IX.1 THE LAPLACE TRANSFORM

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IX.1.1 DEFINITION: The Laplace transform of the function \( f(t) \), \( t \geq 0 \) is defined as

\[
L[f(t)] = \phi(s) = \int_{0}^{\infty} f(t)e^{-st}dt \quad s > 0
\]

There are several traditional notations used for the Laplace transform:

\[
L: f(t) \rightarrow \phi(s) \\
L: y(t) \rightarrow Y(s) \\
L: u(t) \rightarrow \Phi(s)
\]

The function \( e^{-st} \) is the kernel of the transform and \( s \) is the transform variable. The existence condition for the Laplace transform is established for functions growing not faster than an exponential function: if there exist constants \( a > 0 \) and \( M, K > 0 \) such that \( |f(t)| \leq Me^{at} \) for all \( t \geq K \) then the function \( f(x) \) is called of exponential order.

**Theorem 9.1 (sufficient condition for existence of the Laplace transform)**

Let the function \( f(t) \) be piecewise continuous on \([0, \infty)\) and of exponential order with constants \( a > 0 \) and \( M > 0 \), then

1) The Laplace transform \( \phi(s) = \int_{0}^{\infty} f(t)e^{-st}dt \) exists for all \( s > a \)

2) \( \phi(s) \leq \frac{M}{s-a} \)

3) \( \phi(s) \rightarrow 0 \) when \( s \rightarrow \infty \)

4) \( s \phi(s) \) is bounded when \( s \rightarrow \infty \)

**Examples:**

1. \( f(x) = 1 = e^{0x} \quad a = 0, \quad M = 1 \)
   \[
   \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt = -\frac{1}{s}e^{-st}\bigg|_{a}^{\infty} = \frac{1}{s} \quad s > 0
   \]

2. \( f(x) = Me^{ax} \)
   \[
   \mathcal{L}\{f(t)\} = \frac{M}{s-a} \quad s > a
   \]

**Inverse Laplace transform**

We define the inverse Laplace transform of the function \( \phi(s) \) as an operation which yields a function \( f(t) \) such that \( L\{f(t)\} = \phi(s) \):

\[
f(t) = L^{-1}\{\phi(s)\}
\]

Here, we consider the Laplace transform restricted to real values of \( s \). A more general definition is based on the Fourier transform applied to functions which are equal to zero for negative values, and the variable \( s \) is an imaginary frequency: \( s = i\omega, \quad -\infty < \omega < \infty \)

\[
L\{f(s)\} = 2\pi F\{f(t)\} = \int_{0}^{\infty} f(t)e^{-i\omega t}dt = \int_{0}^{\infty} f(t)e^{-st}dt
\]

Then the inverse Laplace transform is defined by

\[
f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \phi(s)e^{st}ds, \quad t > 0, \quad \alpha = \text{Re}(s)
\]
IX.1.2 PROPERTIES

To calculate the Laplace transform and its inverse we will use mostly the table of the Laplace transform and its properties. Let \( L\{f(t)\} = \phi(s) \) then:

1) **Linearity:** Both \( L \) and \( L^{-1} \) are linear:

\[
L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]
\]
\[
L^{-1}[\alpha \phi(s) + \beta \psi(s)] = \alpha L^{-1}[\phi(s)] + \beta L^{-1}[\psi(s)]
\]

2) **Shifting in \( s \):**

\[
L\{e^{at} f(t)\} = \phi(s-a) \quad a \in \mathbb{R}
\]
\[
L\{e^{-at} f(t)\} = \phi(s+a)
\]
\[
L^{-1}\{\phi(s-a)\} = e^{at} f(t)
\]
\[
L^{-1}\{\phi(s+a)\} = e^{-at} f(t)
\]

3) **Shifting in \( t \):** Let \( H(t-a) \) be the Heaviside unit step function

\[
H(t-a) = \begin{cases} 
1 & \text{if } t > a \\
0 & \text{if } t < a
\end{cases}
\]

then

\[
L\{f(t-a)H(t-a)\} = e^{-as}\phi(s) \quad a > 0
\]
\[
L^{-1}\{e^{-as}\phi(s)\} = f(t-a)H(t-a)
\]

4) **Similarity:**

\[
L\{f(at)\} = \frac{1}{a} \phi\left(\frac{s}{a}\right)
\]

5) **Differentiation:**

\[
L\{f'(t)\} = -\frac{d}{ds}\phi(s)
\]

6) **Integration:**

\[
L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \phi(s)ds
\]

7) **Convolution:**

\[
f \ast g = \int_0^t f(t-x)g(x) \, dx \quad \text{definition}
\]

Convolution Theorem

\[
L\{f \ast g\} = F(s)G(s)
\]
\[
L^{-1}\{F(s)G(s)\} = f \ast g
\]

Operational Properties

8) **Transform of derivatives:**

\[
L\{f'(t)\} = s\phi(s) - f(0)
\]
\[
L\{f''(t)\} = s^2\phi(s) - sf(0) - f'(0)
\]

\[
\vdots
\]
\[
L\{f^{(n)}(t)\} = s^n\phi(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)
\]
IX.1.3 EXAMPLES

1) Using the definition, calculate the Laplace transform of \( f(t) = t \):

\[
L\{t\} = \int_{0}^{\infty} te^{-st} \, dt
\]

\[
= -L\left[ \int_{0}^{\infty} te^{-st} \, dt \right]
\]

\[
= -L\left[ \int_{0}^{\infty} t \, e^{-st} \, dt \right]
\]

integration by parts

\[
= -L\left[ \left. \frac{t e^{-st}}{s} \right|_{0}^{\infty} + \int_{0}^{\infty} \frac{e^{-st}}{s} \, dt \right]
\]

\[
= -\left. \frac{t e^{-st}}{s} \right|_{0}^{\infty} + \int_{0}^{\infty} \frac{e^{-st}}{s} \, dt
\]

\[
limit_{t \to \infty} te^{-st} = 0
\]

\[
= 0 + \frac{1}{s}
\]

\[
= \frac{1}{s^2}
\]

2) Derive the property \( L\{f(t-a)H(t-a)\} = e^{-as}\phi(s) \).

According to definition of the Laplace transform

\[
L\{f(t-a)H(t-a)\} = \int_{0}^{\infty} f(t-a)H(t-a)e^{-st} \, dt
\]

\[
= \int_{0}^{\infty} f(t-a)e^{-st} \, dt \quad \text{substitute } t = \tau + a
\]

\[
= \int_{0}^{\infty} f(\tau)e^{-s(\tau+a)} \, d\tau
\]

\[
e^{-as}\int_{0}^{\infty} f(\tau)e^{-s\tau} \, d\tau
\]

\[
= e^{-as}\phi(s)
\]

3) Derive the property \( L\{f'(t)\} = s\phi(s) - f(0) \).

\[
L\{f'(t)\} = \int_{0}^{\infty} f'(t)e^{-st} \, dt
\]

\[
= \int_{0}^{\infty} e^{-st} d[f(t)]
\]

move to differential

\[
= \left[ f(t)e^{-st} \right|_{0}^{\infty} - \int_{0}^{\infty} f(t)d[-e^{-st}] \]

integration by parts

\[
= 0 - f(0) + \int_{0}^{\infty} f(t)e^{-st} \, dt \quad \text{assume } \lim_{t \to \infty} f(t)e^{-st} = 0
\]

\[
= s\phi(s) - f(0)
\]

4) Evaluate \( L\{t^n\} \).

Let \( f(t) = t^n \), then \( f'(t) = 2t \) and \( f(0) = 0 \). Apply the property

\[
L\{f'(t)\} = s\phi(s) - f(0)
\]

\[
L\{2t\} = sL\{t^n\} - f(0)
\]

\[
L\{t^n\} = \frac{1}{s}L\{2t\} = \frac{2}{s}L\{t\} = \frac{2}{s} \cdot \frac{1}{s} = \frac{2}{s^2}
\]

In general, for \( n \), application of property (7) yields

\[
L\{t^n\} = \frac{n!}{s^{n+1}}
\]
5) Inversion of the formula \( L\left\{ t^n \right\} = \frac{n!}{s^{n+1}} \):

\[
L^{-1}\left\{ L\left\{ t^n \right\} \right\} = L^{-1}\left\{ \frac{n!}{s^{n+1}} \right\} = t^n = n! L^{-1}\left\{ \frac{1}{s^{n+1}} \right\} = \frac{t^n}{n!}
\]

6) Inverse transform of rational function (partial fractions decomposition):

Evaluate \( L^{-1}\left\{ \frac{2s+1}{s^2-9} \right\} \).

Convert rational function to partial fractions (see Section):

\[
\frac{2s+1}{s^2-9} = \frac{2s+1}{(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3} = \frac{A(s+3)+B(s-3)}{(s-3)(s+3)}
\]

\[
s = 3 \quad 6 \cdot A = 2 \quad \Rightarrow \quad A = \frac{1}{3}
\]

\[
s = -3 \quad -6 \cdot B = 1 \quad \Rightarrow \quad B = -\frac{1}{6}
\]

Therefore

\[
\frac{2s+1}{s^2-9} = \frac{1}{3} - \frac{1}{6} \frac{1}{s+3}
\]

\[
L^{-1}\left\{ \frac{2s+1}{s^2-9} \right\} = \frac{1}{3} L^{-1}\left\{ \frac{1}{s-3} \right\} - \frac{1}{6} L^{-1}\left\{ \frac{1}{s+3} \right\} = \frac{1}{3} e^{3t} - \frac{1}{6} e^{-3t}
\]

7) Inverse transform of rational function (use convolution theorem):

Evaluate \( L^{-1}\left\{ \frac{1}{s^2(s-1)} \right\} \).

Notice that \( \frac{1}{s^2} = L\{t\} \) and \( \frac{1}{s-1} = L\{e^t\} \). Then by convolution

\[
L^{-1}\left\{ \frac{1}{s^2(s-1)} \right\} = L^{-1}\left\{ L\{t\} L\{e^t\} \right\} = t * e^t
\]

\[
= \int_0^t (t-x)e^x \, dx
\]

\[
= \int_0^t e^x \, dx - \int_0^t xe^x \, dx
\]

\[
= t[e^t]_0^t - [xe^x - e^x]_0^t
\]

\[
= te^t - t - te^t + e^t - 1
\]

\[
= e^t - t - 1
\]
IX.1.4 Solution of IVP for ODE. The Laplace transform eliminates derivatives and it can be used for the solution of differential equations in semi-infinite domains, $0 \leq x < 0$. But because the Laplace transform of the derivatives includes values of the function and its derivatives at zero, the Laplace transform is more suitable for initial value problems rather than for boundary value problems like the Fourier transform.

Example 1  (Solution of IVP by the Laplace Transform)

Consider the 2nd order differential equation

$$y'' + 2y' - 2y = \sin t$$

with two initial conditions at $t = \frac{\pi}{2}$

$$y\left(\frac{\pi}{2}\right) = 0$$

$$y'\left(\frac{\pi}{2}\right) = 0$$

1) Translate initial conditions by the change of the variable $\tau = t - \frac{\pi}{2}$ to $\tau = 0$:

$$y'' + 2y' - 2y = \sin \left(\tau + \frac{\pi}{2}\right)$$

Then initial conditions are at $\tau = 0$:

$$y(0) = 0$$

$$y'(0) = 0$$

2) Apply the Laplace transform

$$Y(s) = \int_0^\infty y(\tau) e^{-\tau s} d\tau$$

to equation ($\diamond$):

$$s^2Y + sY - 2Y = \frac{s}{1 + s^2}$$

Solve for $Y$

$$Y(s) = \frac{s}{(1 + s^2)(s - 1)(s + 2)}$$

3) Then by inverse Laplace transform

$$y(\tau) = \frac{1}{6} e^{\frac{\pi}{2}} + \frac{2}{15} e^{-\tau} - \frac{3}{10} \cos \tau + \frac{1}{10} \sin \tau$$

Use back substitution $\tau = t - \frac{\pi}{2}$ to get the solution of the original IVP:

$$y(t) = \frac{1}{6} e^{\frac{\pi}{2}} + \frac{2}{15} e^{-2(t - \frac{\pi}{2})} - \frac{3}{10} \cos \left(t - \frac{\pi}{2}\right) + \frac{1}{10} \sin \left(t - \frac{\pi}{2}\right)$$

![Graph of y(t)]
Example 2  (Solution of IVP by the Laplace Transform)

Solve the 3rd order differential equation

$$y''' - 2y'' - y' + 2y = e^{-2t}$$  

subject to initial conditions at $t = 0$

$y(0) = 0$

$y'(0) = 0$

$y''(0) = 1$

1) Apply the Laplace transform $Y(s) = \int_{0}^{\infty} y(\tau) e^{-\tau s} d\tau$ to equation (◊◊):

$$[s^3Y - s^2y(0) - sy'(0) - y''(0)] - 2[s^2Y - sy(0) - y'(0)] - [sY - y(0)] + 2Y = \frac{1}{s+2}$$

$$s^3Y - 1 - 2s^2Y - sy + 2Y = \frac{1}{s+2}$$

$$(s^3 - 2s^2 - s + 2)Y = \frac{s + 3}{s + 2}$$

Solve for $Y$

$$Y = \frac{s + 3}{(s + 2)(s - 2)(s - 1)(s + 1)}$$

Convert this expression to partial fractions (see Example 6):

$$Y(s) = \frac{1}{3} \frac{1}{(s + 1)} - \frac{2}{3} \frac{1}{(s - 1)} - \frac{1}{12} \frac{1}{(s + 2)} + \frac{5}{12} \frac{1}{(s - 1)}$$

2) Then the solution of the IVP can be found by inverse Laplace transform

$$y(t) = \frac{1}{3} e^{-t} - \frac{2}{3} e^{t} - \frac{1}{12} e^{-2t} + \frac{5}{12} e^{2t}$$
IX.1.5 SOLUTION OF THE HEAT EQUATION IN SEMI-INFINITE REGIONS

a) Dirichlet Problem
Consider the Heat Equation in the semi-infinite slab for \( u(x,t) \)

\[
\frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial u}{\partial t} \quad x \in (0, \infty), \quad t > 0, \quad \alpha^2 = \frac{1}{\alpha}
\]

\[ u(x,0) = 0 \quad t = 0 \quad \text{initial condition} \]

\[ u(0,t) = f_0(t) \quad t > 0 \quad \text{Dirichlet b.c.} \]

\[ \lim_{x \to \infty} u(x,t) < \infty \quad t > 0 \quad \text{bounded solution} \]

Transformed equation
Apply the Laplace transform in the \( t \) variable

\[ \mathcal{L} \left\{ \frac{\partial u}{\partial t} (x,t) \right\} = sU(x,s) - u(x,0) \]

\[
\frac{\partial^2 U}{\partial x^2} = \alpha^2 sU \quad U(0,s) = F_0(s) \quad \text{where} \quad F_0(s) = \int_0^\infty f_0(t) e^{-st} dt
\]

This is the 2nd order linear ODE with constant coefficients. General solution

\[ U(x,s) = c_1 e^{-ax \sqrt{s}} + c_2 e^{ax \sqrt{s}} \]

Solution of the HE and correspondingly its Laplace transform should be bounded when \( x \to \infty \), therefore, we have to assign \( c_2 = 0 \):

\[ U(x,s) = c_1 e^{-ax \sqrt{s}} \]

Applying boundary conditions, one ends up with the solution for the transformed function

\[ U(x,s) = F_0(s) e^{-ax \sqrt{s}} \]

Note (see Maple Example 7) that the function \( e^{-ax \sqrt{s}} \) is a Laplace transform of the function

\[ g(t) = \frac{ax}{2\sqrt{\pi t^{3/2}}} e^{-\frac{a^2 x^2}{4t}} \]

Then the transformed solution is a product of two Laplace transforms:

\[ U(x,s) = F_0(s) G(s) \]

Apply the inverse Laplace transform using the convolution theorem

\[ u(x,t) = \mathcal{L}^{-1} \left\{ F_0(s) G(s) \right\} = f_0(t) * g(t) = \int_0^t f_0(\tau) g(t-\tau) d\tau \]

Then the solution is given in terms of convolution integral as

\[ u(x,t) = \frac{ax}{2\sqrt{\pi}} \int_0^t f_0(\tau) e^{-\frac{a^2 x^2}{4(t-\tau)}} (t-\tau)^{3/2} d\tau \]

Formal Solution of IVP:
Case of constant b.c. $f_0(t) = f_0$

Consider a case when the boundary condition specifies a constant temperature $f_0(t) = f_0$, then

$$u(x,t) = \frac{af_0x}{2\sqrt{\pi}} \int_0^\infty e^{\frac{-x^2}{4(t-\tau)}} \, d\tau$$

Make the substitution:

\[
\begin{align*}
z &= \frac{ax}{2(t-\tau)^{3/2}} \\
dz &= \frac{ax}{2} \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) (t-\tau)^{-1/2} \, d\tau \\
\tau &= 0 \quad z = \frac{ax}{2t^{3/2}} \\
\tau &= t \quad z = \infty
\end{align*}
\]

\[
\int_0^\infty e^{-z^2} \, dz = \frac{\sqrt{\pi}}{2}
\]

$$u(x,t) = \frac{2f_0}{\sqrt{\pi}} \int_0^\infty e^{-z^2} \, dz = f_0 \left[ 1 - \frac{2}{\sqrt{\pi}} \int_0^\frac{ax}{2\sqrt{t}} e^{-z^2} \, dz \right] = f_0 \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right)$$

see definition of $\text{erf}$ in VII.3, p.485. Then the solution is given by

$$u(x,t) = f_0 \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right)$$

Plot the solution for $a = 1$ and $f_0 = 1$

The temperature distribution in the semi-infinite layer approaches the steady state constant value of the boundary condition $f_0 = 1$.

Case of constant i.c.: $u(x,0) = u_0$

Change the variable: $\theta(x,t) = u(x,t) - u_0$

Initial condition: $\theta(x,0) = u(x,0) - u_0 = 0$

Boundary condition: $\theta(0,t) = u(0,t) - u_0 = f_0 - u_0$

Solution: $\theta(x,t) = (f_0 - u_0) \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right)$

$$u(x,t) = u_0 + (f_0 - u_0) \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right)$$
b) **Robin Problem**

Consider the Heat Equation in the semi-infinite slab for $u(x,t)$

\[
\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \quad x \in (0,\infty) \quad t > 0 \quad a^2 = \frac{1}{\alpha}
\]

\[u(x,0) = u_0 \quad t = 0 \quad \text{initial condition}\]

\[
\left[ -k \frac{\partial u}{\partial x} + hu \right]_{x=0} = hu_\infty \quad t > 0 \quad \text{Robin b.c.}
\]

\[\lim_{x \to \infty} u(x,t) < \infty \quad t > 0 \quad \text{bounded solution}\]

1) **Reduce to IBVP with zero initial condition:**

\[
\theta(x,t) = u(x,t) - u_0
\]

Change of variable: \[\theta(x,t) = u(x,t) - u_0\]

Initial condition: \[\theta(x,0) = u(x,0) - u_0 = 0\]

Boundary condition: \[
\left[ -k \frac{\partial \theta}{\partial x} + h \theta \right]_{x=0} = hu_\infty
\]

Limiting condition: \[\lim_{x \to \infty} \theta(x,t) < \infty\]

Heat Equation: \[
\frac{\partial^2 \theta}{\partial x^2} = a^2 \frac{\partial \theta}{\partial t}
\]

2) **Solution by Laplace transform.** Apply Laplace transform

Transformed equation: \[
\frac{\partial^2 \Theta}{\partial x^2} = a^2 s \Theta
\]

Boundary condition: \[
\left[ -\frac{\partial \Theta}{\partial x} + H \Theta \right]_{x=0} = \frac{H (u_\infty - u_0)}{s}
\]

Limiting condition: \[\lim_{x \to \infty} \Theta(x,s) < \infty\]

Solution: \[
\Theta(x,s) = c_1 e^{a \sqrt{s} x} + c_2 e^{-a \sqrt{s} x} \quad c_2 = 0
\]

\[
\Theta(x,s) = c_1 e^{-a \sqrt{s} x}
\]

Derivative: \[
\frac{\partial}{\partial x} \Theta(x,s) = -a \sqrt{s} c_1 e^{-a \sqrt{s} x}
\]

Substitute into b.c.: \[
c_1 \left[ a \sqrt{s} e^{-a \sqrt{s} x} + H e^{-a \sqrt{s} x} \right]_{x=0} = \frac{H (u_\infty - u_0)}{s}
\]

\[
c_1 \left[ a \sqrt{s} + H \right] = \frac{H (u_\infty - u_0)}{s}
\]

\[
c_1 = \frac{H (u_\infty - u_0)}{as \left( \sqrt{s} + \frac{H}{a} \right)}
\]
Transformed Solution: \[ \Theta(x, s) = (u_\infty - u_0) \frac{H}{s} e^{-ax\sqrt{s}} \]

Use Maple to find the inverse Laplace transform:

\[ \text{invlaplace}(\frac{H}{a} e^{-ax\sqrt{s}} / s / (\sqrt{s} + H/a), s, t); \]

Inverse Laplace transform:

\[ \theta(x, t) = -(u_\infty - u_0) e^{\frac{H}{a} \sqrt{t}} \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right) + (u_\infty - u_0) \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right) \]

Solution:

\[ u(x, t) = u_0 + (u_\infty - u_0) \left[ \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right) - e^{\frac{H}{a} \sqrt{t}} \text{erfc} \left( \frac{ax}{2\sqrt{t}} \right) \right] \]

Example:

\[ u_0 = 10, \; u_\infty = 50, \; a = 10, \; k = 5, \; h = 2 \]

\[ u(x, t) \]
c) Contact Problem

Consider the ideal thermal contact of two semi-infinite slabs of uniform conductivities $k_1$ and $k_2$, and coefficients of thermodiffusivity $\alpha_1$ and $\alpha_2$, correspondingly. Initially slabs are at the temperatures $T_1$ and $T_2$, then the transient temperature distribution is described by $u_1(x,t)$ and $u_2(x,t)$.

\[
\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial u_1}{\partial t} \quad x \in (-\infty, 0) \quad t > 0 \quad a_1^2 = \frac{1}{\alpha_1}
\]

\[
\frac{\partial^2 u_2}{\partial x^2} = \frac{1}{\alpha_2} \frac{\partial u_2}{\partial t} \quad x \in (0, \infty) \quad t > 0 \quad a_2^2 = \frac{1}{\alpha_2}
\]

**Initial conditions**

\[
u_1(x,0) = T_1 \quad t = 0
\]

\[
u_2(x,0) = T_2 \quad t = 0
\]

**Conjugate conditions:**

\[
u_1(0,t) = u_2(0,t) \quad t > 0 \quad \text{continuity}
\]

\[
k_1 \frac{\partial}{\partial x} u_1(0,t) = k_2 \frac{\partial}{\partial x} u_2(0,t) \quad t > 0 \quad \text{conservation of flux}
\]

Symmetric extension of $u_1(x,t)$

Solutions of the Dirichlet problem for the semi-infinite slabs:

\[
u_1(x,t) = T_1 + (T_0 - T_1) \text{erfc} \left( \frac{-a_1 x}{2\sqrt{t}} \right)
\]

\[
u_2(x,t) = T_2 + (T_0 - T_2) \text{erfc} \left( \frac{a_2 x}{2\sqrt{t}} \right)
\]

Here, the condition of temperature continuity at the contact is assumed with a constant temperature of the contact $T_0$:

\[
u_1(0,t) = u_2(0,t) = T_0
\]
Differentiate $u_1$ and $u_2$ with respect to $x$

$$
\frac{\partial}{\partial x} u_1(x,t) = \left( T_0 - T_1 \right) \left( \frac{2}{\sqrt{\pi}} \right) \left( \frac{-a_1}{2\sqrt{t}} \right) \exp \left[ -\left( \frac{a_1 x}{2\sqrt{t}} \right)^2 \right]
$$

$$
\frac{\partial}{\partial x} u_2(x,t) = \left( T_0 - T_2 \right) \left( \frac{2}{\sqrt{\pi}} \right) \left( \frac{a_2}{2\sqrt{t}} \right) \exp \left[ -\left( \frac{a_2 x}{2\sqrt{t}} \right)^2 \right]
$$

and apply condition of conservation of flux at the contact:

$$
k_i \left( T_0 - T_i \right) \left( \frac{1}{\sqrt{\pi}} \right) \left( \frac{a_i}{\sqrt{t}} \right) \exp \left[ -\left( \frac{a_i x}{2\sqrt{t}} \right)^2 \right] = -k_j \left( T_0 - T_j \right) \left( \frac{1}{\sqrt{\pi}} \right) \left( \frac{a_j}{\sqrt{t}} \right) \exp \left[ -\left( \frac{a_j x}{2\sqrt{t}} \right)^2 \right]
$$

at $x = 0$

$$
k_i \left( T_0 - T_i \right) \left( \frac{1}{\sqrt{\pi}} \right) \left( \frac{a_i}{\sqrt{t}} \right) = -k_j \left( T_0 - T_j \right) \left( \frac{1}{\sqrt{\pi}} \right) \left( \frac{a_j}{\sqrt{t}} \right)
$$

$$
a_j k_i \left( T_0 - T_i \right) = -a_i k_j \left( T_0 - T_j \right)
$$

Solve for contact temperature:

**Contact Temperature:**

$$
T_0 = \frac{a_i k_i T_1 + a_j k_j T_2}{a_j k_i + a_i k_j}
$$

Contact temperature is constant for $t > 0$.

**Example:**

![Graph showing temperature profiles over time and distance]

$t = 2$ $t = 10$

$T_0 = 24.3$ $T_1 = 10.0$ $T_2 = 30.0$ $k_j = 4.0$ $a_i = 1.0$ $k_2 = 5.0$ $a_2 = 2.0$
Consider vibration of the semi-infinite string

\[ \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x,t) \quad u(x,t) : \ x \in (0, \infty) \quad t > 0 \]

\[ g(x,t) \quad \text{force per unit length of the string} \]

\[ u(0,t) = f_0(t) \quad t > 0 \quad \text{Dirichlet boundary condition} \]

\[ u(x,0) = u_0(x) \quad t = 0 \quad \text{initial condition} \]

\[ \frac{\partial u(x,0)}{\partial t} = u_1(x) \quad t = 0 \quad \text{initial condition} \]

1) Transformed equation

Apply the Laplace transform in the \( t \) variable

\[ U(x,s) = \int_0^\infty u(x,t)e^{-st} \, dt \]

\[ G(x,s) = \int_0^\infty g(x,t)e^{-st} \, dt \]

\[ F_0(s) = \int_0^\infty f_0(x)e^{-st} \, dt \]

to the WE and boundary condition

\[ s^2 U - su_0 - u_1 = a^2 U^* + G \quad U(0,s) = F_0(s) \]

\[ U^* - \frac{s^2}{a^2} U = Q \quad \text{notation} \]

General solution:

\[ U = c_1 e^{\frac{s}{a} x} + c_2 e^{\frac{-s}{a} x} + U_p \]

where coefficients \( c_1, c_2 \) are, in general, functions of \( s \).

Hence the solution should be bounded, \( c_2 = 0 \), and the general solution becomes

\[ U = c_1 e^{\frac{s}{a} x} + U_p \]

Case \( u_0 = u_1 = G = 0 \)

Consider a case \( Q = 0 \) : initially the string is at rest, there is no external force. Then the solution becomes:

\[ U = F_0(s)e^{\frac{s}{a} x} \]

Note that

\[ L \left\{ \delta \left(t - \frac{x}{a}\right) \right\} = \int_0^\infty \delta \left(t - \frac{x}{a}\right)e^{-st} \, dt = \int_0^\infty \delta \left(t - \frac{x}{a}\right)e^{-\frac{s}{a} x} \, dt = e^{\frac{s}{a} x} \quad \text{for} \ a > 0 \]
2) Inverse transform:

\[ u(x,t) = L^{-1}\left( F_0(x)e^{-\alpha x} \right) = L^{-1}\left( L[f_0(t)] - L\left[ \delta\left( t - \frac{x}{a} \right) \right] \right) = \delta\left( t - \frac{x}{a} \right) * f_0(t) \]

Calculate convolution

\[ \delta\left( t - \frac{x}{a} \right) * f_0(t) = \int_0^t \delta\left( \tau - \frac{x}{a} \right) f_0(t - \tau) \, d\tau \]
\[ = \int_0^t \delta\left( \tau - \frac{x}{a} \right) f_0\left( t - \frac{x}{a} \right) \, d\tau = f_0\left( t - \frac{x}{a} \right) H\left( t - \frac{x}{a} \right) \]

The solution becomes:

\[ u(x,t) = f_0\left( t - \frac{x}{a} \right) H\left( t - \frac{x}{a} \right) \]

Example of time dependent b.c.

3) Consider the case of periodic b.c.: \[ f_0(t) = \sin t \] then

\[ u(x,t) = \sin \left( t - \frac{x}{a} \right) H\left( t - \frac{x}{a} \right) \]

Plot the solution (LT-2.mws):

Exercise

Consider different cases: with gravitational force, fixed end, initial shape and velocity

Coefficients in the Wave Equation:

\[ a^2 = \frac{gT}{w} \left[ \frac{m^2}{s^2} \right] \]

\[ a \] speed of propagation of sound waves in the medium;
\[ g \] acceleration of gravity;
\[ T \] tension;
\[ w \] weight of the string per unit length.
Recall the Volterra integral equation of the 2nd kind (see Chapter XI)
\[ u(x) = \lambda \int_0^x K(x,y)u(y)dy + f(x) \]
If the kernel of the integral equation \( K(x,y) \) is a function of \( x-y \)
\( K(x,y) = g(x-y) \)
then the integral equation is said to be a convolution integral equation:
\[ u(x) = \lambda \int_0^x g(x-y)u(y)dy + f(x) \]

We will consider application of the Laplace transform to the solution of
the convolution integral equation. The method is based on the
convolution theorem for the Laplace transform:
\[ L\{f * g\} = \mathcal{F} \cdot \mathcal{G} \]

Apply the Laplace transform to the convolution integral equation
\[ \bar{u}(s) = \lambda L\left\{ \int_0^x g(x-y)u(y)dy \right\} + \bar{f}(s) \]
\[ \bar{u}(s) = \lambda L\{g \ast u\} + \bar{f}(s) \]
\[ \bar{u}(s) = \lambda \mathcal{G}(s) \bar{u}(s) + \bar{f}(s) \]

Solve for the transformed function \( \bar{u}(s) \)
\[ \bar{u}(s) = \frac{\bar{f}(s)}{1-\lambda \mathcal{G}(s)} \]

Then the formal solution of the integral equation is given by the inverse
Laplace transform
\[ u(x) = \mathcal{L}^{-1}\left\{ \frac{\bar{f}(s)}{1-\lambda \mathcal{G}(s)} \right\} \]

Example 1
(non-homogeneous Volterra integral equation of the 2nd kind)

Solve the integral equation
\[ u(x) = a + \lambda \int_0^x u(y)dy \]

Apply the Laplace transform (use table property 49):
\[ L\left\{ \int_0^x u(y)dy \right\} = \frac{\bar{u}(s)}{s} \]
\[ \bar{u} = \frac{a}{s} + \lambda \frac{\bar{u}}{s} \]
Solve for \( \overline{u} \)
\[
\overline{u} = \frac{a}{s - \lambda}
\]

Then the solution is given by the inverse Laplace transform
\[
u(x) = \mathcal{L}^{-1}\left\{ \frac{a}{s - \lambda} \right\} = ae^{\lambda x}
\]

**Example 2**

*(Abel’s integral equation)*

Solve the integral equation:

\[
\text{Abel integral equation} \quad f(x) = \int_0^x u'(y)(x - y)^{-a} \, dy \quad 0 < a < 1
\]

Rewrite \((AE)\) in the form

\[
f(x) = \int_0^x u'(y)(x - y)^{-a} \, dy
\]

and apply the Laplace transform

Table LT: \[
L\left\{ x^a \right\} = \frac{\Gamma(1-a)s^{-a}}{\Gamma(1-a)s^{-a}}
\]

\[
L\left\{ x^{a-1} \right\} = \Gamma(a)s^{-a}
\]

\[
\mathcal{F}(s) = \mathcal{L}\left\{ \int_0^x u'(y)(x-y)^{-a} \, dy \right\} = \mathcal{L}\{u'(y)x^{-a}\}
\]

\[
= \mathcal{L}\{u'(x)\} L\{x^{-a}\}
\]

\[
= [s\overline{u}(s) - u(0)] \Gamma(1-a)s^{a-1}
\]

Solve for the transformed unknown function

\[
\overline{u}(s) = \frac{\mathcal{F}(s)}{\Gamma(1-a)s^{-a}} + \frac{u(0)}{s}
\]

\[
= \frac{\mathcal{F}(s)}{\Gamma(1-a)} \frac{1}{\Gamma(a)} \left[ \Gamma(a)s^{-a} \right] + \frac{u(0)}{s}
\]

\[
= \frac{1}{\Gamma(1-a)\Gamma(a)} \mathcal{L}\{f(x)\} L\{x^{a-1}\} + \frac{u(0)}{s}
\]

Then application of the inverse Laplace transform and convolution theorem, yields a formal solution for the Abel integral equation

\[
u(x) = u(0) + \frac{1}{\Gamma(1-a)\Gamma(a)} \int_0^x (x-y)^{a-1} f(y) \, dy
\]
IX.1.8 REVIEW QUESTIONS AND EXERCISES

QUESTIONS
1) How is the Laplace transform defined?
2) What condition guarantees the existence of the Laplace transform?
3) How is the inverse Laplace transform defined?
4) What are the main properties of the Laplace transform and the inverse Laplace transform?
5) How can the convolution theorem be applied for evaluation of the inverse Laplace transform?
6) How can the inverse Laplace transform of the rational functions be found?
7) What property allows application of the Laplace transform for solution of differential equations?
8) What are the main steps in the procedure of application of the Laplace transform for solution of differential equations?

EXERCISES
1. a) Derive the similarity property \( L \{ f(at) \} = \frac{L}{a}(\frac{s}{a}) \).

   b) Prove that convolution commutes: \( f \ast g = g \ast f \).

   c) Using integration by parts derive \( L \{ f^{*}(t) \} = s \phi(s) - s f(0) - f'(0) \).

2. Evaluate using the definition of the Laplace transform:
   a) \( L \{ sin(2t) \} \)
   b) \( L \{ te^{2t} \} \)
3. Evaluate using the Table and the properties of the Laplace transform:
   a) \( L\{t^2 e^{3t}\} \)
   b) \( L\{(t+5)^3\} \)
   c) \( L\{(t-1)e^{2t}\} \)
   d) \( L\{te^{-t}\sin t\} \)
   e) \( L\{e^{2t}\cos 2t\} \)
   f) \( L\{f(t)\}, \quad f(t) = \begin{cases} \cos t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases} \)

4. Evaluate the following inverse Laplace transforms using partial fractions:
   a) \( L^{-1}\left[\frac{3s+1}{(s^2-9)(s+1)}\right] \)
   b) \( L^{-1}\left[\frac{2s-1}{(s-3)(s-1)}\right] \)
   c) \( L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] \)
   d) \( L^{-1}\left[\frac{1}{(s^2+9)^2}\right] \)

5. Solve the following Initial Value Problems with the help of Laplace transform and plot the graph of solution:
   a) \( y'' + y' - 6y = 3 \quad y(0) = 0 \quad y'(0) = 2 \)
   b) \( y''' + 2y'' - 2y = \sin 3t \quad y(0) = 0 \quad y'(0) = 0 \quad y''(0) = 1 \)
   c) \( y'' + y = \begin{cases} 1, & 0 \leq t < 3\pi \\ 0, & t \geq 3\pi \end{cases} \quad y(0) = 0 \quad y'(0) = 0 \)
   d) \( y'' + 2y' + 2y = h(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases} \quad y(0) = 0, \quad y'(0) = 1 \)
   e) \( y'' + y = \begin{cases} 0, & 0 \leq t < \pi/2 \\ 1, & t \geq \pi/2 \end{cases} \quad y(0) = 0, \quad y'(0) = 1 \)
   f) \( y'' + 4y = 2\delta\left(t - \frac{\pi}{4}\right) \quad y(0) = 0, \quad y'(0) = 0 \)
   g) \( y'' + 2y' = \delta(t - 1) \quad y(0) = 0, \quad y'(0) = 0 \)

6. Solve the Neumann Initial-Boundary Value Problem for \( u(x,t) \) and plot the graph of solution for your choice of parameters:
   \( \frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial u}{\partial t} \quad x \in (0,\infty) \quad t > 0 \)
   \( u(x,0) = T_0 \)
   \( -k \frac{\partial}{\partial x} u(0,t) = q^*_0 \quad t > 0 \)
   \( \lim_{x \to \infty} u(x,t) < \infty \quad \text{bounded solution} \)
7. Solve the Initial-Boundary Value Problem for \( u(x,t) \) with a time-dependent periodic boundary condition:

\[
\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t} \quad x \in (0, L) \quad t > 0
\]

\[
u(x,0) = T_0
\]

\[
-k \frac{\partial}{\partial x} u(0,t) + h u(0,t) = T_n \sin(\omega t) \quad t > 0
\]

\[
\frac{\partial}{\partial x} u(L,t) = 0
\]

Choose some values for the constants and sketch the solution curves.

8. Exercise on p.713.
IX.1.9 LAPLACE TRANSFORM WITH MAPLE

1) Evaluation of the Laplace Transform using the definition (for $s > 0$):

Example 1:

```maple
> f(t) := t; assume(s > 0);
> phi(s) := int(f(t)*exp(-s*t), t=0..infinity);
```

2) Evaluation of the Laplace Transform and the inverse Laplace transform with the help of commands in the package `inttrans`:

```maple
> with(inttrans):

Example 2:

```maple
> laplace(exp(t), t, s);
```

Example 3:

```maple
> laplace(t^2*sin(t), t, s);
```

Example 4:

```maple
> invlaplace(s/(s^2+9), s, t);
```

Example 5:

```maple
> invlaplace(exp(-2*s)/s, s, t);
```

Example 6:

```maple
> Y := (s+3)/(s+2)/(s-2)/(s-1)/(s+1);
> y(t) := invlaplace(Y, s, t);
```

Example 7:

```maple
> assume(a>0): assume(x>0):
> G(s) := exp(-a*x*sqrt(s));
> invlaplace(G(s), s, t);
```