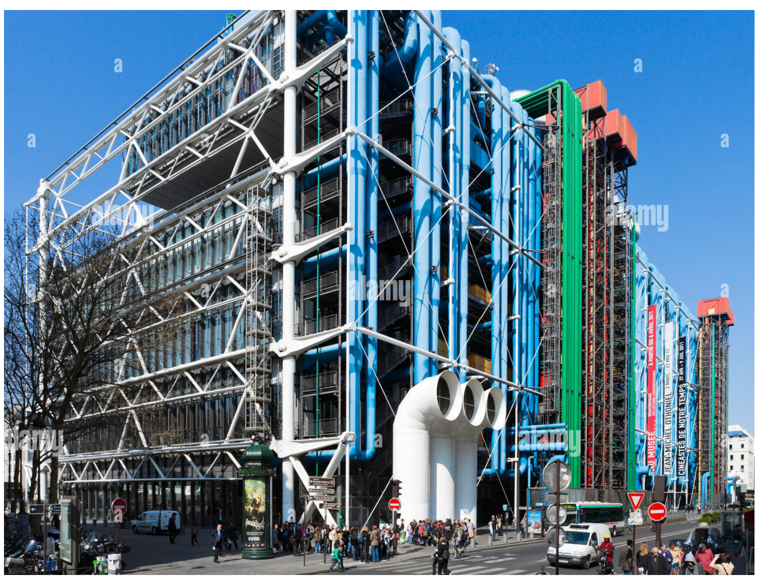
$$\frac{1}{r_1^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$



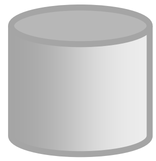
$$u(r,z,t)$$

Thin-wall cylindrical pipe

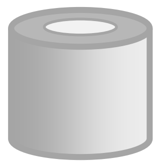




THE HEAT EQUATION Cylindrical Coordinates



$[u]_S = f$



$[u]_S = f$

$$\nabla^2 u + F(r,\theta,z) = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

$$[u]_S = f$$

$$[u]_{t=t_0} = u_0$$

solid cylinder

$(r,\theta,z) \in [0,r_1) \times [-\pi,\pi] \times (0,L) \subset \mathbb{R}^3$

hollow cylinder

$(r,\theta,z) \in (r_1,r_2) \times [-\pi,\pi] \times (0,L) \subset \mathbb{R}^3$

$t > 0$

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

$u_s(r,\theta,z)$ → *STEADY STATE PROBLEM - PE*

$U(r,\theta,z,t)$ → *TRANSIENT PROBLEM - HE*

STEADY STATE PROBLEM - PE

$$\nabla^2 u + F(r,\theta,z) = 0$$

$$[u_s]_S = f$$

TRANSIENT PROBLEM - HE

$$\nabla^2 U = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$[U]_S = 0$$

$$[U]_{t=t_0} = u_0 - u_s = U_0(r,\theta,z)$$

supplemental eigenvalue problems

$\Theta'' = \eta\Theta$ $\Theta(\theta+2\pi) = \Theta(\theta)$	$\overset{SLP}{\Rightarrow}$	$\Theta_n'' = -n^2\Theta_n$ $\quad n = 0,1,2,...$ $\eta_0 = 0 \quad \Theta_0(\theta) = 1$ $\eta_n = -n^2 \quad \Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta)$
$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = \mu R$ $R(0) < \infty$ $R(r_1) = 0$	$\overset{SLP}{\Rightarrow}$	$R_{nm}'' + \frac{1}{r}R_{nm}' - \frac{n^2}{r^2}R_{nm} = -\lambda_{nm}^2 R$ $\mu_{nm} = -\lambda_{nm}^2$ $r^2R_{nm}'' + rR_{nm}' + \left(r^2\lambda_{nm}^2 - n^2\right)R_{nm} = 0$ $R_{nm}(r) = J_n(\lambda_{nm}r) \quad n = 0,1,2,... \quad m = (0),1,2,...$
$Z'' = \gamma Z$ $[Z]_{z=0} = 0$ $[Z]_{z=K} = 0$	$\overset{SLP}{\Rightarrow}$	$Z_k'' = -\omega_k^2 Z_k$ $\gamma_k = -\omega_k^2$ $Z_k(z)$

SEPARATION OF VARIABLES

$$U(r,\theta,z,t) = \Phi(r,\theta,z)\,T(t)$$

HELMHOLTZ EQUATION

$$\frac{\nabla^2\Phi}{\Phi} = \beta$$

$$\Phi(r,\theta,z) = R(r)\Theta(\theta)Z(z)$$

$$\beta_{nmk} = -\left(\lambda_{nm}^2 + \omega_k^2\right)$$

$$R_{nm},\ \Theta_n,\ Z_k$$

$$\frac{1}{\alpha}\frac{T'}{T} = \beta$$

$$T = e^{-\alpha\left(\lambda_{nm}^2 + \omega_k^2\right)t}$$

TRANSIENT SOLUTION

$$U(r,\theta,z,t) = \Phi(r,\theta,z)T(t)$$

see p.654 for the case of solid cylinder, and p.658 for the case of hollow cylinder

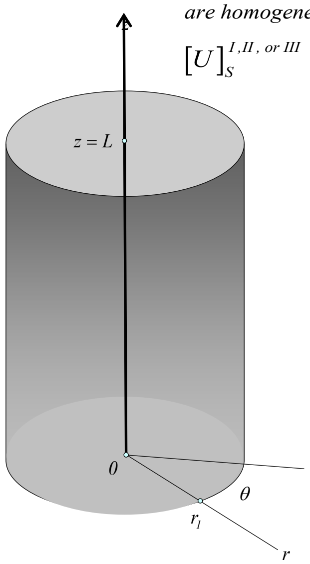
SOLUTION OF IBVP

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

VIII.3.3.5 BASIC IBVP FOR HEAT EQUATION IN FINITE SOLID CYLINDER 3-D

all boundary conditions are homogeneous

$$\left[U\right]_S^{I,II,\text{ or }III} = 0$$



initial condition:

$$U\left(r,\theta,z,t=0\right) = U_0\left(r,\theta,z\right)$$

Consider the ***Basic IBVP for the 3-D Heat Equation***:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$\left(r,\theta,z\right) \in \left[0,r_1\right) \times \left[0,2\pi\right] \times \left(0,L\right) \ , \quad t > 0$$

$$\left(r,\theta,z\right) \in \left[0,r_1\right) \times \left[-\pi,\pi\right] \times \left(0,L\right)$$

Separation of variables:

$$U\left(r,\theta,z,t\right) = \Phi\left(r,\theta,z\right) T\left(t\right)$$

Separated equation:

$$\frac{\dfrac{\partial^2 \Phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \Phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \Phi}{\partial \theta^2} + \dfrac{\partial^2 \Phi}{\partial z^2}}{\Phi} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

Separated equation yields the Helmholtz Equation:

Helmholtz Equation

$$\frac{\nabla^2 \Phi}{\Phi} = \beta \qquad\qquad \nabla^2 \Phi = \beta \Phi$$

The solution of the Helmholtz Equation subject to boundary conditions can be obtained by the eigenfunction expansion method.

Assume

$$\Phi\left(r,\theta,z\right) = R\left(r\right)\Theta\left(\theta\right)Z\left(z\right)$$

Substitute into the Helmholtz Equation

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = \beta$$

boundary conditions Separation of variables in the boundary conditions yields:

$r = r_1$ $\quad \left[\Phi\right]_{r=r_1} = \left[R\left(r_1\right)\right]\Theta\left(\theta\right)Z\left(z\right) = 0 \quad \Rightarrow \quad \left[R\right]_{r=r_1} = 0$

$z = 0$ $\quad \left[\Phi\right]_{z=0} = R\left(r\right)\Theta\left(\theta\right)\left[Z\left(0\right)\right] = 0 \quad \Rightarrow \quad \left[Z\right]_{z=0} = 0$

$z = L$ $\quad \left[\Phi\right]_{z=K} = R\left(r\right)\Theta\left(\theta\right)\left[Z\left(K\right)\right] = 0 \quad \Rightarrow \quad \left[Z\right]_{z=L} = 0$

From physical consideration, we need

bounded solution $\quad r = 0 \quad \left[\Phi\right]_{r=0} = \left[R\left(0\right)\right]\Theta\left(\theta\right)Z\left(z\right) < \infty \quad \Rightarrow \quad \left[R\right]_{r=0} < \infty$

2π *-periodic solution* $\quad \Phi\left(r,\theta+2\pi,z\right) = \Phi\left(r,\theta+2\pi,z\right) \quad \Rightarrow \quad \Theta\left(\theta+2\pi\right) = \Theta\left(\theta\right)$

Separate variables

$$\frac{\Theta''}{\Theta} = r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = \eta$$

1st equation

$$\Theta'' - \eta\Theta = 0$$

that is the SLP without boundary conditions, with condition of periodicity $\Theta(\theta + 2\pi) = \Theta(\theta)$ (see also the section VIII.3.6).

It can be considered in the interval $-\pi \le \theta < \pi$ with the condition $\Theta(-\pi) = \Theta(\pi)$

The case $\eta = 0$ yields the linear solution

$\Theta_0 = c_1\theta + c_2$

The only periodic linear function is a constant function, therefore,

$\Theta_0 = 1$

can be taken as an eigenfunction corresponding to $\eta_0 = 0$.

For positive eigenvalues, the separation constant has to be $\eta = -\mu^2$, then the general solution is

$\Theta_0 = c_1 \cos\mu\theta + c_2 \sin\mu\theta$

A function with a period $T = \frac{2\pi}{n}$ is also a 2π-periodic. Therefore, for 2π –periodic solution, the frequency μ can be any positive integer

$\mu = \frac{2\pi}{T} = \frac{2\pi}{2\pi}n = n$.

So, for $\eta_n = -n^2$, the corresponding eigenfunctions are

$\Theta_n = c_1 \cos n\theta + c_2 \sin n\theta$

That is consistent with the standard Fourier series over symmetric 2π-interval $(-\pi,\ \pi)$, which is based on the complete set of mutually orthogonal functions:

$\{1,\ \cos(n\theta),\ \sin(n\theta)\}$

Therefore, solution of the first equation can be summarized as:

$\eta_0 = 0$	$\Theta_0(\theta) = 1$	
$\eta_n = -n^2$	$\Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$	$n = 1, 2, \ldots$

2nd equation

$$r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = -n^2 \qquad n = 0, 1, 2, \ldots$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{n^2}{r^2} = \beta - \frac{Z''}{Z} = \mu$$ separate variables

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{n^2}{r^2} = \mu$$ consider equation for R

$$r^2 R'' + rR' - n^2 R - \mu r^2 R = 0$$

That is the Sturm-Liouville problem for Bessel Equation of order n

$$r^2 R'' + rR' + \left[(-\mu) r^2 - n^2\right] R = 0 \qquad (rR')' + \left[\frac{-n^2}{r} + (-\mu) r\right] R = 0$$

$$\mu = -\lambda^2$$

$$r^2 R'' + rR' + \left[\lambda^2 r^2 - n^2\right] R = 0$$ (see sectionVII.12, p.507)

$$[R]_{r=0} < \infty$$

$$[R]_{r=r_1} = 0$$

$$R_n(r) = c_{1,n} J_n(\lambda r) + c_{2,n} Y_n(\lambda r)$$ *general solution*

$$[R]_{r=0} < \infty \quad \Rightarrow \quad c_{2,n} = 0$$

$$R_n(r) = c_{1,n} J_n(\lambda r)$$

$$[R]_{r=r_1} = 0 \quad \Rightarrow \quad \left[J_n(\lambda r_1)\right] = 0 \quad \Rightarrow \quad \lambda_{mn} \qquad n = 0,1,2,\ldots \quad m = (0),1,2,\ldots$$

$$R_n(r) = c_{1,n} J_n(\lambda r)$$

$$\left[J_n(\lambda r_1)\right] = 0 \quad \Rightarrow \quad \lambda_{mn} \qquad n = 0,1,2,\ldots \quad m = (0),1,2,\ldots$$

n comes from the order of the Bessel functions $J_n(\lambda r)$.
Eigenvalues λ_{mn} should be found for each $n = 0,1,2,\ldots$

3rd equation

$$\beta - \frac{Z''}{Z} = -\lambda_{mn}^2$$

$$\frac{Z''}{Z} = \lambda_{mn}^2 + \beta = \gamma$$ combine constants to a single parameter γ

$$Z'' - \gamma Z = 0$$

$$[Z]_{z=0} = 0 \quad \overset{SLP}{\Longrightarrow} \quad \gamma = -\omega_k^2 \qquad k = (0),1,2,\ldots$$

$$[Z]_{z=K} = 0 \qquad Z_k(z)$$ eigenfunctions

Then the second part of the last equation becomes

$$\lambda_{mn}^2 + \beta = -\omega_k^2$$

and the constant of separation is

$$\beta_{mnk} = -\left(\lambda_{mn}^2 + \omega_k^2\right)$$

Solution for $T(t)$

$$T_{nmk}(t) = e^{-\alpha\left(\lambda_{mn}^2+\omega_k^2\right)t}$$

The solution of the Basic IBVP for the Heat Equation is:

$$U(r,\theta,z,t) = \sum_m \sum_k A_{0mk} J_0(\lambda_{0m} r) Z_k(z) e^{-\alpha\left(\lambda_{0m}^2+\omega_k^2\right)t} + \sum_{n=1} \sum_m \sum_k \left[A_{nmk} \cos(n\theta) + B_{nmk} \sin(n\theta) \right] J_n(\lambda_{nm} r) Z_k(z) e^{-\alpha\left(\lambda_{nm}^2+\omega_k^2\right)t}$$

The coefficients in this solution should be found to satisfy the initial condition:

$$U(r,\theta,z,0) = U_0(r,\theta,z)$$

$$U_0(r,\theta,z) = \sum_m \sum_k A_{0mk} J_0(\lambda_{0m} r) Z_k(z) + \sum_{n=1} \left[\sum_m \sum_k A_{nmk} Z_k(z) J_n(\lambda_{nm} r) \right] \cos(n\theta) + \sum_{n=1} \left[\sum_m \sum_k B_{nmk} Z_k(z) J_n(\lambda_{nm} r) \right] \sin(n\theta)$$

The following cascade of expansions over the eigenfunctions yields equations for calculation of coefficients. First, they are calculated as the coefficients of the standard Fourier series over interval $(-\pi,\pi)$:

$$\sum_m \left[\sum_k A_{0mk} Z_k(z) \right] J_0(\lambda_{0m} r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} U_0(r,\theta,z)\, d\theta$$

$$\sum_m \left[\sum_k A_{nmk} Z_k(z) \right] J_n(\lambda_{nm} r) = \frac{1}{\pi} \int_{-\pi}^{+\pi} U_0(r,\theta,z) \cos(n\theta)\, d\theta$$

$$\sum_m \left[\sum_k B_{nmk} Z_k(z) \right] J_n(\lambda_{nm} r) = \frac{1}{\pi} \int_{-\pi}^{+\pi} U_0(r,\theta,z) \sin(n\theta)\, d\theta$$

Second, as the coefficients of expansion into Fourier-Bessel series:

$$\sum_k A_{0mk} Z_k(z) = \frac{1}{2\pi \left\| J_0(\lambda_{0m}) \right\|^2} \int_{-\pi}^{+\pi} \int_0^{r_1} U_0(r,\theta,z)\, J_0(\lambda_{0m} r)\; r dr\, d\theta$$

$$\sum_k A_{nmk} Z_k(z) = \frac{1}{\pi \left\| J_n(\lambda_{nm}) \right\|^2} \int_{-\pi}^{+\pi} \int_0^{r_1} U_0(r,\theta,z)\, J_n(\lambda_{nm} r)\, \cos(n\theta)\, r dr\, d\theta$$

$$\sum_k B_{nmk} Z_k(z) = \frac{1}{\pi \left\| J_n(\lambda_n) \right\|^2} \int_{-\pi}^{+\pi} \int_0^{r_1} U_0(r,\theta,z)\, J_n(\lambda_{nm} r)\, \sin(n\theta)\, r dr\, d\theta$$

and, finally, by expansion into Generalized Fourier series, the coefficients for solution of the Basic IBVP are defined

$$A_{0mk} = \frac{1}{2\pi \left\| J_0(\lambda_{0m} r) \right\|^2 \left\| Z_k(z) \right\|^2} \int_0^L \int_{-\pi}^{+\pi} \int_0^{r_1} U_0(r,\theta,z)\, J_0(\lambda_{0m} r)\, Z_k(z)\, r dr\, d\theta\, dz$$

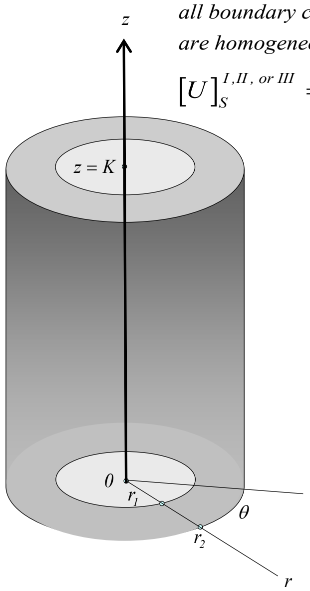
$$A_{nmk} = \frac{1}{\pi \left\| J_n(\lambda_{nm} r) \right\|^2 \left\| Z_k(z) \right\|^2} \int_0^L \int_{-\pi}^{+\pi} \int_0^{r_1} U_0(r,\theta,z)\, J_n(\lambda_{nm} r)\, Z_k(z)\, \cos(n\theta)\, r dr\, d\theta\, dz$$

$$B_{nmk} = \frac{1}{\pi \left\| J_n(\lambda_{nm} r) \right\|^2 \left\| Z_k(z) \right\|^2} \int_0^L \int_{-\pi}^{+\pi} \int_0^{r_1} U_0(r,\theta,z)\, J_n(\lambda_{nm} r)\, Z_k(z)\, \sin(n\theta)\, r dr\, d\theta\, dz$$

VIII.3.3.6 BASIC IBVP FOR HEAT EQUATION IN FINITE HOLLOW CYLINDER 3-D

all boundary conditions are homogeneous

$$\left[U\right]_S^{I,II,\ or\ III} = 0$$



initial condition:

$$U(r,\theta,z,t=0) = U_0(r,\theta,z)$$

Consider the Helmholtz equation which appears in the separation of variables in the ***Basic IBVP for the 3-D Heat Equation***:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$(r,\theta,z) \in (r_1,r_2)\times[0,2\pi]\times(0,L)\ ,\quad t>0$$

$$(r,\theta,z) \in (r_1,r_2)\times[-\pi,\pi]\times(0,L)$$

Separation of variables:

$$U(r,\theta,z,t) = \Phi(r,\theta,z)T(t)$$

Separated equation:

$$\frac{\dfrac{\partial^2 \Phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \Phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \Phi}{\partial \theta^2} + \dfrac{\partial^2 \Phi}{\partial z^2}}{\Phi} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

Separated equation yields the Helmholtz Equation:

Helmholtz Equation

$$\frac{\nabla^2 \Phi}{\Phi} = \beta \qquad \nabla^2 \Phi = \beta \Phi$$

The solution of the Helmholtz Equation subject to boundary conditions can be obtained by the eigenfunction expansion method.

Assume

$$\Phi(r,\theta,z) = R(r)\Theta(\theta)Z(z)$$

Substitute into the Helmholtz Equation

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = \beta$$

boundary conditions

Separation of variables in the boundary conditions yields:

$r = r_1$ $\quad [\Phi]_{r=r_1} = [R(r_1)]\Theta(\theta)Z(z) = 0 \quad \Rightarrow \quad [R]_{r=r_1} = 0$

$r = r_2$ $\quad [\Phi]_{r=r_2} = [R(r_2)]\Theta(\theta)Z(z) = 0 \quad \Rightarrow \quad [R]_{r=r_2} = 0$

$z = 0$ $\quad [\Phi]_{z=0} = R(r)\Theta(\theta)[Z(0)] = 0 \quad \Rightarrow \quad [Z]_{z=0} = 0$

$z = L$ $\quad [\Phi]_{z=K} = R(r)\Theta(\theta)[Z(K)] = 0 \quad \Rightarrow \quad [Z]_{z=K} = 0$

From physical consideration, we need

2π -periodic solution $\quad \Phi(r,\theta+2\pi,z) = \Phi(r,\theta+2\pi,z) \quad \Rightarrow \quad \Theta(\theta+2\pi) = \Theta(\theta)$

Separate variables

$$\frac{\Theta''}{\Theta} = r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = \eta$$

1st equation

$$\Theta'' - \eta\Theta = 0$$

that is the SLP without boundary conditions, with condition of periodicity $\Theta(\theta + 2\pi) = \Theta(\theta)$ (see also the section VIII.3.6).

It can be considered in the interval $-\pi \le \theta < \pi$ with the condition $\Theta(-\pi) = \Theta(\pi)$

The case $\eta = 0$ yields the linear solution

$\Theta_0 = c_1\theta + c_2$

The only periodic linear function is a constant function, therefore,

$\Theta_0 = 1$

can be taken as an eigenfunction corresponding to $\eta_0 = 0$.

For positive eigenvalues, the separation constant has to be $\eta = -\mu^2$, then the general solution is

$\Theta_0 = c_1 \cos\mu\theta + c_2 \sin\mu\theta$

A function with a period $T = \dfrac{2\pi}{n}$ is also a 2π -periodic. Therefore, for 2π –periodic solution, the frequency μ can be any positive integer

$\mu = \dfrac{2\pi}{T} = \dfrac{2\pi}{2\pi}n = n$.

So, for $\eta_n = -n^2$, the corresponding eigenfunctions are

$\Theta_n = c_1 \cos n\theta + c_2 \sin n\theta$

That is consistent with the standard Fourier series over symmetric 2π -interval $(-\pi, \pi)$, which is based on the complete set of mutually orthogonal functions:

$\{1,\ \cos(n\theta),\ \sin(n\theta)\}$

Therefore, solution of the first equation can be summarized as:

$$\eta_0 = 0 \qquad \Theta_0(\theta) = 1$$

$$\eta_n = -n^2 \qquad \Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta) \qquad n = 1,2,\dots$$

2nd equation

$$r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = -n^2 \qquad n = 0,1,2,\dots$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{n^2}{r^2} = \beta - \frac{Z''}{Z} = \mu$$ separate variables

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{n^2}{r^2} = \mu$$ consider equation for R

$$r^2R''+rR'-n^2R-\mu r^2R=0$$

That is the Sturm-Liouville problem for Bessel Equation of order n

$$r^2R''+rR'+\left[(-\mu)r^2-n^2\right]R=0 \qquad (rR')'+\left[\frac{-n^2}{r}+(-\mu)r\right]R=0$$

$$\mu=-\lambda^2$$

$$r^2R''+rR'+\left[\lambda^2r^2-n^2\right]R=0$$ (see sectionVII.12, p.515)

$$[R]_{r=r_1}=0$$

$$[R]_{r=r_2}=0$$

$$R_n(r)=c_{1,n}J_n(\lambda r)+c_{1,n}Y_n(\lambda r)$$ *general solution*

See solution of the Sturm-Liouville problem for the Bessel equation in the annular domain (Section VII.12, p.515):
For each $n=0,1,2,...$, there infinitely many eigenvalues λ_{nm} and corresponding eigenfunctions (orthogonal w.r.t weight $p(r)=r$):

$$R_{nm}(r)=c_{1,n}J_n(\lambda_{nm}r)+c_{1,n}Y_n(\lambda_{nm}r)$$

$$[characteristic\ eqn]=0 \quad\Rightarrow\quad \lambda_{mn} \qquad n=0,1,2,... \quad m=(0),1,2,...$$

n comes from the order of the Bessel functions $J_n(\lambda r)$ and $Y_n(\lambda r)$.
Eigenvalues λ_{mn} should be found for each $n=0,1,2,...$
The square of the norm of eigenfunctions is denoted as $\left\|R_{nm}(r)\right\|^2$

3rd equation

$$\beta-\frac{Z''}{Z}=-\lambda_{mn}^2$$

$$\frac{Z''}{Z}=\lambda_{mn}^2+\beta=\gamma$$ combine constants to a single parameter γ

$$Z''-\gamma Z=0$$

$$[Z]_{z=0}=0 \quad \overset{SLP}{\Longrightarrow} \quad \gamma=-\omega_k^2 \qquad k=(0),1,2,...$$

$$[Z]_{z=K}=0 \qquad Z_k(z)$$ eigenfunctions

Then the second part of the last equation becomes

$$\lambda_{mn}^2+\beta=-\omega_k^2$$

and the constant of separation is

$$\beta_{mnk}=-\left(\lambda_{mn}^2+\omega_k^2\right)$$

Solution for $T(t)$

$$T_{nmk}(t) = e^{-\alpha\left(\lambda_{mn}^2+\omega_k^2\right)t}$$

The solution of the Basic IBVP for the Heat Equation is:

$$R_{0m}(r) = J_0\left(\lambda_{0mr}\right)$$

$$U(r,\theta,z,t) = \sum_m\sum_k A_{0mk}R_{0m}(r)Z_k(z)e^{-\alpha\left(\lambda_{0m}^2+\omega_k^2\right)t} + \sum_{n=1}\sum_m\sum_k\left[A_{nmk}\cos(n\theta)+B_{nmk}\sin(n\theta)\right]R_{nm}(\lambda_{nm}r)Z_k(z)e^{-\alpha\left(\lambda_{nm}^2+\omega_k^2\right)t}$$

The coefficients in this solution should be found to satisfy the initial condition:

$$U(r,\theta,z,0) = U_0(r,\theta,z)$$

$$U_0(r,\theta,z) = \sum_m\sum_k A_{0mk}R_{0m}(\lambda_{0m}r)Z_k(z) + \sum_{n=1}\left[\sum_m\sum_k A_{nmk}Z_k(z)R_{nm}(\lambda_{nm}r)\right]\cos(n\theta) + \sum_{n=1}\left[\sum_m\sum_k B_{nmk}Z_k(z)R_{nm}(\lambda_{nm}r)\right]\sin(n\theta)$$

The following cascade of expansions over the eigenfunctions yields equations for calculation of coefficients. First, they are calculated as the coefficients of the standard Fourier series over interval $(-\pi,\pi)$:

$$\sum_m\left[\sum_k A_{0mk}Z_k(z)\right]R_{0m}(\lambda_{0m}r) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}U_0(r,\theta,z)\,d\theta$$

$$\sum_m\left[\sum_k A_{nmk}Z_k(z)\right]R_{nm}(\lambda_{nm}r) = \frac{1}{\pi}\int_{-\pi}^{+\pi}U_0(r,\theta,z)\cos(n\theta)\,d\theta$$

$$\sum_m\left[\sum_k B_{nmk}Z_k(z)\right]R_{nm}(\lambda_{nm}r) = \frac{1}{\pi}\int_{-\pi}^{+\pi}U_0(r,\theta,z)\sin(n\theta)\,d\theta$$

Second, as the coefficients of expansion into Fourier-Bessel series:

$$\sum_k A_{0mk}Z_k(z) = \frac{1}{2\pi\left\|R_{0m}(\lambda_{0m})\right\|^2}\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,R_{0m}(\lambda_{0m}r)\,r\,dr\,d\theta$$

$$\sum_k A_{nmk}Z_k(z) = \frac{1}{\pi\left\|R_{nm}(\lambda_{nm})\right\|^2}\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,R_{nm}(\lambda_{nm}r)\,\cos(n\theta)\,r\,dr\,d\theta$$

$$\sum_k B_{nmk}Z_k(z) = \frac{1}{\pi\left\|R_{nm}(\lambda_{nm})\right\|^2}\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,R_{nm}(\lambda_{nm}r)\,\sin(n\theta)\,r\,dr\,d\theta$$

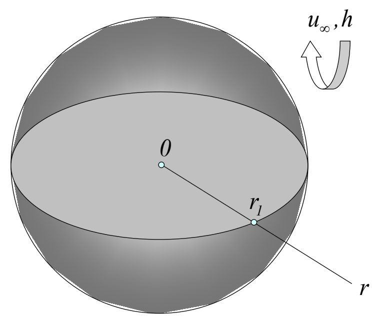
and, finally, by expansion into Generalized Fourier series, the coefficients for solution of the Basic IBVP are defined

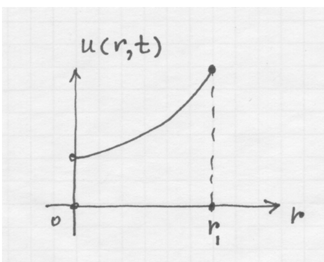
$$A_{0mk} = \frac{1}{2\pi\left\|R_{0m}(\lambda_{0m}r)\right\|^2\left\|Z_k(z)\right\|^2}\int_0^L\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,R_{0m}(\lambda_{0m}r)\,Z_k(z)\,r\,dr\,d\theta\,dz$$

$$A_{nmk} = \frac{1}{\pi\left\|R_{nm}(\lambda_{nm}r)\right\|^2\left\|Z_k(z)\right\|^2}\int_0^L\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,R_{nm}(\lambda_{nm}r)\,Z_k(z)\,\cos(n\theta)\,r\,dr\,d\theta\,dz$$

$$B_{nmk} = \frac{1}{\pi\left\|R_{nm}(\lambda_{nm}r)\right\|^2\left\|Z_k(z)\right\|^2}\int_0^L\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,R_{nm}(\lambda_{nm}r)\,Z_k(z)\,\sin(n\theta)\,r\,dr\,d\theta\,dz$$

VIII.3.4 SOLID SPHERE





Consider **heat conduction** in the sphere with angular symmetry:

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial \theta} = 0$$

he non-stationary temperature field $u(r,t)$ depends only on the radial variable *r* and time variable *t*.

Initial-boundary value problem:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{\dot{q}(r)}{k} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r \in [0, r_1) \qquad t > 0$$

initial condition:

$$u(r,0) = u_0(r) \qquad r \in [0, r_1]$$

boundary conditions:

$$k\left.\frac{\partial u}{\partial r}\right|_{r=r_1} = h\left(u_\infty - u\big|_{r=r_1}\right) \qquad t > 0$$

$$u\big|_{r=0} < \infty \qquad t > 0$$

where u_∞ is the ambient temperature and h is a convective coefficient. Rewrite the boundary condition in the standard form

$$\left[\frac{\partial u}{\partial r} + \frac{h}{k}u\right]_{r=r_1} = \frac{hu_\infty}{k}$$

1) Superposition of Steady State and Transient Solutions:

$$u(r,t) = u_s(r) + U(r,t)$$

2) Steady State Solution:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_s}{\partial r}\right) + \frac{\dot{q}(r)}{k} = 0$$

boundary conditions:

$$\left[\frac{\partial u_s}{\partial r} + \frac{h}{k}u_s\right]_{r=r_1} = \frac{hu_\infty}{k} \qquad t > 0$$

$$u_s\big|_{r=0} < \infty \qquad t > 0$$

General solution:

$$u_s(r) = -\int\left[\frac{1}{r^2}\int\frac{\dot{q}(r)}{k}r^2\,dr\right]dr - \frac{c_1}{r} + c_2$$

For the solid sphere (bounded solution at $r = 0$):

$$u_s(r) = -\int\left[\frac{1}{r^2}\int\frac{\dot{q}(r)}{k}r^2\,dr\right]dr + c_2$$

For uniform heat generation ($\dot{q} = const$):

$$u_s(r) = \frac{\dot{q}}{6k}\left(r_1^2 - r^2\right) + \frac{\dot{q}r_1}{3h} + u_\infty$$

3) Transient Solution:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad r\in\left[0,r_1\right) \quad t>0$$

initial condition:

$$U\left(r,0\right)=u_0\left(r\right)-u_s\left(r\right) \qquad r\in\left[0,r_1\right]$$

boundary conditions:

$$\left[\frac{\partial U}{\partial r}+\frac{h}{k}U\right]_{r=r_1}=0 \qquad t>0$$

$$U\Big|_{r=0}<\infty \qquad t>0$$

Reduction to Cartesian coordinates

Introduce the new dependent variable (reduction to Cartesian case):

$$V\left(r,t\right) \;=\; r\;U\left(r,t\right)$$

Write $\quad U \quad =\frac{1}{r}V$

Evaluate l.h.s. $\quad \frac{\partial}{\partial r}U \quad =-\frac{1}{r^2}V+\frac{1}{r}\frac{\partial}{\partial r}V$

$$r^2\frac{\partial}{\partial r}U=-V+r\frac{\partial}{\partial r}V$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}U\right)=-\frac{\partial}{\partial r}V+\frac{\partial}{\partial r}V+r\frac{\partial^2}{\partial r^2}V$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}U\right)=\frac{1}{r}\frac{\partial^2}{\partial r^2}V$$

Evaluate r.h.s. $\quad \frac{1}{\alpha}\frac{\partial U}{\partial t} \quad =\frac{1}{\alpha}\frac{1}{r}\frac{\partial V}{\partial t}$

Into equation: $\quad \frac{1}{r}\frac{\partial^2}{\partial r^2}V \quad =\frac{1}{\alpha}\frac{1}{r}\frac{\partial V}{\partial t}$

$$\frac{\partial^2 V}{\partial r^2}=\frac{1}{\alpha}\frac{\partial V}{\partial t}$$

which formally is the 1-d Heat Equation for r in the finite interval $r\in\left[0,r_1\right)$, which requires two boundary conditions.

The first condition at $r=0$ is obtained directly from the equation used for a change of variable:

$$V\Big|_{r=0}=rU\Big|_{r=0}=0 \qquad \textit{Dirichlet}$$

Consider the second boundary condition at $r=r_1$:

$$\left[\frac{\partial U}{\partial r}+\frac{h}{k}U\right]_{r=r_1}=0$$

$$\left[\frac{\partial}{\partial r}\frac{V}{r}+\frac{h}{k}\frac{V}{r}\right]_{r=r_1}=0$$

$$\left[\frac{1}{r}\frac{\partial V}{\partial r}-\frac{V}{r^2}+\frac{h}{k}\frac{V}{r}\right]_{r=r_1}=0$$

$$\left[\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r}\left(\frac{h}{k}-\frac{1}{r}\right)V\right]_{r=r_1}=0$$

$$\left[\frac{\partial V}{\partial r}+\left(\frac{h}{k}-\frac{1}{r_1}\right)V\right]_{r=r_1}=0$$

$$\left[\frac{\partial V}{\partial r}+HV\right]_{r=r_1}=0 \qquad H=\frac{h}{k}-\frac{1}{r_1} \qquad \textit{Robin}$$

Initial-boundary value problem:

$$\frac{\partial^2 V}{\partial r^2}=\frac{1}{\alpha}\frac{\partial V}{\partial t}$$

$$V(r,0)=rU(r,0)=r\left[u_0(r)-u_s(r)\right]$$

$$V\big|_{r=0}=rU\big|_{r=0}=0 \qquad D$$

$$\left[\frac{\partial V}{\partial r}+HV\right]_{r=r_1}=0 \qquad H=\frac{h}{k}-\frac{1}{r_1} \qquad R$$

4) Sturm-Liouville Problem corresponding to the case of Dirichlet-Robin boundary conditions (table SLP):

Eigenvalues λ_n are the positive roots of the equation:

$$\lambda\cos\lambda r_1+H\sin\lambda r_1=0$$

Eigenfunctions $X_n=\sin\lambda_n r$

$$\|X_n\|^2=\frac{r_1}{2}-\frac{\sin(2\lambda_n r_1)}{4\lambda_n}$$

Solution (see Example 2, p.?):

$$V(r,t)=\sum_{n=1}^{\infty}c_n\sin(\lambda_n r)e^{-\alpha\lambda_n^2 t}$$

$$c_n=\frac{\int_0^{r_1}r\left[u_0(r)-u_s(r)\right]\sin(\lambda_n r)\,dr}{\|X_n\|^2}$$

5) Solution:

$$u(r,t)=u_s(r)+\frac{1}{r}V(r,t)$$

6) Example (*turkey-3.mws*) **Roasting of a turkey**

The turkey ($W = 15\ lb$) is assumed to be a sphere with the uniform initial temperature $u_0 = 10\,^oC$. It is exposed to the convective environment at $u_\infty = 150\,^oC$ with the convective coefficient $h = 10\ \dfrac{W}{m^2 K}$. The turkey is considered to be done when its minimum temperature reaches $u_{done} = 75\,^oC$ (Standard for California). Thermophysical properties of turkey meat used for calculation are from the Table (Section VIII.1.15, p.580).

```
> restart;with(plots):
> alpha:=0.13e-6;rho:=1050;cp:=3540;k:=0.5;
```

$$\alpha := 0.13\ 10^{-6}$$

$$\rho := 1050$$

$$cp := 3540$$

$$k := 0.5$$

```
> h:=10;
```

$$h := 10$$

```
> qdot:=0.0;
```

$$qdot := 0.$$

```
> W:=15.0;VOL:=W/rho;r1:=fsolve(4/3*Pi*r^3=VOL,r=0..1);
```

$$W := 15.0$$

$$VOL := 0.01428571429$$

$$r1 := 0.1505235493$$

```
> H[2]:=h/k-1/r1;
```

$$H_2 := 13.35652126$$

Specified Temperatures:

```
> uinf:=150;ud:=75;
```

$$uinf := 150$$

$$ud := 75$$

```
> u0:=10;
```

$$u0 := 10$$

Steady State Solution:

```
> us:=qdot*(r1^2-r)/6/k+qdot*r1/3/h+uinf;
```

$$us := 150.$$

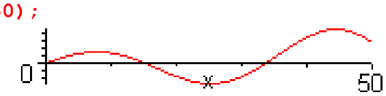
Transient Solution:

characteristic equation:

```
> w(x):=x*cos(x*r1)+H[2]*sin(x*r1);
```

$$w(x) := x\cos(0.1505235493\ x) + 13.35652126\ \sin(0.1505235493\ x)$$

```
> plot(w(x),x=0..50);
```



Eigenvalues:

```
> n:=1: for m from 1 to 20 do y:=fsolve(w(x)=0,x=10*m..10*(m+1)): if type(y,float)
then lambda[n]:=y: n:=n+1 fi od:
> for i to 4 do lambda[i] od;
```

$$15.22059059$$

$$33.80636804$$

$$53.79455908$$

$$74.23157321$$

```
> N:=n-1;
```

$$N := 10$$

Eigenfunctions:

```
> X[n]:=sin(lambda[n]*r);
```

$$X_n := \sin(\lambda_n\, r)$$

```
> NX2[n]:=r1/2-sin(2*lambda[n]*r1)/4/lambda[n];
```

$$NX2_n := 0.07526177465 - \frac{1}{4}\,\frac{\sin(0.3010470986\,\lambda_n)}{\lambda_n}$$

```
> c[n]:=int(r*(u0-us)*X[n],r=0..r1)/NX2[n];
```

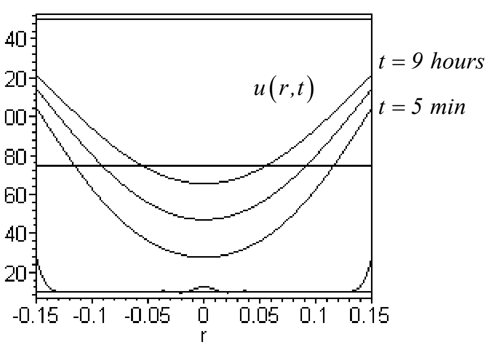
```
> u(r,t):=us+(1/r)*sum(c[n]*X[n]*exp(-alpha*lambda[n]^2*t),n=1..N);
```

$$\begin{aligned}
\mathrm{u}(r,t) := 150. + (&-14.93682091\,\sin(15.22059059\,r)\,\mathbf{e}^{(-0.00003011662913\,t)} \\
&+ 4.270369272\,\sin(33.80636804\,r)\,\mathbf{e}^{(-0.0001485731676\,t)} \\
&- 1.825389731\,\sin(53.79455908\,r)\,\mathbf{e}^{(-0.0003762010963\,t)} \\
&+ 0.9848499073\,\sin(74.23157321\,r)\,\mathbf{e}^{(-0.0007163424399\,t)} \\
&- 0.6104851654\,\sin(94.84937974\,r)\,\mathbf{e}^{(-0.001169532629\,t)} \\
&+ 0.4138908063\,\sin(115.5555678\,r)\,\mathbf{e}^{(-0.001735901602\,t)} \\
&- 0.2985392385\,\sin(136.3110699\,r)\,\mathbf{e}^{(-0.002415492011\,t)} \\
&+ 0.2252900588\,\sin(157.0967601\,r)\,\mathbf{e}^{(-0.003208320964\,t)} \\
&- 0.1759525722\,\sin(177.9022284\,r)\,\mathbf{e}^{(-0.004114396373\,t)} \\
&+ 0.1411724164\,\sin(198.7213411\,r)\,\mathbf{e}^{(-0.005133722283\,t)})/r
\end{aligned}$$

Solution:

Symmetric Extension:

```
> u2(r,t):=subs(r=-r,u(r,t)):
> t1:=0.5*60*10:t2:=3*60*60:t2:=5*60*60:t3:=7*60*60:t4:=9*60*60:
> z1:=subs(t=t1,u2(r,t)):z2:=subs(t=t2,u2(r,t)):z3:=subs(t=t3,u2(r,t)):z4:=subs(t=t4,u2(r,t)):
> plot({u0,us,ud,z1,z2,z3,z4},r=-r1..r1,color=black,axes=boxed);
```

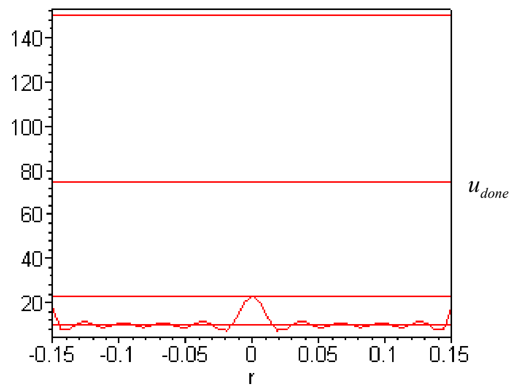


Temperature at the center:

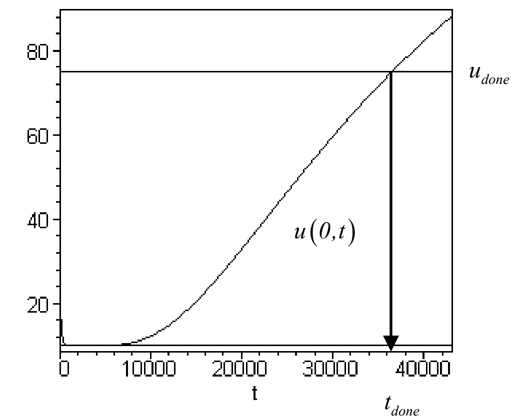
```
> uc:=limit(u2(r,t),r=0);
```

$$\begin{aligned}
uc := 150. &- 227.3472358\,\mathbf{e}^{(-0.00003011662913\,t)} + 35.39233832\,\mathbf{e}^{(-0.003208320964\,t)} \\
&+ 144.3656753\,\mathbf{e}^{(-0.0001485731676\,t)} - 31.30235469\,\mathbf{e}^{(-0.004114396373\,t)} \\
&- 98.19603573\,\mathbf{e}^{(-0.0003762010963\,t)} + 28.05397191\,\mathbf{e}^{(-0.005133722283\,t)} \\
&+ 73.10695799\,\mathbf{e}^{(-0.0007163424399\,t)} - 57.90413928\,\mathbf{e}^{(-0.001169532629\,t)} \\
&+ 47.82738713\,\mathbf{e}^{(-0.001735901602\,t)} - 40.69420301\,\mathbf{e}^{(-0.002415492011\,t)}
\end{aligned}$$

```
> animate({u2(r,t),uc,ud,u0,us},r=-r1..r1,t=0..11*3600,frames=200,axes=boxed);
```




```
> plot({uc,u0,ud},t=0..12*3600,axes=boxed,color=black);
```

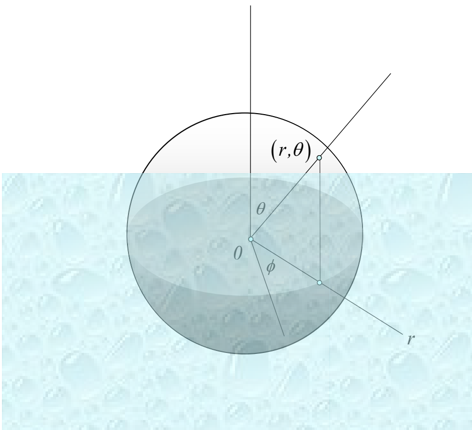


VIII.3.4.2 Heat Equation in Spherical Coordinates: $u(r,\phi,\theta,t)$

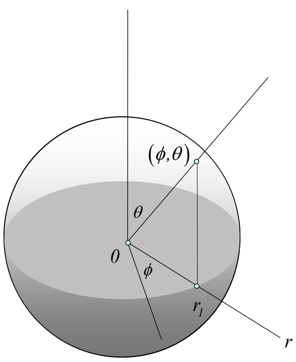
$$\frac{\partial^2 u}{\partial r^2}+2r\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$

1) Sphere with angular symmetry $\left(\frac{\partial u}{\partial\phi}=0\right)$: $\quad\frac{\partial^2 u}{\partial r^2}+2r\frac{\partial u}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$

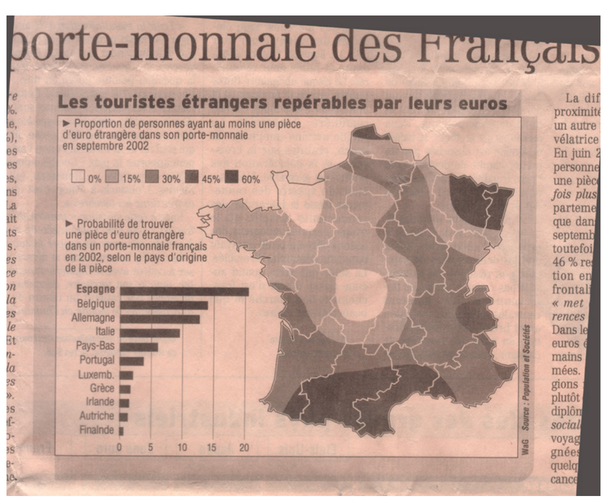
Example: Floating ball



2) Spherical surface of fixed radius r_1 $\left(\frac{\partial u}{\partial r}=0\right)$: $\quad\frac{1}{r_1^2\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{r_1^2\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$



Example: Diffusion of foreign mint coins in France

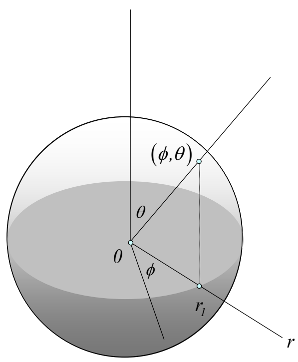


VIII.3.4.3 PDE in spherical coordinates

Consider a BVP generated by separation of variables in a PDE in spherical coordinates. We will only see what the Sturm-Liouville problems are in this case.

1. Laplace's Equation

Recall the general form of Laplace's Equation in spherical coordinates for the function $u(r,\phi,\theta),\ \ r\in D$:



$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0 \tag{1}$$

or with differential operators written in self-adjoint form:

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right]+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial\theta}+\frac{\partial^2 u}{\partial\theta^2}=0 \tag{2}$$

separation of variables

Assume

$$u(r,\varphi,\theta)=R(r)\,\Phi(\varphi)\,\Theta(\theta) \tag{3}$$

Substitute into equation (1)

$$R''\Phi\Theta+\frac{2}{r}R'\Phi\Theta+\frac{1}{r^2\sin^2\theta}R\Phi''\Theta+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}R\Phi\Theta'+\frac{1}{r^2}R\Phi\Theta''=0$$

Multiply the equation by $\dfrac{r^2}{R\Phi\Theta}$

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}+\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}=0$$

Consider the axisymmetric case ($\dfrac{\partial}{\partial\phi}=0$):

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}=0$$

Separate variables and set both sides of the equation equal to the same constant

$$r^2\frac{R''}{R}+2r\frac{R'}{R}=-\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}-\frac{\Theta''}{\Theta}=\mu$$

<u>It yields two equations:</u>

1) $\quad r^2\dfrac{R''}{R}+2r\dfrac{R'}{R}=\mu$

which can be rewritten in the form

$r^2R''+2rR'-\mu R=0$ (Euler-Cauchy equation)

or in the self-adjoint Sturm-Liouville form

$$-\frac{1}{1}\left(r^2R'\right)'=\left(-\mu\right)R \tag{4}$$

Solutions of this equation are sought in the form $R=r^m$

2) $\Theta'' + \dfrac{\cos\theta}{\sin\theta}\Theta' + \mu\Theta = 0$

Use change of independent variable $x = \cos\theta$, then

$$\Theta' = \frac{d\Theta}{d\theta} = \frac{d\Theta}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d\Theta}{dx}$$

$$\Theta'' = \frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) = \frac{d}{d\theta}\left(-\sin\theta\frac{d\Theta}{dx}\right) = -\cos\theta\frac{d\Theta}{dx} - \sin\theta\frac{d}{d\theta}\left(\frac{d\Theta}{dx}\right)$$

$$= -\cos\theta\frac{d\Theta}{dx} - \sin\theta\frac{d}{dx}\left(\frac{d\Theta}{dx}\right)\frac{dx}{d\theta} = -\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2}$$

Substitute into equation

$$-\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2} - \frac{\cos\theta}{\sin\theta}\sin\theta\frac{d\Theta}{dx} + \mu\Theta = 0$$

$$\sin^2\theta\frac{d^2\Theta}{dx^2} - 2\cos\theta\frac{d\Theta}{dx} + \mu\Theta = 0$$

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \mu\Theta = 0$$

or in self-adjoint Sturm-Liouville form:

$$-\frac{d}{dx}\left[\left(1-x^2\right)\frac{d\Theta}{dx}\right] = \mu\Theta \tag{5}$$

This equation is called Legendre's differential equation. It happens that its solution is bounded only if the separation constant is a non-negative integer of the form

$$\mu = n(n+1) \qquad n = 0,1,2,\ldots$$

Its solution consists of Legendre polynomials $P_n(x)$ (see Sec. 5.7).

2. Heat Equation

Consider the axisymmetric heat equation for $u(r,t)$, $r \in D$, $t > 0$ in spherical coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = a^2\frac{\partial u}{\partial t} \tag{6}$$

Separation of variables

$$u(r,t) = R(r)T(t)$$

Substitute into equation (6)

$$R''T + \frac{2}{r}R'T = a^2RT'$$

divide by RT and separate variables

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} = a^2\frac{T'}{T} = \mu$$

It yields two ordinary differential equations. Equation for R is

$$r^2R'' + 2rR' - \mu r^2R = 0 \tag{7}$$

which is a spherical Bessel equation of zero order (see equation (25) in Sec. 5.6 with $n = 0$, AAEM-II).

Eigenvalue problem:

$$LR \equiv \frac{1}{r^2}\left(r^2R'\right)' = \mu R$$

Its solutions are given by spherical Bessel functions

$$j_0(r) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}(r)}{\sqrt{r}}$$

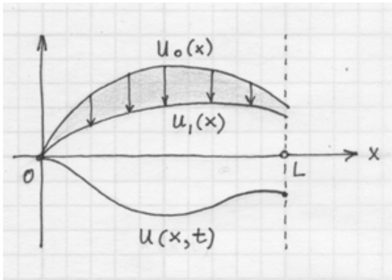
$$y_0(r) = \sqrt{\frac{\pi}{2}}\frac{Y_{1/2}(r)}{\sqrt{r}}$$

Why we cannot be completely satisfied with the method of separation of variables?

How about the time dependent boundary conditions, for example?

VIII.3.5 THE WAVE EQUATION

VIII.3.5.1 1-D Cartesian BASIC homogeneous equation with homogeneous boundary conditions



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t), \quad x \in (0,L), \ t > 0$$

Initial conditions: $u(x,0) = u_0(x)$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x)$$

Boundary conditions: $u(0,t) = 0$, $t > 0$ (I)

$$k_2 \frac{\partial u(L,t)}{\partial x} + h_2 u(x,t) = 0, \quad t > 0 \qquad \text{(III)}$$

Denote $H_2 = \frac{h_2}{k_2} > 0$

1. Separation of variables we assume that the function $u(x,t)$ can be represented as a product of two functions each of a single variable

$u(x,y) = X(x)\,T(t)$ substitute into equation

$a^2 X''(x)\,T(t) = X(x)\,T''(t)$ after separation of variables, one gets

$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = \mu$ with a separation constant μ

boundary conditions:

$\underline{x=0}$ $X(0)\,T(t) = 0 \Rightarrow X(0) = 0$

$\underline{x=L}$ $X'(L)\,T(t) + H_2 X(L)\,T(t) = 0 \Rightarrow X'(L) + H_2 X(L) = 0$

2. Sturm-Liouville problem $X'' - \mu X = 0$

This Sturm-Liouville problem has solution with $\mu_n = -\lambda_n^2$:

eigenvalues λ_n are positive roots of equation $\lambda \cos \lambda L + H_2 \sin \lambda_n L = 0$

eigenfunctions $X_n(x) = \sin \lambda_n x$

Then solutions of the second differential equation $T'' + \lambda_n^2 a^2 T = 0$ are

$T_n(t) = c_1 \cos \lambda_n at + c_2 \sin \lambda_n at$

Solution: $u_n(x,t) = X_n T_n = \sin(\lambda_n x)(c_1 \cos \lambda_n at + c_2 \sin \lambda_n at)$

Then solution of the wave equation is a superposition

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\lambda_n x)(b_n \cos \lambda_n at + d_n \sin \lambda_n at)$$

initial conditions: $\underline{t=0}$

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin(\lambda_n x)=u_0(x)$$

which is a generalized Fourier series expansion of the function $f(x)$ over the interval $(0,L)$ with coefficients

$$b_n=\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}=\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}}$$

The derivative with respect to *t* of the assumed solution is

$$\frac{\partial u(x,t)}{\partial t}=\sum_{n=1}^{\infty}\lambda_n a \sin\lambda_n x\left(-b_n \sin\lambda_n at+d_2\cos\lambda_n at\right)$$

Then the second initial condition yields

$\underline{t=0}$

$$\frac{\partial u(x,0)}{\partial t}=\sum_{n=1}^{\infty} d_n\lambda_n a\sin\lambda_n x=u_1(x)$$

It can be treated as a Fourier series with coefficients

$$d_n\lambda_n a=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}}$$

then

$$d_n=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\lambda_n a\left(\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}\right)}$$

Then the solution of the initial-boundary value problem is:

3. Solution

$$u(x,t)=\sum_{n=1}^{\infty}\left[b_n\cos(\lambda_n a t)+d_n\sin(\lambda_n a t)\right]\sin(\lambda_n x)$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{\sin(\lambda_n x)}{\left(\frac{L}{2}-\frac{\sin(2\lambda_n L)}{4\lambda_n}\right)}\left\{\left[\int_0^L u_0(x)\sin(\lambda_n x)dx\right]\cos\lambda_n at+\left[\frac{\int_0^L u_1(x)\sin(\lambda_n x)dx}{\lambda_n a}\right]\sin(\lambda_n at)\right\}$$

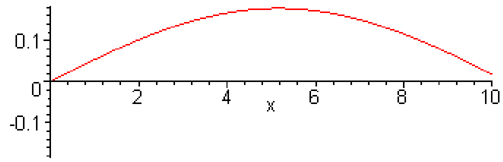
4. Normal modes of string vibration

The solution of the Wave Equation is obtained as a sum of terms

$$u_n(x,t) = X_n T_n = sin(\lambda_n x)(c_1 \cos \lambda_n at + c_2 \sin \lambda_n at)$$

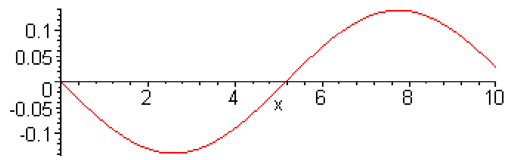
which we call the basic solutions. However, in the context of contributions to the vibration of a string, these functions are known as **normal modes**.
In our example, for $n = 1,2,3,4,...$, they have the following shapes
(see the Maple the file for animation):

```
> m1:=subs(n=1,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m1},x=0..L,t=0..9);
```



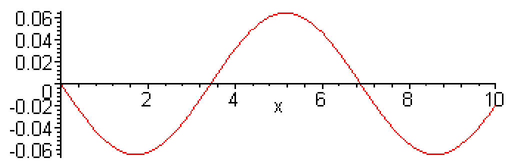
fundamental mode

```
> m2:=subs(n=2,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m2},x=0..L,t=0..9);
```



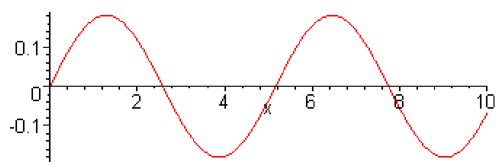
first overtone

```
> m3:=subs(n=3,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m3},x=0..L,t=0..9);
```



second overtone

```
> m4:=subs(n=4,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m4},x=0..L,t=0..9);
```



third overtone

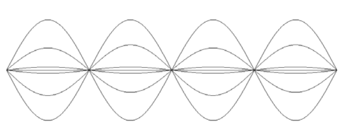
overtones

The first of these normal modes is called the **fundamental mode**, while the others are referred to as the **first overtone**, the **second overtone**, and so on. The **frequency** of oscillation of the normal mode increases with its number and is determined by the corresponding eigenvalue λ_n and coefficient a, which has a physical meaning related to the speed of wave propagation (speed of sound). Fixed points exist in the vibration of overtones.

The entire motion of the string is a superposition of vibration of all overtones with a different amplitude. The participation of different modes in the string's vibration is determined by the initial conditions.

If representing the initial shape of the string at rest requires the use of different modes, then all of them will be present in the undamped vibration of the string.

standing waves



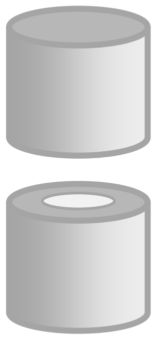
However, if the initial shape of the string exactly matches one of the overtones, then only that mode will be present in the string's vibration.

This phenomenon is known as **standing waves**. Standing waves do not propagate; they only oscillate, maintaining the same shape.

ALLEGORY OF GEOMETRY
Museum of Louvre, Paris

Rene Descartes University, Paris

THE WAVE EQUATION Cylindrical Coordinates



$[u]_S = f$

$[u]_S = f$

$$\nabla^2 u + F(r,\theta,z) = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$[u]_S = f \qquad t > 0$ *boundary conditions*

$[u]_{t=t_0} = u_0$

initial conditions

$\left[\frac{\partial}{\partial t}u\right]_{t=t_0} = u_0$

solid cylinder

$(r,\theta,z) \in [0,r_1)\times[-\pi,\pi]\times(0,L) \subset \mathbb{R}^3$

hollow cylinder

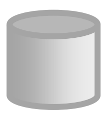
$(r,\theta,z) \in (r_1,r_2)\times[-\pi,\pi]\times(0,L) \subset \mathbb{R}^3$

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

STEADY STATE PROBLEM - PE

$$\nabla^2 u_s + F(r,\theta,z) = 0$$

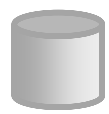
$[u_s]_S = f$



TRANSIENT PROBLEM - HE

$$\nabla^2 U = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}$$

$[U]_S = 0$



initial conditions

$[U]_{t=t_0} = u_0 - u_s$

$\left[\frac{\partial}{\partial t}U\right]_{t=t_0} = u_1$

supplemental eigenvalue problems

$\Theta'' = \eta\Theta$, $\Theta(\theta+2\pi) = \Theta(\theta)$ $\overset{SLP}{\Rightarrow}$ $\Theta_n'' = -n^2\Theta_n \quad n = 0,1,2,\ldots$

$\eta_0 = 0 \quad \Theta_0(\theta) = 1$

$\eta_n = -n^2 \quad \Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta)$

$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = \mu R$, $R(0) < \infty$, $R(r_1) = 0$ $\overset{SLP}{\Rightarrow}$ $R_{nm}'' + \frac{1}{r}R_{nm}' - \frac{n^2}{r^2}R_{nm} = -\lambda_{nm}^2 R$

$\mu_{nm} = -\lambda_{nm}^2$

$r^2R_{nm}'' + rR_{nm}' + (r^2\lambda_{nm}^2 - n^2)R_{nm} = 0$

$R_{nm}(r) = J_n(\lambda_{nm}r) \quad n = 0,1,2,\ldots \quad m = (0),1,2,\ldots$

$Z'' = \gamma Z$, $[Z]_{z=0} = 0$, $[Z]_{z=K} = 0$ $\overset{SLP}{\Rightarrow}$ $Z_k'' = -\omega_k^2 Z_k$

$\gamma_k = -\omega_k^2$

$Z_k(z)$

SEPARATION OF VARIABLES

$$U(r,\theta,z,t) = \Phi(r,\theta,z)T(t) \qquad \frac{\nabla^2\Phi}{\Phi} = \frac{1}{\alpha}\frac{T''}{T} = \beta$$

HELMHOLTZ EQUATION

$\nabla^2\Phi = \beta\Phi$

$\Phi(r,\theta,z) = R(r)\Theta(\theta)Z(z)$

$R_{nm},\ \Theta_n,\ Z_k$

$\beta_{nmk} = -(\lambda_{nm}^2 + \omega_k^2)$

$\frac{1}{v^2}\frac{T''}{T} = \beta$

$T_{nmk}(t) = c_1\cos(v^2\beta_{nmk}t) + c_2\sin(v^2\beta_{nmk}t)$

STEADY STATE SOLUTION

TRANSIENT SOLUTION

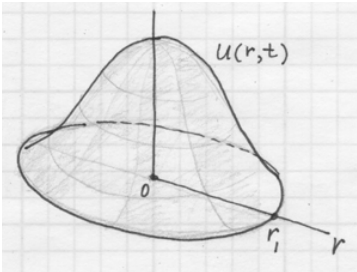
$$U(r,\theta,z,t) = \Phi(r,\theta,z)T(t)$$

see p.654 for the case of solid cylinder, and p.658 for the case of hollow cylinder

SOLUTION OF IBVP

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

VIII.3.5.2 1-D polar coordinates — *Wave Equation in polar coordinates with angular symmetry*



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$u(r,t),\ \ 0\le r<r_1\ ,\ t>0$

Initial conditions: $u(r,0)=u_0(r)$

$$\frac{\partial u(r,0)}{\partial t}=u_1(r)$$

Boundary condition: $u(r_1,t)=0 \qquad t>0$ (Dirichlet)

$u(0,t)<\infty$

1. Separation of variables

Assume

$u(r,y)=R(r)\,T(t)$

Substitute into the equation

$$R''T+\frac{1}{r}RT'=\frac{1}{v^2}RT''$$

After separation of variables (division by *RT*), we receive

$$\frac{R''}{R}+\frac{1}{r}\frac{R}{R}=\frac{1}{v^2}\frac{T''}{T}=\mu$$ with a separation constant μ .

boundary condition

$r=r_1$ $\qquad u(r_1,t)=R(r_1)\,T(t)=0 \ \Rightarrow\ R(r_1)=0$

2. Solution of Sturm-Liouville problem

Consider the equation for $R(r)$ for which we have a homogeneous boundary condition:

$$R''+\frac{1}{r}R'-\mu R=0 \qquad R(r_1)=0$$

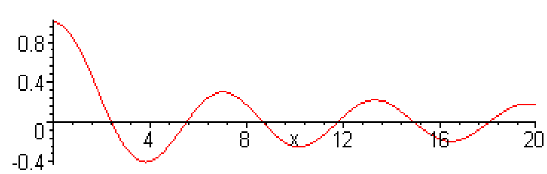
That is the Eigenvalue problem for the Bessel equation of 0^{th} order, solution for which is presented in VII.2, p.509.

Separation constant $\mu_n=-\lambda_n^2$

Eigenfunctions $R_n(r)=J_0(\lambda_n r)$

Eigenvalues are the roots of $J_0(\lambda_n r_1)=0$

The figure shows the graph of the function $w(\lambda)=J_0(\lambda r_1)$ with $r_1=1$



The weight function $p(r)=r$

Orthogonality $\int_0^{r_1} J_0(\lambda_n r)J_0(\lambda_m r)\,r\,dr=0$ for $n\ne m$

Norm $\|R_n(r)\|_p^2=r_1^2J_1^2(\lambda_n r_1)/2$

solution for T

The result of a negative separation constant $\mu = -\lambda^2$ agrees with a physical sense of solution for $T(t)$. Equation for T

$$\frac{1}{a^2}\frac{T''}{T} = \mu = -\lambda_n^2$$

Then solutions $T_n(t)$ with determined eigenvalues are

$$T_n(t) = a_n \cos(v\lambda_n t) + b_n \sin(v\lambda_n t)$$

3. Solution

$$u(r,t) = \sum_{n=1} \left[a_n \cos(v\lambda_n t) + b_n \sin(v\lambda_n t) \right] J_0(\lambda_n r)$$

We will choose the values of coefficients in such a way that initial conditions are satisfied.

4. Initial conditions

Consider the first initial condition

$$u(r,0) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) = u_0(r)$$

then coefficients for the generalized Fourier series are defined as

$$a_n = \frac{\int_0^{r_1} u_0(r) J_0(\lambda_n r)\, r\, dr}{\int_0^{r_1} J_0^2(\lambda_n r)\, r\, dr}$$

$$a_n = \frac{\int_0^{r_1} r u_0(r) J_0(\lambda_n r)\, dr}{r_1^2 J_1^2(\lambda_n r_1)/2}$$

The second condition for the derivative with respect to time

$$\frac{\partial u(r,t)}{\partial t} = \sum_{n=1}^{\infty} J_0(\lambda_n r)\left(-a_n \lambda_n v \sin \lambda_n vt + b_n \lambda_n v \cos \lambda_n vt\right)$$

becomes

$$\frac{\partial u(r,0)}{\partial t} = \sum_{n=1}^{\infty} b_n \lambda_n v\, J_0(\lambda_n r) = u_1(r)$$

Then coefficients in this generalized Fourier expansion are

$$b_n \lambda_n v = \frac{\int_0^{r_1} u_1(r) J_0(\lambda_n r)\, r\, dr}{\int_0^{r_1} r J_0^2(\lambda_n r)\, r\, dr} \quad \Rightarrow \quad b_n = \frac{\int_0^{r_1} r u_1(r) J_0(\lambda_n r)\, dr}{v\lambda_n r_1^2 J_1^2(\lambda_n r_1)/2}$$

Then solution of the initial-boundary value problem is

5. Solution

$$u(r,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\lambda_n vt) + b_n \sin(\lambda_n vt) \right] J_0(\lambda_n r)$$

$$u(r,t) = \frac{r_1^2}{2} \sum_{n=1}^{\infty} \left\{ \left[\int_0^{r_1} u_0(r) J_0(\lambda_n r)\, r\, dr \right] \cos(\lambda_n vt) + \left[\frac{1}{v\lambda_n} \int_0^{r_1} u_1(r) J_0(\lambda_n r)\, r\, dr \right] \sin(\lambda_n vt) \right\} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_1)}$$

VIII.3.6 SINGULAR STURM_LIOUVILLE PROBLEM – CIRCULAR STRING

We studied a regular Sturm-Liouville Problem in which the ordinary differential equation is set in the finite interval and both boundary conditions do not vanish. In a singular Sturm-Liouville problem not all of these conditions hold. Usually, the interval is not finite, and one or both boundary conditions are missing. Instead of boundary conditions, when the solution may not exist at the boundaries, the eigenfunctions should satisfy some limiting conditions. One of such requirements can be the following:
Let y_1 and y_2 be eigenfunctions corresponding to two distinct eigenvalues λ_1 and λ_2, correspondingly. Then they have to satisfy the following condition:

$$\lim_{x\to x_2^-} p(x)\left[y_1(x)y_2'(x)-y_2(x)y_1'(x)\right] = \lim_{x\to x_1^+} p(x)\left[y_1(x)y_2'(x)-y_2(x)y_1'(x)\right]$$

In the other cases the absence of boundary conditions is because of the periodical or cycled domain, when we demand that the solution should be continuous and smooth
$y(x_1)=y(x_2)$ and $y'(x_1)=y'(x_2)$
In this case, it is still possible to have the orthogonal set of solutions $\{y_n(x)\}$ on $[x_1,x_2]$.
We will not study the formal approach to solution of such problems, but rather discuss the practical examples of its application.
Here, we consider an interesting example of a singular SLP in a cycled domain with no boundary conditions. Physical demonstration of this example can be seen on the ceiling of the hall of the Eyring Science Building.

Example 1 Consider vibration of a thin closed ring string of radius r described in polar coordinates by deflection over the plane $z=0$
$u(\theta,t),\ \theta\in[0,2\pi]$, $t>0$
The Wave Equation reduces to

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=a^2\frac{\partial^2 u}{\partial t^2} \qquad r=const$$

with initial conditions

$$u(\theta,0)=u_0(\theta)$$

$$\frac{\partial u}{\partial t}(\theta,0)=u_1(\theta)$$

There are no boundaries for a closed string, but rather a physical condition for a continuous and smooth string:

$$u(0,t)=u(2\pi,t) \quad t>0$$

$$\frac{\partial u}{\partial\theta}(0,t)=\frac{\partial u}{\partial\theta}(2\pi,t) \quad t>0$$

separation of variables

Assume $u(\theta,t)=\Theta(\theta)\,T(t)$

Substitute into equation $\frac{1}{r^2}\Theta''T=a^2\,\Theta\,T''$

Separate variables $\frac{\Theta''}{\Theta}=a^2r^2\frac{T''}{T}=\mu$ $\qquad\mu$ is a separation constant

Consider

$$\frac{\Theta''}{\Theta}=\mu$$

$$\Theta''-\mu\Theta=0$$

We already have experience with solution of this special equation for regular Sturm-Liouville Problems and know that in all cases except the case of both boundary conditions of Neumann type, only a negative separation constant ,

$\mu = -\lambda^2$, generates eigenvalues and eigenfunctions. General solution in this case is

$\Theta(\theta) = c_1 \cos \lambda\theta + c_2 \sin \lambda\theta$

This solution suits our problem because it is periodic. The values of λ which satisfy periodicity on the interval $\theta \in [0,2\pi]$, are

$$\lambda_n = \frac{2n\pi}{2\pi} = n$$

Therefore, solutions are

$\Theta_n(\theta) = c_{1,n} \cos n\theta + c_{2,n} \sin n\theta$

Obviously, that for all $n = 0,1,2,...$ 2π is a period for this solution and for its derivative

$\Theta_n'(\theta) = -c_{1,n} n \sin n\theta + c_{2,n} n \cos n\theta$

With these values of the separation constant, $\mu_n = -\lambda_n^2 = -n^2$, $n = 0,1,2,...$ consider the equation for $T(t)$:

$$a^2 r^2 \frac{T''}{T} = -n^2$$

$$T'' + \frac{n^2}{a^2 r^2} T = 0$$

which also has a periodic (in t) general solution

$$T_n(t) = c_{3,n} \cos \frac{n}{ar} t + c_{4,n} \sin \frac{n}{ar} t$$

Then periodic solution of the wave equation can be constructed in the form of an infinite series:

$$\begin{aligned} u(\theta,t) = \Theta(\theta)T(t) &= \sum_{n=0}^{\infty} \Theta_n(\theta) T_n(t) \\ &= \sum_{n=0}^{\infty} \left(c_{1,n} \cos n\theta + c_{2,n} \sin n\theta\right)\left(c_{3,n} \cos \frac{n}{ar} t + c_{4,n} \sin \frac{n}{ar} t\right) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(c_{1,n} c_{3,n} \cos n\theta \cos \frac{n}{ar} t + c_{1,n} c_{4,n} \cos n\theta \sin \frac{n}{ar} t + c_{2,n} c_{3,n} \sin n\theta \cos \frac{n}{ar} t + c_{2,n} c_{4,n} \sin n\theta \sin \frac{n}{ar} t \right)$$

$$= \sum_{n=0}^{\infty} \left(b_{1,n} \cos n\theta \cos \frac{n}{ar} t + b_{2,n} \cos n\theta \sin \frac{n}{ar} t + b_{3,n} \sin n\theta \cos \frac{n}{ar} t + b_{4,n} \sin n\theta \sin \frac{n}{ar} t \right)$$

where coefficients b are new arbitrary constants which can be chosen in such a way that this solution will satisfy the initial conditions.

Consider the first initial condition:

$$\begin{aligned} t = 0 \qquad u(\theta,0) = u_0(\theta) &= \sum_{n=0}^{\infty} \left(b_{1,n} \cos n\theta + b_{3,n} \sin n\theta\right) \\ &= b_{1,0} + \sum_{n=1}^{\infty} \left(b_{1,n} \cos n\theta + b_{3,n} \sin n\theta\right) \end{aligned}$$

which can be treated as a standard Fourier series expansion of the function $u_0(\theta)$ on the interval $[0,2\pi]$. Therefore, the coefficients of this expansion are

$$b_{1,0} = \frac{1}{2\pi} \int_0^{2\pi} u_0(\theta) d\theta$$

$$b_{1,n} = \frac{1}{\pi} \int_0^{2\pi} u_0(\theta) \cos n\theta d\theta$$

$$b_{3,n} = \frac{1}{\pi} \int_0^{2\pi} u_0(\theta) \sin n\theta d\theta$$

For the second initial condition, differentiate the solution first with respect to t

$$\frac{\partial u}{\partial t}(\theta,t)=\sum_{n=0}^{\infty}\left(-b_{1,n}\frac{n}{ar}\cos n\theta\sin\frac{n}{ar}t+b_{2,n}\frac{n}{ar}\cos n\theta\cos\frac{n}{ar}t-b_{3,n}\frac{n}{ar}\sin n\theta\sin\frac{n}{ar}t+b_{4,n}\frac{n}{ar}\sin n\theta\cos\frac{n}{ar}t\right)$$

then apply the second initial condition

$$\frac{\partial u}{\partial t}(\theta,0)=u_1(\theta)\qquad=\sum_{n=0}^{\infty}\left(b_{2,n}\frac{n}{ar}\cos n\theta+b_{4,n}\frac{n}{ar}\sin n\theta\right)$$

$$=b_{2,n}\cdot 0+\sum_{n=1}^{\infty}\left(b_{2,n}\frac{n}{ar}\cos n\theta+b_{4,n}\frac{n}{ar}\sin n\theta\right)$$

Where the coefficients are determined as

$$b_{2,0}\cdot 0=\frac{1}{2\pi}\int_0^{2\pi}u_1(\theta)d\theta$$

$$b_{2,n}=\frac{ar}{n}\frac{1}{\pi}\int_0^{2\pi}u_1(\theta)\cos n\theta d\theta$$

$$b_{4,n}=\frac{ar}{n}\frac{1}{\pi}\int_0^{2\pi}u_1(\theta)\sin n\theta d\theta$$

Coefficient $b_{2,0}$ can be any constant, it will not influence the initial speed of the string, but not to influence the initial shape of the string it has to be chosen equal to zero (otherwise, initially the string will shifted by $b_{2,0}$ and will not be centered over the plane $z=0$):

$b_{2,0}=0$

Therefore, solution of the problem is given by the infinite series

$$u(\theta,t)=b_{1,0}+\sum_{n=1}^{\infty}\left(b_{1,n}\cos n\theta\cos\frac{n}{ar}t+b_{2,n}\cos n\theta\sin\frac{n}{ar}t+b_{3,n}\sin n\theta\cos\frac{n}{ar}t+b_{4,n}\sin n\theta\sin\frac{n}{ar}t\right)$$

where coefficients are determined according to abovementioned formulas.

Consider particular cases (Maple examples):

1) isolated wave

2) standing waves

VIII.3.7 REVIEW QUESTIONS, EXAMPLES AND EXERCISES

REVIEW QUESTIONS

1. What is the main assumption in the method of ***separation of variables***?
2. What is a separation constant?
3. How does the Sturm-Liouville problem manifest in the method of separation of variables?
4. What is the form of the solution of the initial value problem (IVBP) in the method of separation of variables?
5. How many terms are required in the truncated infinite series for an accurate representation of the solution?
6. Can you provide an example when the solution of the IBVP is described just by a single-term trigonometric function? How does this occur?

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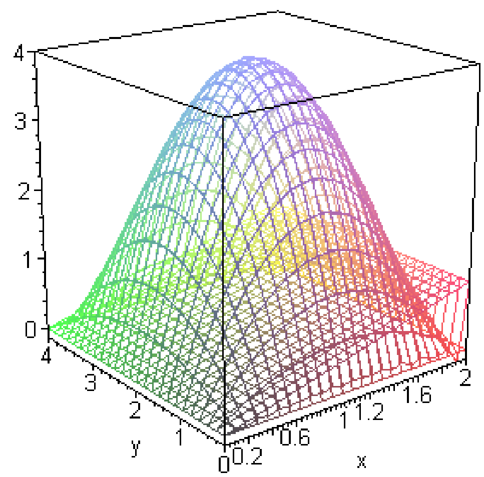
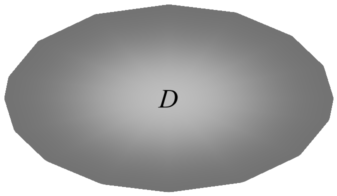


EXAMPLES AND EXERCISES 1. Let $D \subset \mathbb{R}^3$ be a ***domain*** (open connected set), and let $S = \overline{D} \setminus D$ be the ***boundary*** of D (recall Section VIII.1.11, p.568).

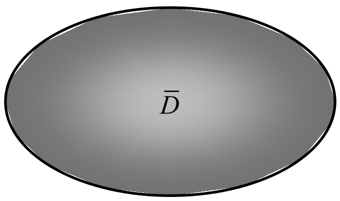
domain D is an open set



Show that if $\mathbf{r} \in S$ is a point of the boundary of D, then any open ball $B(\mathbf{r},R)$ with a radius $R > 0$ includes points both from D and $\mathbb{R}^3 \setminus D$, i.e. intersection of any $B(\mathbf{r},R)$ with the domain and with the surroundings is not empty:

$B(\mathbf{r},R) \cap D \neq \varnothing$ and $B(\mathbf{r},R) \cap \left(\mathbb{R}^3 \setminus \overline{D}\right) \neq \varnothing$.

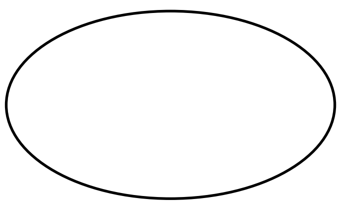
$\overline{D}$ is a closure of the domain D
$\overline{D}$ is a closed set



Remark: this property is usually used as the more general definition of the boundary:
If $A \subset \mathbb{R}^n$ is an arbitrary subset of $\mathbb{R}^n$ (***not necessarily domain***),
then $x \in \mathbb{R}^n$ is called *a* ***boundary point*** of A if for any radius $R > 0$:
$B(x,R) \cap A \neq \varnothing$ and $B(x,R) \cap \left(\mathbb{R}^n \setminus A\right) \neq \varnothing$.

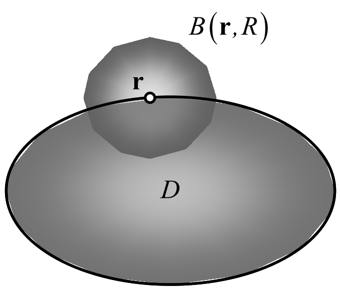
Then the set $\partial A = \left\{ x \in \mathbb{R}^n \,\middle|\, x \text{ is boundary point of } A \right\}$ is called the ***boundary*** of A in $\mathbb{R}^n$.

boundary $S = \overline{D} \setminus D$



If $S = \overline{D} \setminus D$ is the ***boundary*** of domain D , the S is the boundary of S in general sense too.

Examples of the boundary in general sense:



a) $\partial (0,1] = \{0,1\}$
b) $\partial \{a\} = \{a\}$ (the boundary of an insulated point is the point itself)
c) $\partial\, \mathbb{Q} = \mathbb{R}$
d) $\partial\, \mathbb{Z} = \mathbb{Z}$
e) $\partial\, \varnothing = \varnothing$
f) $\partial\, \mathbb{R}^n = \varnothing$
g) $\partial \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0\}$

2. a) Solve the Dirichlet problem for the Heat Equation:

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in [0,L],\ t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $u(0,t) = 0$, $t > 0$ (Dirichlet)
$u(L,t) = 0,$ $t > 0$ (Dirichlet)

b) Sketch the graph of solution for $L=3$ and $a=0.1$ and initial conditions:

i) $u_0(x) = 1$
ii) $u_0(x) = x(L-x)$
iii) $u_0(x) = \sin 2x$

3. The Superposition Principle for Non-Homogeneous Heat Equation with Non-Homogeneous Boundary Condition.:

Heat Equation:

$$\frac{\partial^2 u}{\partial x^2} + F(x) = a^2 \frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in (0,L),\ t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $u(0,t) = g_0,\ t > 0$ (Dirichlet)

$$\frac{\partial u(L,t)}{\partial x} = g_L, \quad t > 0 \qquad \text{(Neumann)}$$

Supplemental problems **a) steady state solution:**

$$\frac{\partial^2 u_s}{\partial x^2} + F(x) = 0 \qquad u_s(x): \quad x \in (0,L)$$

$$u_s(0) = g_0$$

$$\frac{\partial u_s}{\partial x}(L) = f_L$$

b) transient solution:

$$\frac{\partial^2 U}{\partial x^2} = a^2 \frac{\partial U}{\partial t} \qquad U(x,t): \ x \in (0,L),\ t > 0$$

$$U(x,0) = u_0(x) - u_s(x)$$

$$U(0,t) = 0 \qquad t > 0$$

$$U(L,t) = 0 \qquad t > 0$$

First supplemental problem is a BVP for ODE.
The second supplemental problem is an IBVP problem for the homogeneous Heat Equation with homogeneous boundary conditions.

Show that $u(x,t) = U(x,t) + u_s(x)$ is a solution of the non-homogeneous IBVP.

Solve the problem with

$F(x) = 5,\ g_0 = 1,\ g_L = 3$ and $u_0(x) = x(4 - x)$.

Sketch the graph of the solution.

4. a) Solve the IBVP:

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} + F(x) \qquad u(x,t), \quad x \in (0,L), \ t > 0$$

initial condition: $u(x,0) = u_0(x)$

boundary conditions: $u(0,t) = f_1 \qquad t > 0 \qquad$ (I)

$$k\frac{\partial u(L,t)}{dx} + hu(L,t) = f_2 \qquad t > 0 \qquad \text{(III)}$$

b) Sketch the graph of solution with

$L = 4$, $a = 0.5$, $k = 2.0$,

$u_0(x) = x^2 - \frac{L}{2}x + 5$, $f_1 = 10$, $f_2 = 1$, $F(x) = x$

5. a) Solve the IBVP for the Heat Equation in the plane wall with distributed heat generation:

$$\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{1}{\alpha}\frac{\partial u}{\partial t}, \qquad u(x,t), \quad x \in (0,L), \quad t > 0, \qquad F(x) = \frac{\dot{q}}{k}x$$

Initial Condition: $u(x,0) = u_0(x)$

Boundary Conditions: $\frac{\partial u(0,t)}{\partial x} = 0 \qquad t > 0$

$$k\frac{\partial u(L,t)}{dx} = h_2\left[T_\infty - u(L,t)\right] \qquad t > 0$$

b) Sketch the graph of solution with

$L = 0.5$, $\alpha = 0.0005$, $k = 150$,

$u_0(x) = 200$, $T_\infty = 10$, $h_2 = 250$, $\dot{q} = 200000$

6. a) Solve the Heat Equation in the cylindrical domain with angular symmetry

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = a^2 \frac{\partial u}{\partial t} \qquad u(r,z): \qquad 0 \le r < r_1, \ t > 0$$

Boundary condition: $u(r_1,t) = 0 \qquad t > 0$

Initial condition $u(r,0) = u_0(r)$

b) Sketch the graph of the solution for

$r_1 = 0.5$

$a = 3$

$u_0(r) = 6r^2 + 1$

7. a) Solve the Heat Equation in the cylindrical domain with angular symmetry

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=a^2\frac{\partial u}{\partial t} \qquad u(r,z), \qquad 0\le r<r_1, \qquad t>0$$

Boundary condition: $u(r_1,t)=f_1$ $\qquad t>0$

Initial condition $u(r,0)=u_0(r)$

b) Display some creativity in visualization of solution for

$r_1=0.5$

$a=3000$

$f_1=70$

$u_0(r)=25r^2+20$

c) Give some physical interpretation of the problem

8. Solve the IBVP for the Heat Equation in polar coordinates with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=a^2\frac{\partial u}{\partial t} \qquad u(r,t), \quad r\in[0,r_1), \quad t>0$$

Initial conditions: $u(r,0)=u_0(r)$

Boundary condition: $k\dfrac{\partial u(r_1,t)}{\partial r}+hu(r_1,t)=f_1$ $\qquad t>0$

And sketch the graph of solution for

$r_1=2$, $a=0.5$, $k=0.1$, $h=12$, $f_1=2$, and $u_0(r)=(r-r_1)^2$

(hint: first, find the steady state solution)

9. a) Solve the Heat Equation in the annular domain with angular symmetry (cylindrical wall with uniform heat generation)

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(r,t): \qquad r_1<r<r_2, \quad t>0$$

Boundary condition: $u(r_1,t)=T_1$ $\qquad t>0$

$u(r_2,t)=T_2$ $\qquad t>0$

Initial condition: $u(r,0)=u_0(r)$ $\qquad r_1<r<r_2$

b) Display some creativity in visualization of the solution for

$r_1=0.5$ $\qquad T_1=50$

$r_2=0.6$ $\qquad T_2=10$

$k=150$ $\qquad u_0(r)=10$

$\alpha=0.00001$ $\qquad \dot{q}=500000$

10. EXAMPLE Radiation Induced Thermal Stratification in Surface Layers of Stagnant Water

Professor Raymond Viskanta (on the left)
Antalya, Turkey, June 2001

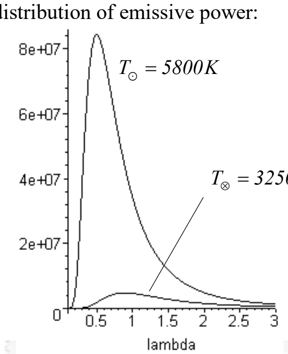
Radiation Induced Thermal Stratification in Surface Layers of Stagnant Water

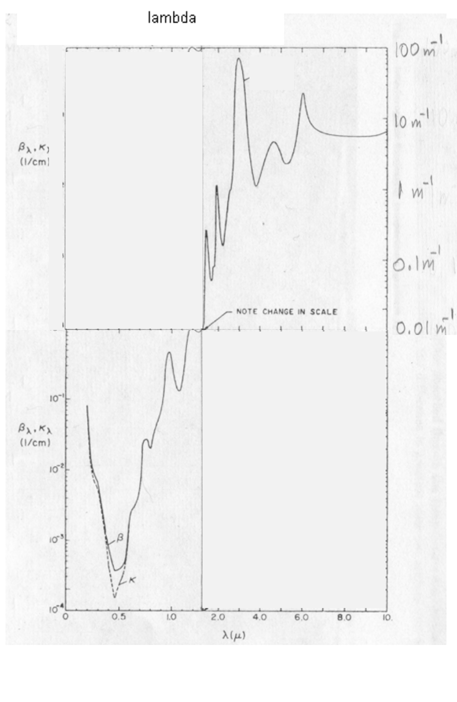
Based on papers: [1] D.M.Snider, R.Viskanta *Radiation Induced Thermal Stratification in Surface Layers of Stagnant Water,* ASME Journal of Heat Transfer, Feb 1975, pp.35-40.
[2] R.Viskanta, J.S.Toor *Radiant Energy Transfer in Waters,* Water Resources Research, Vol. 8, No.3, June 1972, pp. 595-608.

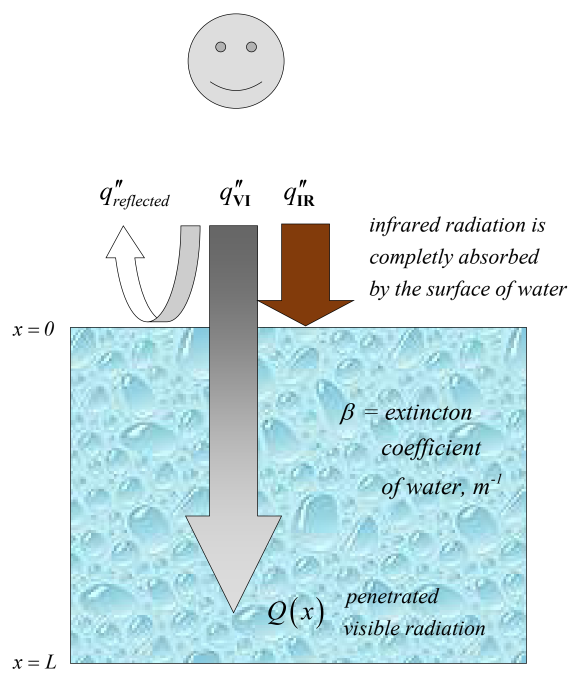
Introduction: The vertical temperature distribution in a body of water have important effects on chemical and physical properties, dissolved oxygen content, water quality, aquatic life and ecological balance as well as mixing processes in water.
Solar radiation is recognized as the principle natural heat load in waters. Some investigators have considered the radiation to be absorbed at the water surface (i.e. opaque) and others treated the water as being semitransparent but ignored the spectral nature of radiation. Since the ultraviolet (UV) and infrared (IR) parts of the incoming solar radiation are largely absorbed within the first centimeters of the water and the visible part (VI) penetrates more deeply and carries significant energy to depths, the modeling of water as a gray medium is open to question and needs to be examined.
In the works of Raymond Viskanta (Purdue University) and coworkers, analysis for the time dependent thermal stratification of in surface layers of stagnant water by solar radiation was developed. The transient temperature distribution is obtained by solving the one-dimensional energy equation for combined conduction and radiation energy transfer using a ***finite difference method***. Experimentally, solar heating ($T_\odot = 5800K$) of water is simulated using tungsten filament lamps ($T_\otimes = 3250K$) in parabolic reflectors of known spectral characteristics.

Our Objective: ***Analytical*** investigation of transient combined conduction-radiation heat transfer with two band spectral model (VI-IR) of incident radiation.

Spectral distribution of emissive power:







β_λ, $\left[\frac{1}{cm}\right]$ *spectral absorption coefficient of liquid water*

Model:

Heat equation: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{Q(x)}{k} = \dfrac{1}{\alpha}\dfrac{\partial u}{\partial t}$

Initial condition: $u(x,0) = u_0(x) = T_0$

Boundary conditions: $\left[-k\dfrac{\partial u}{dx} = -h_{eff}\left(u - T_\infty\right) + q''_{\mathbf{IR}} \right]_{x=0}$

$\left[u\right]_{x=L} = T_L$

Source function
(radiant energy absorption rate): $Q(x) = q''_{\mathbf{VI}}\beta e^{-\beta x}$

Water Properties:

Extinction coefficient	$\beta = 70$	m^{-1}
Density	$\rho = 1000$	$\dfrac{kg}{m^3}$
Specific heat	$c_p = 4180$	$\dfrac{J}{kg \cdot K}$
Conductivity	$k = 0.6$	$\dfrac{W}{m \cdot K}$

Data:

Length	$L = 0.381$	m
Temperature	$T_0 = T_{inf} = T_L = 25$	^oC
Visible irradiation	$q''_{\mathbf{VI}} = 850$	$\dfrac{W}{m^2}$
Infrared irradiation	$q''_{\mathbf{IR}} = 150$	$\dfrac{W}{m^2}$
Efficient convective coefficient	$h_{eff} = 12$	$\dfrac{W}{m^2 \cdot K}$

a. Solve the given IBVP:

$u(x,t) =$
$u(0.05m, 3000s) =$ $28.3^o C$ *particular value*

b. Sketch the graph of the solution for $t = 5, 10, 15, 30, 60, 90\ min$ and compare with Viskanta's results.

c. Your view on the problem. How can the accuracy of the model be improved?
What have you learned from this problem?

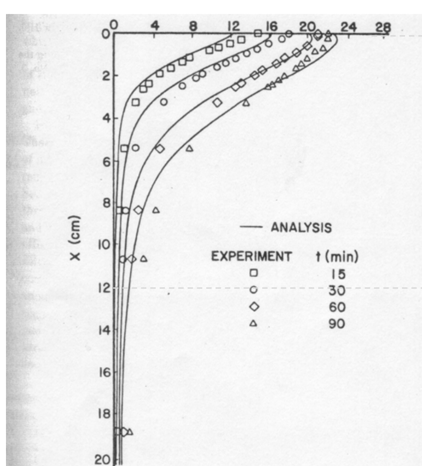


Fig. 4 Comparison of measured and predicted temperature distributions for lamps at 3250 K, $T_0 = T_\infty = 25$ C and $L = 38.1$ cm

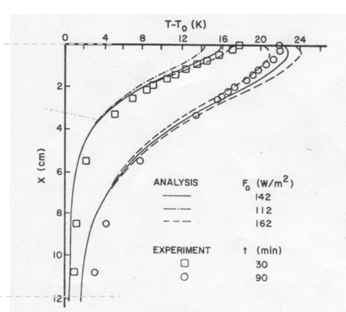


Fig. 6 Effect of variation in radiant energy flux considered absorbed at the surface on predicted temperature distribution for lamps at 3250 K, $T_0 = T_\infty = 25$ C, and $L = 38.1$ cm

Solution:

Heat equation: $$\frac{\partial^2 u}{\partial x^2}+\frac{Q(x)}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad Q(x)=q''_{\mathbf{VI}}\beta e^{-\beta x}$$

Initial condition: $$u(x,0)=u_0(x)=T_0$$

Boundary conditions: $$\left[-k\frac{\partial u}{dx}=-h_{eff}(u-T_\infty)+q''_{\mathbf{IR}}\right]_{x=0}=0$$

$$[u]_{x=L}=T_L$$

$$\left[-k\frac{\partial u}{dx}+h_{eff}u\right]_{x=0}=h_{eff}T_\infty+q''_{\mathbf{IR}}$$

$$\left[-\frac{\partial u}{dx}+\frac{h_{eff}}{k}u\right]_{x=0}=\frac{h_{eff}T_\infty+q''_{\mathbf{IR}}}{k}$$

$$\left[-\frac{\partial u}{dx}+Hu\right]_{x=0}=f_0 \qquad H=\frac{h_{eff}}{k}\,,\ f_0=\frac{h_{eff}T_\infty+q''_{\mathbf{IR}}}{k}$$

$$\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad x\in(0,L)$$

$$\left[-\frac{\partial u}{dx}+Hu\right]_{x=0}=f_0$$

$$[u]_{x=L}=T_L$$

$$u(x,0)=u_0(x)=T_0$$

$$F(x)=\frac{q''_{\mathbf{VI}}\beta}{k}e^{-\beta x}$$

I Steady State Solution:

$$\frac{\partial^2 u_s}{\partial x^2}+F(x)=0 \qquad x\in(0,L)$$

$$\left[-\frac{\partial u_s}{dx}+Hu_s\right]_{x=0}=f_0$$

$$[u_s]_{x=L}=T_L$$

$$\frac{\partial^2 u_s}{\partial x^2}+F(x)=0$$

$$\frac{\partial^2 u_s}{\partial x^2}=-F(x)=-\frac{q''_{\mathbf{VI}}\beta}{k}e^{-\beta x}$$

$$\frac{\partial^2 u_s}{\partial x^2}=-\frac{q''_{\mathbf{VI}}\beta}{k}\int e^{-\beta x}dx$$

$$\frac{\partial^2 u_s}{\partial x^2}=-\frac{q''_{\mathbf{VI}}\beta}{k}\frac{1}{(-\beta)}\int e^{-\beta x}d(-\beta x)$$

$$\frac{\partial u_s}{\partial x}=\frac{q''_{\mathbf{VI}}}{k}e^{-\beta x}+c_1$$

$$u_s=\frac{q''_{\mathbf{VI}}}{k}\int e^{-\beta x}dx+c_1x+c_2$$

$$u_s = -\frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta x} + c_1 x + c_2$$

Boundary conditions:

$$x = 0 \qquad \left[-\left(\frac{q''_{\mathbf{VI}}}{k} e^{-\beta x} + c_1 \right) + H\left(-\frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta x} + c_1 x + c_2 \right) \right]_{x=0} = f_0$$

$$-\left(\frac{q''_{\mathbf{VI}}}{k} + c_1 \right) + H\left(-\frac{q''_{\mathbf{VI}}}{k\beta} + c_2 \right) = f_0$$

$$-c_1 + Hc_2 = \frac{q''_{\mathbf{VI}}}{k}\left(1 + \frac{H}{\beta} \right) + f_0 \qquad f_0 = \frac{h_{eff} T_\infty + q''_{\mathbf{IR}}}{k}$$

$$-c_1 + Hc_2 = \frac{1}{k}\left[\left(1 + \frac{H}{\beta} \right) q''_{\mathbf{VI}} + h_{eff} T_\infty + q''_{\mathbf{IR}} \right]$$

$$-kc_1 + h_{eff} c_2 = \left(1 + \frac{h_{eff}}{k\beta} \right) q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff} T_\infty$$

$$x = L \qquad -\frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta L} + c_1 L + c_2 = T_L$$

$$c_1 L + c_2 = \frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta L} + T_L$$

In matrix form:

$$\begin{bmatrix} -k & h_{eff} \\ L & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \left(1 + \frac{h_{eff}}{k\beta} \right) q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff} T_\infty \\ \frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta L} + T_L \end{bmatrix}$$

Use Cramer's Rule:

$$det \begin{bmatrix} -k & h_{eff} \\ L & 1 \end{bmatrix} = -\left(k + h_{eff} L \right)$$

$$c_1 = \frac{det \begin{bmatrix} \left(1 + \frac{h_{eff}}{k\beta} \right) q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff} T_\infty & h_{eff} \\ \frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta L} + T_L & 1 \end{bmatrix}}{det \begin{bmatrix} -k & h_{eff} \\ L & 1 \end{bmatrix}} = \frac{\left(1 + \frac{h_{eff}}{k\beta} \right) q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff} T_\infty - h_{eff} \frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta L} - h_{eff} T_L}{-\left(k + h_{eff} L \right)}$$

$$c_1 = \frac{\left(1 + \frac{h_{eff}}{k\beta} \right) q''_{\mathbf{VI}} + q''_{\mathbf{IR}} + h_{eff}\left(T_\infty - T_L \right) - h_{eff} \frac{q''_{\mathbf{VI}}}{k\beta} e^{-\beta L}}{-\left(k + h_{eff} L \right)} \qquad c1 := \frac{h\ Tinf + qir + q0 + \frac{q0\ h}{k\ \beta} - \left(TL + \frac{q0\ \mathbf{e}^{(-\beta\ L)}}{k\ \beta} \right) h}{-k - L\ h}$$

$$c_2 = \frac{det\begin{bmatrix} -k & \left(1+\frac{h_{eff}}{k\beta}\right)q''_{VI} + q''_{IR} + h_{eff}T_\infty \\ L & \frac{q''_{VI}}{k\beta}e^{-\beta L} + T_L \end{bmatrix}}{det\begin{bmatrix} -k & h_{eff} \\ L & 1 \end{bmatrix}} = \frac{-k\frac{q''_{VI}}{k\beta}e^{-\beta L} - kT_L - L\left(1+\frac{h_{eff}}{k\beta}\right)q''_{VI} - q''_{IR}L - h_{eff}T_\infty L}{-\left(k+h_{eff}L\right)}$$

$$c_2 = \frac{\frac{q''_{VI}}{\beta}e^{-\beta L} + kT_L + L\left(1+\frac{h_{eff}}{k\beta}\right)q''_{VI} + q''_{IR}L + h_{eff}T_\infty L}{\left(k+h_{eff}L\right)}$$

$$c2 := \frac{-k\left(TL + \frac{q0\, \mathbf{e}^{(-\beta L)}}{k\beta}\right) - L\left(h\, Tinf + qir + q0 + \frac{q0\, h}{k\beta}\right)}{-k - L\, h}$$

$$u_s = -\frac{q''_{VI}}{k\beta}e^{-\beta x} + \left[\frac{\left(1+\frac{h_{eff}}{k\beta}\right)q''_{VI} + q''_{IR} + h_{eff}\left(T_\infty - T_L\right) - h_{eff}\frac{q''_{VI}}{k\beta}e^{-\beta L}}{-\left(k+h_{eff}L\right)}\right]x + \frac{\frac{q''_{VI}}{\beta}e^{-\beta L} + kT_L + L\left(1+\frac{h_{eff}}{k\beta}\right)q''_{VI} + q''_{IR}L + h_{eff}T_\infty L}{\left(k+h_{eff}L\right)}$$

II Transient Solution: $U(x,t) = u(x,t) - u_s(x)$

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad x \in (0,L)$$

$$\left[-\frac{\partial U}{dx} + HU\right]_{x=0} = 0 \qquad \mathbf{R}$$

$$\left[U\right]_{x=L} = 0 \qquad \mathbf{D}$$

$$U(x,0) = u_0(x) - u_s(x) = U_0(x)$$

Supplemental SLP (RD):

λ_n roots of characteristic equation:

$X_n = sin\left[\lambda_n (x-L)\right]$ $\qquad \left\|X_n\right\|^2$

Solution: $U(x,t) = \sum_{n=1} a_n X_n e^{-\alpha\lambda_n^2 t}$ $\qquad a_n = \frac{1}{\left\|X_n\right\|^2}\int_0^L \left[u_0(x) - u_s(x)\right]X_n(x)\,dx$

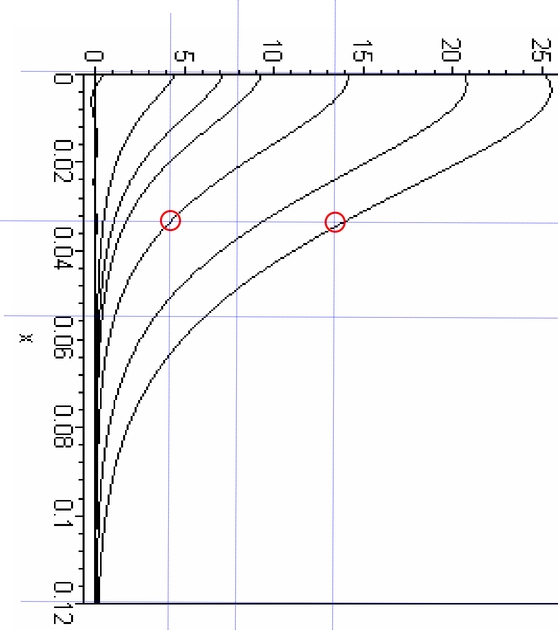
III Solution: $u(x,t) = u_s(x) + U(x,t)$

```
> U0:=subs(t=0,u(x,t)):
> U1:=subs(t=300,u(x,t)):
> U2:=subs(t=600,u(x,t)):
> U3:=subs(t=900,u(x,t)):
> U4:=subs(t=1800,u(x,t)):        30 min
> U5:=subs(t=3600,u(x,t)):
> U6:=subs(t=5400,u(x,t)):        90 min
```

Comparison

Current analytical solution

Experimant and numerical solution [Viskanta]



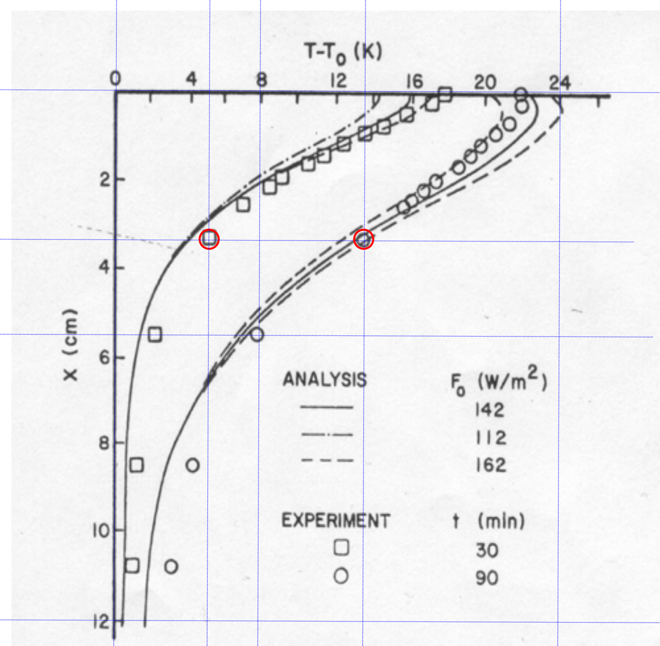
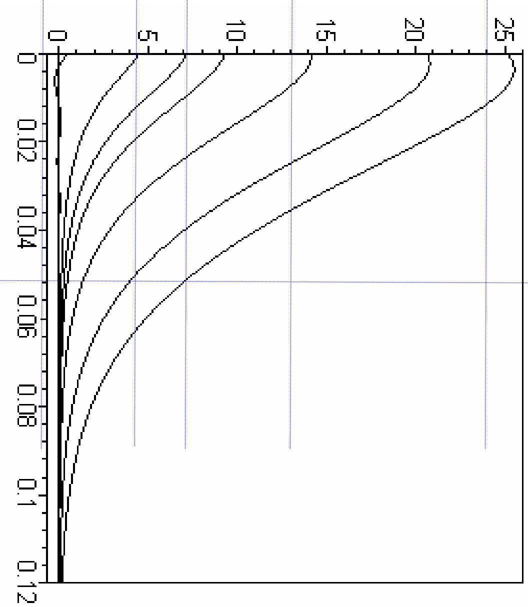


Fig. 6 Effect of variation in radiant energy flux considered absorbed at the surface on predicted temperature distribution for lamps at 3250 K, $T_0 = T_\infty = 25$ C, and $L = 38.1$ cm



11. Find the solution of the IBVP for the Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\quad x\in(0,L),\ t>0$$

initial condition: $u(x,0)=u_0(x)$

$\frac{\partial u(x,0)}{\partial t}=u_1(x)$

boundary conditions: $u(0,t)=0$ $\quad t>0$ (Dirichlet)

$u(L,t)=0$ $\quad t>0$ (Dirichlet)

Sketch the graph of solution with $L=2$, $a=0.5$, and

a) $u_1(x)=-0.1$, $u_0(x)=x^2(L-x)^2$

b) $u_1(x)=0$, $u_0(x)=\sin\frac{6\pi}{L}x$

(observe the phenomena called standing waves)

12. Find the solution of the IBVP for the Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\quad x\in(0,L),\ t>0$$

initial condition: $u(x,0)=u_0(x)$

$\frac{\partial u(x,0)}{\partial t}=u_1(x)$

boundary conditions: $-u'(0,t)+H_1u(0,t)=0$, $\quad t>0$ (Robin)

$u(L,t)=0$ $\quad t>0$ (Dirichlet)

Sketch the graph of solution with $L=5$, $a=2.0$, and

a) $u_1(x)=0.2$, $u_0(x)=(L-x)^2$

b) $u_1(x)=0$, $u_0(x)=X_5(x)$ (eigenfunction)

13. a) Solve the IBVP:

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial u}{\partial t}+F(x) \qquad u(x,t),\quad x\in(0,L),\ t>0$$

initial condition: $u(x,0)=u_0(x)$

boundary conditions: $u(0,t)=f_1$ $\quad t>0$ (Dirichlet)

$k\frac{\partial u(L,t)}{dx}+hu(L,t)=f_2$ $\quad t>0$ (Robin)

b) Sketch the graph of solution with

$L=4$, $a=0.5$, $k=2.0$, $u_0(x)=x(x-L/2)+5$, $f_1=10$, $f_2=1$, $F(x)=x$

14A. Find the solution for vibration of the annular membrane with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=a^2\frac{\partial^2 u}{\partial t^2} \qquad u(r,t),\quad r\in(r_1,r_2),\quad t>0$$

Initial conditions: $u(r,0)=u_0(r)$

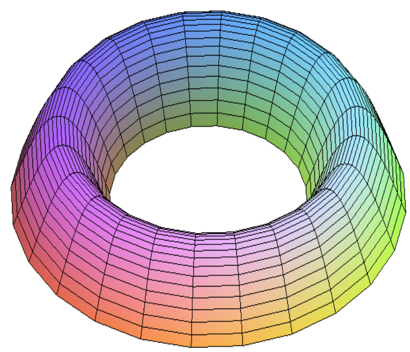
$$\frac{\partial u}{\partial t}(r,0)=u_1(r)$$

Boundary condition: $u(r_1,t)=0 \qquad t>0$

$$u(r_2,t)=0 \qquad t>0$$

And sketch the graph of solution for

$r_1=1$, $r_2=2$ $a=0.5$, $u_0(r)=(r-r_1)(r_2-r)$, and $u_1(r)=0$.



14B. Heavy membrane

Find the solution for vibration of the annular membrane with angular symmetry:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(r)=a^2\frac{\partial^2 u}{\partial t^2} \qquad u(r,t),\quad r\in(r_1,r_2),\quad t>0$$

Initial conditions: $u(r,0)=u_0(r)$

$$\frac{\partial u}{\partial t}(r,0)=u_1(r)$$

Boundary condition: $u(r_1,t)=0 \qquad t>0$

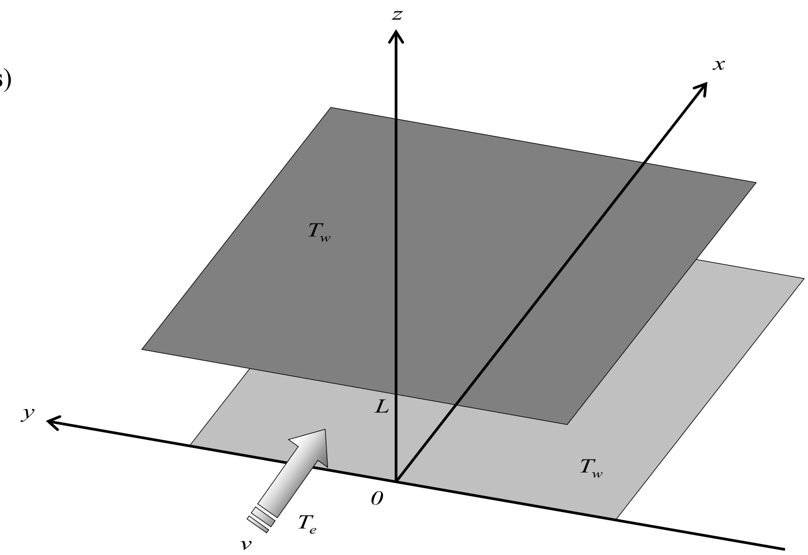
$$u(r_2,t)=0 \qquad t>0$$

And sketch the graph of solution for

$r_1=1$, $r_2=2$ $a=0.5$, $F=-1.5$, $u_0(r)=(r-r_1)(r_2-r)$, and $u_1(r)=0$.

Non-Classical IBVPs

15. (**Flow Between Two Plates**)



$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}-\frac{\rho c_p}{k}v\frac{\partial T}{\partial x}+\frac{\dot{q}}{k}=\frac{\rho c_p}{k}\frac{\partial T}{\partial t}$$

$x = 0 \quad T = T_e$

$x \to \infty \quad T < \infty$

$z = 0 \quad T = T_w$

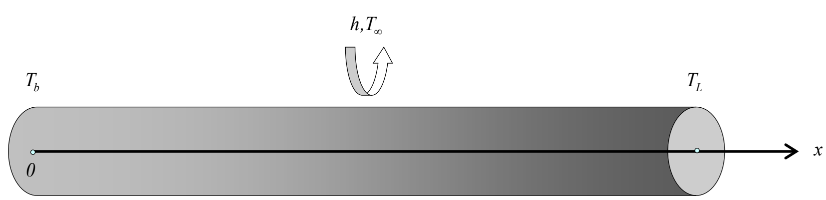
$z = L \quad T = T_w$

$t = 0 \quad T = T_0$

Find steady state solution for $\dot{q} = 0$.

Sketch the graph for $T_e = 80$, $T_w = 10$, $v = 2$, $L = 0.02$, fluid is water.

16. (**Transient Conduction in the Fin**)



$$\frac{\partial^2 T}{\partial x^2}-\frac{hP}{kA_c}\left(T-T_\infty\right)+\frac{\dot{q}}{k}=\frac{\rho c_p}{k}\frac{\partial T}{\partial t}$$

$x = 0 \quad T = T_b$

$x = L \quad T = T_L$

$t = 0 \quad T = T_0$

Find transient state solution for $\dot{q} = 0$.

circular copper fin ($D = 0.005$)

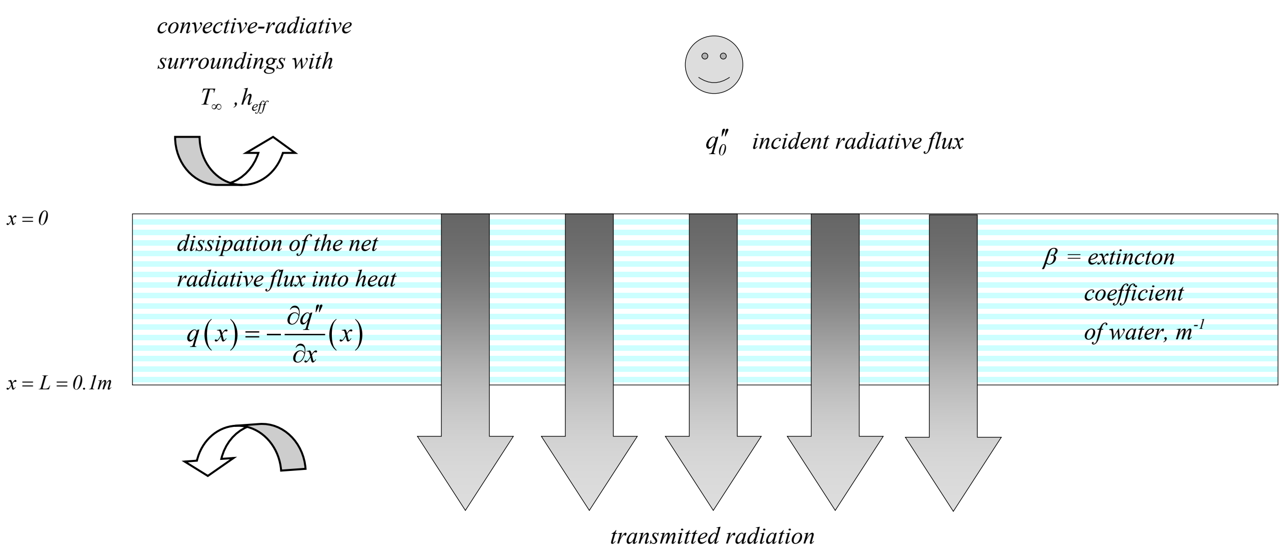
Sketch the graph for $T_b = 200$, $T_L = 50$, $T_\infty = 10$, $T_0 = 10$, $h = 150$, $L = 0.2$,

17. [Based on Nellis&Klein, p.37] Absorption in a lens

Analytical investigation of transient combined conduction-radiation heat transfer with a gray spectral model of incident radiation.

A lens is used to focus the illumination radiation that is required to develop the resist in a lithographic manufacturing process

The lens is not perfectly transparent but rather absorbs some of the illumination energy that passes through it.



Model:	Heat equation:	$\frac{\partial^2 u}{\partial x^2} + \frac{Q(x)}{k} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$	
	Initial condition:	$u(x,0) = u_0(x) = T_0$	
	Boundary conditions:	$\left[-k\frac{\partial u}{dx} = -h_{eff}(u - T_\infty)\right]_{x=0}$	
		$\left[k\frac{\partial u}{dx} = -h_{eff}(u - T_\infty)\right]_{x=L}$	
	Dissipation source function (radiant energy absorption rate):	$Q(x) = q_0''\beta e^{-\beta x}$	
The Lens Properties:	Extinction coefficient	$\beta = 100$	m^{-1}
	Density	$\rho = 2500$	$\frac{kg}{m^3}$
	Specific heat	$c_p = 750$	$\frac{J}{kg\cdot K}$
	Conductivity	$k = 1.5$	$\frac{W}{m\cdot K}$
Data:	Length	$L = 0.1$	m
	Temperature	$T_{inf} = T_0 = 20$	^oC
	Incident radiative flux	$q_0'' = 1000$	$\frac{W}{m^2}$
	Efficient convective coefficient	$h_{eff} = 20$	$\frac{W}{m^2\cdot K}$

a. Solution of the given IBVP:

$u(x,t) =$

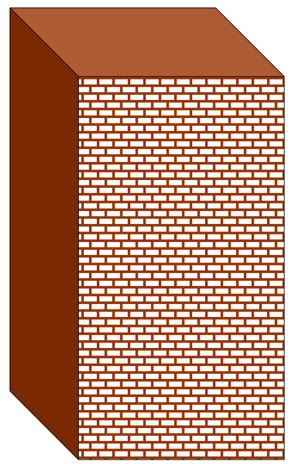
b. Steady State Solution:

$U(x) =$

c. Sketch the graph.

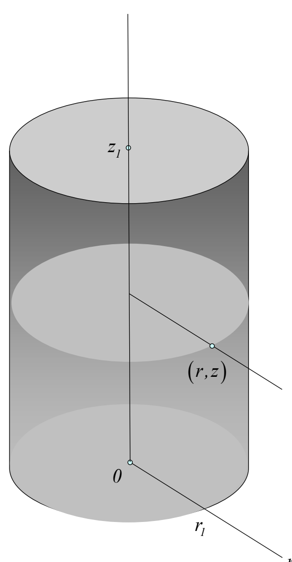
18. Investigate the temperature field in the long column of square cross-section two adjacent sides of which are thermally insulated and two others are maintained at temperatures $T_1 = 100^o C$ and $T_2 = 500^o C$ if initially it was of uniform temperature $T_0 = 20^o C$. Sketch the temperature surfaces.

$L = 2m$



19. Use separation of variables for solution of IBVP for long cylinder with angular symmetry. $\left(\dfrac{\partial u}{\partial \theta} = 0 \right)$:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$



20. Set up a mathematical model (choose an appropriate coordinate system and dimension of the problem, write the governing equation and corresponding initial and boundary conditions) for the following engineering models (do not solve the problem):

a) A very thin long wire dissipates energy in the massive layer of the stagnant media with the rate per unit length q, $\left[\frac{W}{m}\right]$. The media has a thermal conductivity $k, \left[\frac{W}{m \cdot K}\right]$. Determine the stationary temperature distribution in the media.

b) In the massive layer of homogeneous material (with thermal properties k, ρ, c_p) which was initially at the uniform temperature T_0, a localized heat source spontaneously started to dissipate energy with the rate q $[W]$. Determine the development of the temperature field in the material.

c) A very long tree trunk of radius R in the forest is exposed to the surrounding air (average wind speed is v $\left[\frac{m}{s}\right]$), but the dense crown prevents the direct sun radiation of the trunk. Set up the mathematical model describing the temperature distribution in the tree trunk during the day. Conductivity in the tree depends on direction: it is much higher along the tree than in the radial direction.

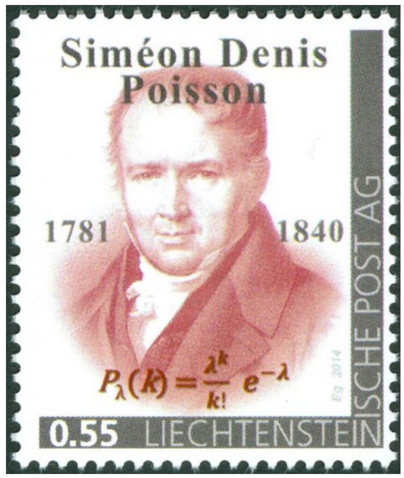
d) A wide reservoir of water of L meters deep is exposed to the solar irradiation G_0, $\left[\frac{W}{m^2}\right]$ incident at the angle θ. Penetration of the solar radiative flux along the path s is described by the Lambert-Beer Law $G(x) = G_0 \cos\theta \, e^{-\kappa s}$, where $\kappa, \left[\frac{1}{m}\right]$ is the gray absorption coefficient of water. Then the solar energy dissipated in water (radiative dissipation source or the divergence of radiative flux) is determined by $Q(s) = -\frac{dG(x)}{dx}$, $\left[\frac{W}{m^3}\right]$. Set up the mathematical model describing the equilibrium temperature field in the water layer.

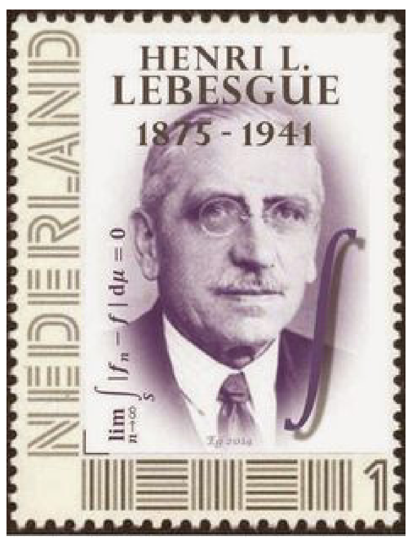
e) Two opposite sides of the long column are insulated. There is an intensive condensation of the water steam on one of the other sides. The last side is exposed to the convective environment at temperature T_∞ and convective coefficient $h, \left[\frac{W}{m^2 \cdot K}\right]$. Due to some chemical reaction there is production energy in the column with the volumetric rate $\dot{q}, \left[\frac{W}{m^3}\right]$. Initially, column was at the uniform temperature T_0. Describe the transient temperature distribution inside of the column.

Stanislaw Mazur and Per Enflo

Stanislaw Mazur was a close collaborator with Banach at Lwów and was a member of the Lwów School of Mathematics, where he participated in the mathematical activities at the Scottish Café.
On 6 November 1936, he posed the "basis problem" of determining whether every Banach space has a Schauder basis, with Mazur promising a "live goose" as a reward: Thirty seven years later, a live goose was awarded by Mazur to Per Enflo in a ceremony that was broadcast throughout Poland.

Lvov in 2009


Siméon Denis
Poisson
1781
1840
$P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$
Eg 2014
0.55 LIECHTENSTEIN
ISCHE POST AG


HENRI L.
LEBESGUE
1875 - 1941
NEDERLAND
$\lim_{n\to\infty} \int_S |f_n - f| \, d\mu = 0$
Eg 2014
1


冨嶽三十六景
神奈川沖
浪裏