IX.3 Finite Fourier Transform

IX.3.1 FINITE FOURIER TRANSFORM IN CARTESIAN COORDINATES

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1. Differential operator

The unknown function $u(x,y,z,t)$ is generally a function of spatial variables and the time variable t. Let the variable x belong to a finite interval

$$0 \le x \le L$$

Our objective is to construct an integral transform that eliminates the second-order derivative with respect to the variable x

Differential operator

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} \tag{1}$$

To do so, we need to establish the ***operational property*** of transforming the considered differential operator Lu in the case of non-homogeneous boundary conditions imposed on the function u

boundary conditions

$$x = 0 \qquad \left[-k_1 \frac{\partial u}{\partial x} + h_1 u \right]_{x=0} = f_0 \tag{2}$$

$$x = L \qquad \left[+k_2 \frac{\partial u}{\partial x} + h_2 u \right]_{x=L} = f_L \tag{3}$$

where functions $f_0(y,z,t)$ and $f_L(y,z,t)$ can depend on the other variables involved in the problem.

We will follow the outline of construction of the generalized finite integral transform (Section IX.5.5, p.862).

2. Sturm-Liouville Problem

Consider the supplemental eigenvalue problem for operator L

$$X'' = \lambda X \tag{4}$$

for function $X(x)$ subject to homogeneous boundary conditions of the same kind as those for the function u (Equations 2 and 3)

$$x = 0 \qquad -k_1 X(0) + h_1 X(0) = 0 \tag{5}$$

$$x = L \qquad +k_2 X(L) + h_2 X(L) = 0 \tag{6}$$

The Sturm-Liouville form of the equation (4) is

$$(1 \cdot X')' + (0 - \lambda \cdot 1) X = 0 \tag{7}$$

where the coefficients can be identified as $r(x) = 1,\ q(x) = 0,\ p(x) = 1$.

According to the Sturm-Liouville Theorem, for the existence of non-trivial solutions to equation (7), the parameter λ should be non-negative

$$\lambda = -\mu^2$$

eigenvalue problem

Therefore, eigenvalue problem is reduced to solution of the equation

$$X'' + \mu^2 X = 0$$

subject to homogeneous boundary conditions (5 and 6).

Eigenvalues and eigenfunctions

This eigenvalue problem (4-6) generates infinitely many eigenvalues $0 = \mu_0 < \mu_1 < \mu_2 < ...$ and corresponding eigenfunctions $X_0 = 1, X_1, X_2, ...$

Solution of eigenvalue problem for different combination of boundary conditions are summarized in the Table Sturm-Liouville Problem (p.448).

Inner product with the weight function $p(x) = 1$ is defined as

Inner product

$$(u,v) = \int_0^L u(x)v(x)dx \tag{8}$$

Then the square of the *norm* of the functions is defined by

$$\| u(x) \|^2 = (u,u) = \int_0^L u^2(x)dx \tag{9}$$

Vector Space

$L_2(0,L)$ with the defined inner product and norm is a Hilbert space.

Orthogonality of eigenfunctions $X_0, X_1, X_2, ...$ in terms of inner product:

Orthogonality

$$(X_n, X_m) = \int_0^L X_n(x) X_m(x) dx = 0 \text{ if } n \neq m$$

Generalized Fourier series representation of the functions $f \in L_2(0,L)$ based on the eigenfunctions X_n is as follows:

Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n X_n(x), \qquad a_n = \frac{(f, X_n)}{(X_n, X_n)^2} = \frac{(f, X_n)}{\| X_n \|^2} \tag{10}$$

3. Finite Fourier Transform

The integral transform pair of the direct and inverse transforms of the functions $u(x) \in L_2(0,L)$ is based on the Fourier series representation (10) and can be defined in the following form

Finite Fourier Transform	$F\{u(x)\} = \int_0^L u(x) X_n(x) dx = \bar{u}_n$		(11)
Inverse Transform	$F^{-1}\{\bar{u}_n\} = \sum_n^{\infty} \bar{u}_n \frac{X_n(x)}{\| X_n \|^2} = u(x)$		(12)

Particular form of the Finite Fourier transform and its operational properties for different types of boundary conditions are summarized in the following table.

Finite Fourier Transform	
	$\bar{u}_n = \int_0^L u(x)\, X_n(x)\, dx$ *direct transform*
	$u(x) = \sum_{n=1}^{\infty} \bar{u}_n \dfrac{X_n(x)}{\lVert X_n(x) \rVert^2}$ *inverse transform*

Boundary Conditions for $u(x)$	**Eigenfunctions** $X_n(x)$ are solutions of the eigenvalue problem
$\left[-k_1 u' + h_1 u\right]_{x=0} = f_0 \qquad H_1 = \dfrac{h_1}{k_1}$	$X'' + \mu^2 X = 0\,, \qquad -k_1 X'(0) + h_1 X(0) = 0$
$\left[\ k_2 u' + h_2 u\right]_{x=L} = f_L \qquad H_2 = \dfrac{h_2}{k_2}$	$+k_1 X'(L) + h_1 X(L) = 0$

Boundary conditions	Eigenvalues μ_n	Eigenfunctions $X_n(x)$	Norm $\lVert X_n(x) \rVert^2$	Operational property $\int_0^L \left(\dfrac{\partial^2 u}{\partial x^2}\right) X_n(x)\, dx$
D $u(0) = f_0$ **D** $u(L) = f_L$	$\dfrac{n\pi}{L}\,,\ n = 1,2,\ldots$	$\sin(\mu_n x)$	$\dfrac{L}{2}$	$-\mu_n^2 \bar{u}_n + f_0 X_n'(0) - f_L X_n'(L)$ $X_n'(0) = \mu_n$ $X_n'(L) = \mu_n \cos(\mu_n L)$
N $u'(0) = f_0$ **D** $u(L) = f_L$	$\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}\,,\ n = 0,1,\ldots$	$\cos(\mu_n x)$	$\dfrac{L}{2}$	$-\mu_n^2 \bar{u}_n - f_0 X_n(0) - f_L X_n'(L)$ $X_n(0) = 1$ $X_n'(L) = -\mu_n \sin(\mu_n L)$
D $u(0) = f_0$ **N** $u'(L) = f_L$	$\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}\,,\ n = 0,1,\ldots$	$\sin(\mu_n x)$	$\dfrac{L}{2}$	$-\mu_n^2 \bar{u}_n + f_0 X_n'(0) + f_L X_n(L)$ $X_n'(0) = \mu_n$ $X_n(L) = \sin(\mu_n L)$
N $u'(0) = f_0$ **N** $u'(L) = f_L$	$\mu_0 = 0$ $\dfrac{n\pi}{L}\,,\ n = 1,2,\ldots$	$X_0 = 1$ $\cos(\mu_n x)$	$L, \quad n = 0$ $\dfrac{L}{2}\,,\ n = 1,2,\ldots$	$f_L - f_0\,, \qquad n = 0$ $-\mu_n^2 \bar{u}_n - f_0 X_n(0) + f_L X_n(L)$ $X_n(0) = 1$ $X_n(L) = \cos(\mu_n L)$

D $u(0)=f_0$ **R** $k_2u'(L)+h_2u(L)=f_L$	μ_n *are positive root of* $\mu\cos\mu L+H_2\sin\mu L=0$	$\sin(\mu_n x)$	$\frac{L}{2}-\frac{\sin 2\mu_n L}{4\mu_n}$	$-\mu_n^2\bar{u}_n+f_0X_n'(0)+\frac{f_L}{k_2}X_n(L)$ $X_n'(0)=\mu_n$ $X_n(L)=\sin(\mu_n L)$
R $-k_1u'(0)+h_1u(0)=f_0$ **D** $u(L)=f_L$	μ_n *are positive root of* $\mu\cos\mu L+H_1\sin\mu L=0$	$\sin(\mu_n(x-L))$	$\frac{L}{2}-\frac{\sin 2\mu_n L}{4\mu_n}$	$-\mu_n^2\bar{u}_n+\frac{f_0}{k_1}X_n(0)-f_LX_n'(L)$ $X_n'(0)=-\sin(\mu_n L)$ $X_n'(L)=\mu_n$
N $u'(0)=f_0$ **R** $k_2u'(L)+h_2u(L)=f_L$	μ_n *are positive root of* $\mu\sin\mu L-H_2\cos\mu L=0$	$\cos\mu_n x$	$\frac{L}{2}+\frac{\sin(2\mu_n L)}{4\mu_n}$	$-\mu_n^2\bar{u}_n+\frac{f_0}{k_1}X_n(0)+f_LX_n(L)$ $X_n(0)=1$ $X_n(L)=\cos(\mu_n L)$
R $-k_1u'(0)+h_1u(0)=f_0$ **N** $u'(L)=f_L$	μ_n *are positive root of* $\mu\sin\mu L-H_1\cos\mu L=0$	$\cos\mu_n(x-L)$	$\frac{L}{2}+\frac{\sin(2\mu_n L)}{4\mu_n}$	$-\mu_n^2\bar{u}_n+f_LX_n(L)+\frac{f_0}{k_1}X_n(0)$ $X_n(0)=\cos(\mu_n L)$ $X_n(L)=1$
R $-k_1u'(0)+h_1u(0)=f_0$ **R** $k_2u'(L)+h_2u(L)=f_L$	*Eigenvalues* μ_n *are positive roots of* $(H_1H_2-\mu^2)\sin\mu L+(H_1+H_2)\mu\cos\mu L=0$ *Eigenfunctions:* $X_n=\mu_n\cos\mu_n x+H_1\sin\mu_n x$ $\lVert X_n\rVert^2=\frac{(\mu_n^2+H_1^2)}{2}\left(L+\frac{H_2}{\mu_n^2+H_2^2}\right)+\frac{H_1}{2}$			$-\mu_n^2\bar{u}_n+\frac{f_0}{k_1}X_n(0)+\frac{f_L}{k_2}X_n(L)$ $X_n(0)=\mu_n$ $X_n(L)=\mu_n\cos\mu_n L+H_1\sin\mu_n L$

4. Derivation of the Operational Properties of the Finite Fourier Transform

$$F\left\{\frac{\partial^2}{\partial x^2}u(x)\right\}=\int_0^L\left[\frac{\partial^2 u(x)}{\partial x^2}\right]X_n(x)\,dx \quad =\int_0^L X_n(x)\,d\left[\frac{\partial u(x)}{\partial x}\right]$$

$$=\left[X_n(x)\frac{\partial u(x)}{\partial x}\right]_0^L-\int_0^L\frac{\partial u(x)}{\partial x}X_n'(x)\,dx$$

$$=\left[X_n(x)\frac{\partial u(x)}{\partial x}\right]_0^L-\int_0^L X_n'(x)\,du(x)$$

$$=\left[X_n(x)\frac{\partial u(x)}{\partial x}\right]_0^L-\left[X_n'(x)u(x)\right]_0^L+\int_0^L u(x)X_n''(x)\,dx$$

$$=\left[X_n(x)\frac{\partial u(x)}{\partial x}\right]_0^L-\left[X_n'(x)u(x)\right]_0^L-\mu_n^2\int_0^L u(x)X_n(x)\,dx$$

$$=\left[X_n(x)\frac{\partial u(x)}{\partial x}\right]_0^L-\left[X_n'(x)u(x)\right]_0^L-\mu_n^2\,\overline{u}_n$$

$$\int_0^L\left[\frac{\partial^2 u(x)}{\partial x^2}\right]X_n(x)\,dx \qquad =X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+X_n'(0)u(0)-\mu_n^2\,\overline{u}_n \tag{14}$$

Derivation of operational properties based on equation (14) with non-homogeneous boundary conditions for the function $u(x)$ from the table:

D $X_n(0)=0$
D $X_n(L)=0$

$$\int_0^L\left[\frac{\partial^2 u(x)}{\partial x^2}\right]X_n(x)\,dx \;=X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+X_n'(0)u(0)\;-\mu_n^2\,\overline{u}_n$$

$$=\cancel{X_n(L)}\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+\cancel{X_n'(0)}u(0)\;-\mu_n^2\,\overline{u}_n$$

$$=-f_0X_n(0)-f_L\,X_n'(L)-\mu_n^2\,\overline{u}_n$$

N $X_n'(0)=0$
D $X_n(L)=0$

$$\int_0^L\left[\frac{\partial^2 u(x)}{\partial x^2}\right]X_n(x)\,dx \;=X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+X_n'(0)u(0)\;-\mu_n^2\,\overline{u}_n$$

$$=\cancel{X_n(L)}\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+\cancel{X_n'(0)}u(0)\;-\mu_n^2\,\overline{u}_n$$

$$=-f_0X_n(0)-f_L\,X_n'(L)-\mu_n^2\,\overline{u}_n$$

D $X_n(0)=0$

N $X_n'(L)=0$

$$\int_0^L \left[\frac{\partial^2 u(x)}{\partial x^2}\right] X_n(x)\,dx = f_L X_n(L) - \cancel{X_n(0)}\frac{\partial u(0)}{\partial x} - \cancel{X_n'(L)}u(L) + f_0 X_n'(0) - \mu_n^2\,\overline{u}_n$$

$$= f_L X_n(L) + f_0 X_n'(0) - \mu_n^2\,\overline{u}_n$$

N $X_n'(0)=0$

N $X_n'(L)=0$ $\quad \int_0^L \left[\frac{\partial^2 u(x)}{\partial x^2}\right] X_0(x)\,dx = 0$

$$\int_0^L \left[\frac{\partial^2 u(x)}{\partial x^2}\right] X_n(x)\,dx = f_L X_n(L) - f_0 X_n(0) - \cancel{X_n'(L)}u(L) + \cancel{X_n'(0)}u(0) - \mu_n^2\,\overline{u}_n$$

$$= f_L X_n(L) - f_0 X_n(0) - \mu_n^2\,\overline{u}_n \qquad n = 1,2,\dots\ ,$$

D $X_n(0)=0$

R $X_n'(L)+\frac{h_2}{k_2}X_n(L)=0 \;\Rightarrow\; X_n'(L)=-\frac{h_2}{k_2}X_n(L)$

$$\int_0^L \left[\frac{\partial^2 u(x)}{\partial x^2}\right] X_n(x)\,dx = X_n(L)\frac{\partial u(L)}{\partial x} - X_n(0)\frac{\partial u(0)}{\partial x} - X_n'(L)u(L) + X_n'(0)u(0) - \mu_n^2\,\overline{u}_n$$

$$= X_n(L)\left[\frac{\partial u(L)}{\partial x} + \frac{h_2}{k_2}u(L)\right] - X_n(0)\frac{\partial u(0)}{\partial x} + X_n'(0)u(0) - \mu_n^2\,\overline{u}_n$$

$$= \frac{1}{k_2}X_n(L)\underbrace{\left[k_2\frac{\partial u(L)}{\partial x} + h_2 u(L)\right]}_{f_2} - \cancel{X_n(0)}\frac{\partial u(0)}{\partial x} + X_n'(0)\underbrace{u(0)}_{f_1} - \mu_n^2\,\overline{u}_n$$

$$= \frac{f_2}{k_2}X_n(L) + f_1 X_n'(0) - \mu_n^2\,\overline{u}_n$$

R $-X_n'(0)+\frac{h_1}{k_1}X_n(0)=0 \;\Rightarrow\; X_n'(0)=\frac{h_1}{k_1}X_n(0)$

D $X_n(L)=0$

$$\int_0^L \left[\frac{\partial^2 u(x)}{\partial x^2}\right] X_n(x)\,dx = X_n(L)\frac{\partial u(L)}{\partial x} - X_n(0)\frac{\partial u(0)}{\partial x} - X_n'(L)u(L) + X_n'(0)u(0) - \mu_n^2\,\overline{u}_n$$

$$= X_n(L)\frac{\partial u(L)}{\partial x} - X_n(0)\frac{\partial u(0)}{\partial x} - X_n'(L)u(L) + \frac{h_1}{k_1}X_n(0)u(0) - \mu_n^2\,\overline{u}_n$$

$$= \cancel{X_n(L)}\frac{\partial u(L)}{\partial x} + \frac{1}{k_1}X_n(0)\underbrace{\left[-k_1\frac{\partial u(0)}{\partial x} + h_1 u(0)\right]}_{f_0} - X_n'(L)\underbrace{u(L)}_{f_L} - \mu_n^2\,\overline{u}_n$$

$$= \frac{f_0}{k_1}X_n(0) - f_L X_n'(L) - \mu_n^2\,\overline{u}_n$$

N $X_n'(0)=0$

R $X_n'(L)+\frac{h_2}{k_2}X_n(L)=0 \Rightarrow X_n'(L)=-\frac{h_2}{k_2}X_n(L)$

$$\begin{aligned}\int_0^L\left[\frac{\partial^2u(x)}{\partial x^2}\right]X_n(x)dx &= X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+X_n'(0)u(0)-\mu_n^2\bar{u}_n\\ &= X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}+\frac{h_2}{k_2}X_n(L)u(L)+X_n'(0)u(0)-\mu_n^2\bar{u}_n\\ &= \frac{1}{k_2}X_n(L)\underbrace{\left[k_2\frac{\partial u(L)}{\partial x}+h_2u(L)\right]}_{f_L}-X_n(0)\underbrace{\frac{\partial u(0)}{\partial x}}_{f_0}+\cancel{X_n'(0)}u(0)-\mu_n^2\bar{u}_n\\ &= \frac{f_L}{k_2}X_n(L)-f_0X_n(0)-\mu_n^2\bar{u}_n\end{aligned}$$

R $-X_n'(0)+\frac{h_1}{k_1}X_n(0)=0 \Rightarrow X_n'(0)=\frac{h_1}{k_1}X_n(0)$

N $X_n'(L)=0$

$$\begin{aligned}\int_0^L\left[\frac{\partial^2u(x)}{\partial x^2}\right]X_n(x)dx &= X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+X_n'(0)u(0)-\mu_n^2\bar{u}_n\\ &= X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+\frac{h_1}{k_1}X_n(0)u(0)-\mu_n^2\bar{u}_n\\ &= X_n(L)\underbrace{\frac{\partial u(L)}{\partial x}}_{f_L}+\frac{1}{k_1}X_n(0)\underbrace{\left[-k_1\frac{\partial u(0)}{\partial x}+h_1u(0)\right]}_{f_0}-\cancel{X_n'(L)}u(L)-\mu_n^2\bar{u}_n\\ &= f_LX_n(L)+\frac{f_0}{k_1}X_n(0)-\mu_n^2\bar{u}_n\end{aligned}$$

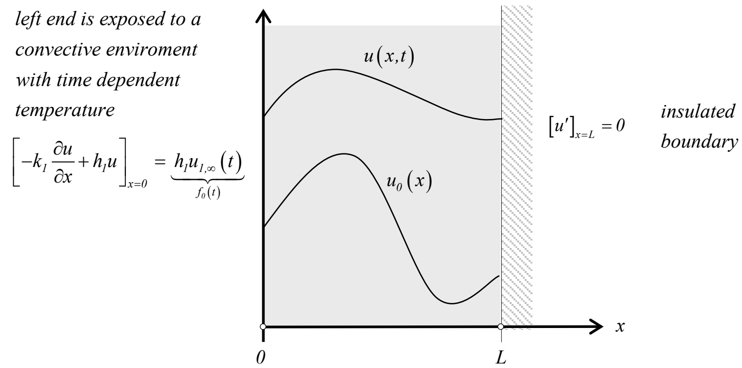
R $-X_n'(0)+\frac{h_1}{k_1}X_n(0)=0 \Rightarrow X_n'(0)=\frac{h_1}{k_1}X_n(0)$

R $X_n'(L)+\frac{h_2}{k_2}X_n(L)=0 \Rightarrow X_n'(L)=-\frac{h_2}{k_2}X_n(L)$

$$\begin{aligned}\int_0^L\left[\frac{\partial^2u(x)}{\partial x^2}\right]X_n(x)dx &= X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}-X_n'(L)u(L)+X_n'(0)u(0)-\mu_n^2\bar{u}_n\\ &= X_n(L)\frac{\partial u(L)}{\partial x}-X_n(0)\frac{\partial u(0)}{\partial x}+\frac{h_2}{k_2}X_n(L)u(L)+\frac{h_1}{k_1}X_n(0)u(0)-\mu_n^2\bar{u}_n\\ &= \frac{1}{k_2}X_n(L)\underbrace{\left[k_2\frac{\partial u(L)}{\partial x}+h_2u(L)\right]}_{f_L}+\frac{1}{k_1}X_n(0)\underbrace{\left[-k_1\frac{\partial u(0)}{\partial x}+h_1u(0)\right]}_{f_0}-\mu_n^2\bar{u}_n\\ &= \frac{f_L}{k_2}X_n(L)+\frac{f_0}{k_1}X_n(0)-\mu_n^2\bar{u}_n\end{aligned}$$

IX.3.2 Example Heat Equation

Heat conduction in the 1-dimensional slab with heat generation



Equation: $$\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + S(x,t), \qquad 0 < x < L \qquad t > 0$$

Initial condition: $$u(x,0) = u_0(x) \qquad 0 \le x \le L$$

Boundary conditions: $$\left[-k_1 \frac{\partial u}{\partial x} + h_1 u\right]_{x=0} = h_1 u_{1,\infty}(t) \quad = f_0(t) \qquad \textit{Robin}$$

$$\left[\frac{\partial}{\partial x} u\right]_{x=L} = f_L(t) \quad = 0 \qquad \textit{Neumann}$$

1) Integral transform

According to the table FFT, the kernel of the integral transform (eigenfunctions) corresponding to Robin and Neumann boundary conditions is:

$X_n = \cos \mu_n (x - L)$,

where eigenvalues μ_n are the positive roots of characteristic equation

$\mu \sin \mu L - H_1 \cos \mu L = 0$

2) Transformed equation

According to operational property from the table FFT (R-N), the second derivative of $u(x,t)$ w.r.t. x is transformed to

$$\int_0^L \left[\frac{\partial^2 u}{\partial x^2}\right] X_n(x)\,dx \qquad = -\mu_n^2 \bar{u}_n + f_L X_n(L) + \frac{f_0}{k_1} X_n(0)$$

Apply the Finite Fourier integral transform to the Heat Equation:

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\bar{u}_n(t) = \cancel{f_L} X_n(L) + \frac{f_0(t)}{k_1} X_n(0) - \mu_n^2 \bar{u}_n(t) + \bar{S}_n(t) \qquad \text{(T)}$$

and

$$\bar{S}_n(t) = \int_0^L S(x,t) X_n(x)\,dx$$

is the transformed source function.

Transformation of the initial condition:

$$\overline{u}_n(0)=\int_0^L u_0(x)X_n(x)dx=\overline{u}_{0,n}$$

Apply Laplace transform to the equation (T):

$$\frac{s}{\alpha}U_n(s)-\frac{1}{\alpha}\overline{u}_{0,n}=\frac{1}{k_1}\hat{f}_0(s)X_n(0)-\mu_n^2U_n(s)+\hat{\overline{S}}_n(s),\qquad \text{where } X_n(0)=cos(\mu_nL)$$

where the Laplace transformation is denoted by $U_n(s)=L\{\overline{u}_n(t)\}$, $\hat{f}_0(s)=L\{f_0(t)\}$ $\hat{\overline{S}}_n(s)=L\{\overline{S}_n(t)\}$.

3) Solve for transformed function

$$U_n(s)=\overline{u}_{0,n}\frac{1}{s+\alpha\mu_n^2}+\frac{\alpha}{k_1}X_n(0)\hat{f}_0(s)\frac{1}{s+\alpha\mu_n^2}+\alpha\hat{\overline{S}}_n(s)\frac{1}{s+\alpha\mu_n^2}$$

Note that $\frac{1}{s+\alpha\mu_n^2}=L\left\{e^{-\alpha\mu_n^2t}\right\}$.

Apply inverse Laplace transform

$$\overline{u}_n(t)=L\{U_n(s)\}=\overline{u}_{0,n}e^{-\alpha\mu_n^2t}+\frac{\alpha}{k_1}X_n(0)\left[f_0(t)*e^{-\alpha\mu_n^2t}\right]+\alpha\left[\overline{S}_n(t)*e^{-\alpha\mu_n^2t}\right]$$

which using convolution integral can be written as

$$\overline{u}_n(t)=L^{-1}\{U_n(s)\}=\overline{u}_{0,n}e^{-\alpha\mu_n^2t}+\frac{\alpha}{k_1}X_n(0)\left[\int_{\tau=0}^{t}f_0(\tau)e^{-\alpha\mu_n^2(t-\tau)}d\tau\right]+\alpha\left[\int_{\tau=0}^{t}\overline{S}_n(\tau)e^{-\alpha\mu_n^2(t-\tau)}d\tau\right]$$

Then the solution of the given initial-boundary value problem is found by the inverse Finite Fourier Transform:

$$u(x,t)=\sum_{n=1}^{\infty}\overline{u}_n\frac{X_n(x)}{\|X_n(x)\|^2}$$

Example The particular case of periodic surrounding temperature and zero source function and initial condition

$f_0(t)=h_1f_0\sin\omega t$, $S(x,t)=0$, $u_0(x)=0$, and $f_L=0$

Then transformed solution can be written as

$$\overline{u}_n(t)=\frac{\alpha}{k_1}X_n(0)\left[\int_{\tau=0}^{t}f_0h_1\sin(\omega\tau)e^{-\alpha\mu_n^2(t-\tau)}d\tau\right]$$

After evaluation of convolution integral the transformed function becomes

$$\overline{u}_n(t)=\frac{\alpha}{k_1}f_0h_1X_n(0)\frac{\alpha\mu_n^2\sin(\omega t)-\omega\cos(\omega t)+\omega e^{-\alpha\mu_n^2t}}{\omega^2+\alpha^2\mu_n^4}$$

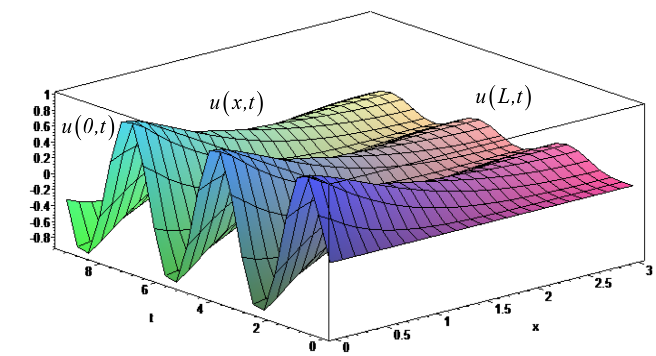
which reaches the periodic quasi-steady state

$$\overline{u}_n(t)=\frac{\alpha}{k_1}f_0\,h_1\cos(\mu_nL)\,\frac{\alpha\mu_n^2\sin(\omega t)-\omega\cos(\omega t)}{\omega^2+\alpha^2\mu_n^4}$$

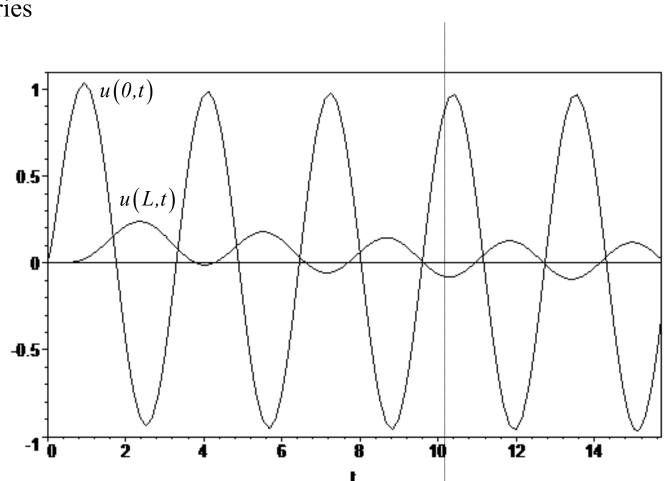
Solution of this problem is shown on the next figures.

FIT-01 RN .mws **<u>Heat Equation in the finite layer, Robin - Neumann boundary conditions</u>**

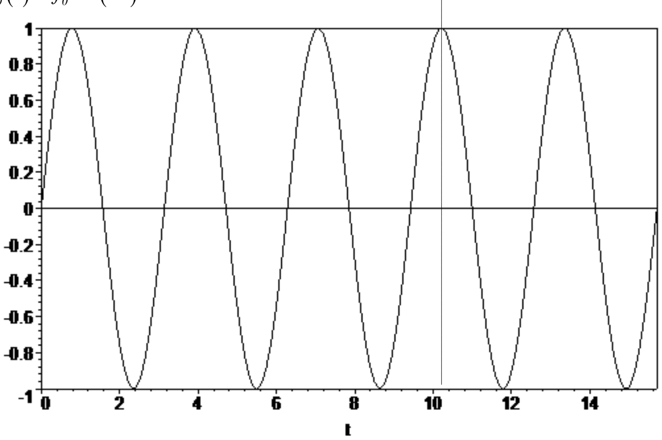
Maple solution with $L = 3,\ k_1 = 2,\ h_1 = 4,\ \alpha = 1,\ f_0(t) = f_0 \sin \omega t,\ f_0 = 1,\ \omega = 2$



Temperature at the boundaries



External temperature $f_0(t) = f_0 \sin(\omega t)$



IX.3.3 Flash Method – Estimation directe de propriétés thermophysiques de matériaux orthotropes

Institut Supérieur de l'Aeronautique et de l'Espace

Université de Poitiers

École Nationale Supérieure de Mécanique et d'Aérotechnique

Estimation directe de propriétés thermophysiques de matériaux orthotropes

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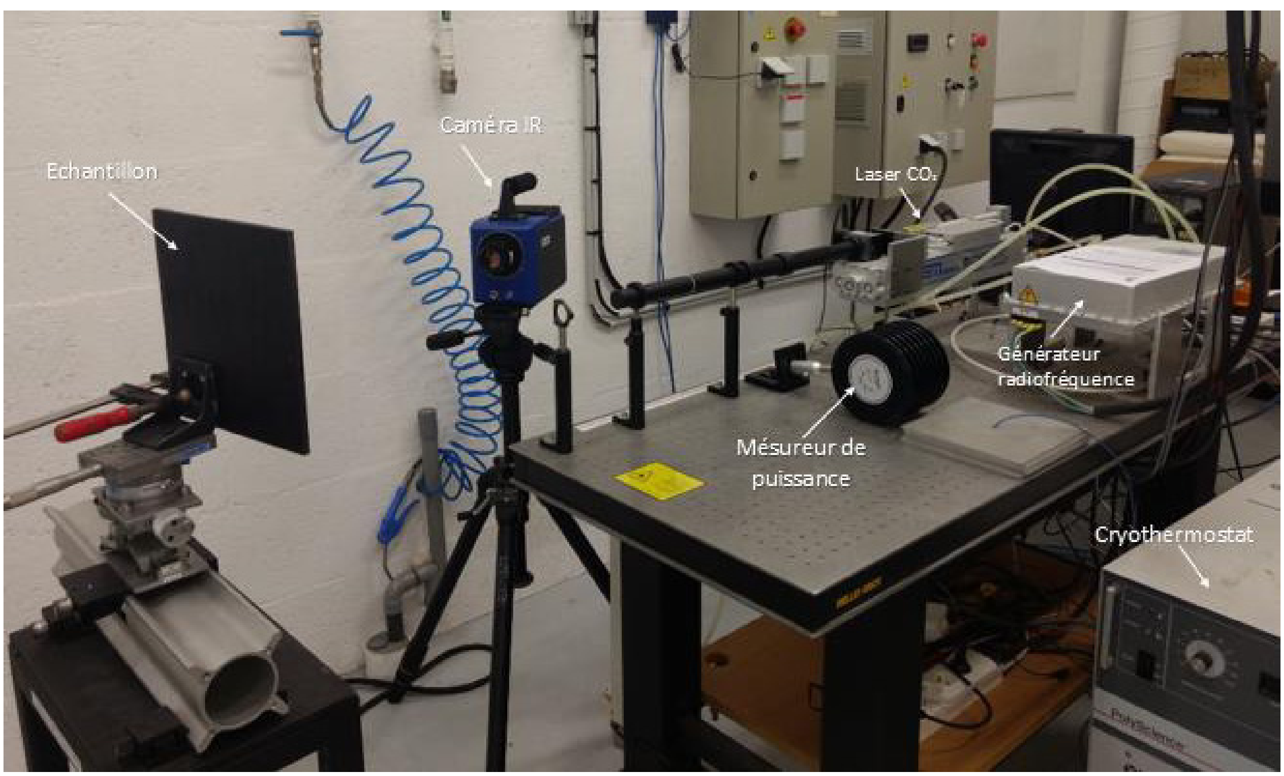


Figure 3: Dispositif expérimental de la méthode flash face avant

This example is based on publication:

Elissa El Rassy, Yann Billaud, Didier Saury, “Simultaneous and direct identification of thermophysical properties for orthotropic materials,” Measurement, 135 (2019) 199–212. J. Homepage: www.elsevier.com/locate/measurement

Abstract: A direct and simultaneous estimation method of the main three-dimensional thermal diffusivity tensor (ax; ay; az) of orthotropic opaque materials, is presented in this paper. This method consists in coupling a non-intrusive and unique 3D flash experiment with a transient nonlinear inverse heat transfer technique. A short and non-uniform excitation is applied on the surface of an orthotropic material using a CO2 laser, while the front face temperature cartography is measured over time by an IR camera. The inverse problem developed in the present study is based on the minimization of the least-squares criterion between the outputs of a 3D thermal quadrupoles model, and the experimental measurements. In order to properly estimate the thermal diffusivities, parameters related to the thermal excitation, in terms of shape and intensity, should be also estimated. In addition to that increase in the number of unknown parameters, the discontinuity nature of the excitation justifies the choice of an analytical model. Considering the large number of parameters to estimate, as well as the non-linear nature of the problem, a hybrid optimization algorithm combining a stochastic method and deterministic one, is applied. The identification method proposed in this work, named as DSEH (Direct and Simultaneous Estimation using Harmonics), is validated using an isotropic opaque material of known properties. Finally, the method is used on an orthotropic carbon fiber composite, commonly used in industries, thanks to its thermal and mechanical characteristics. The results are compared to other methods and shown to be in a good agreement with the literature values. The parameters identification is then completed by a sensitivity analysis, and evaluated in terms of robustness, accuracy, and time consumption.

Objective is to find the analytical solution of the front surface temperature for comparison with the experimental measurements. Then fit the experimental results to find the components of the coefficient of conduction $\left(k_1, k_2, k_3\right)$.

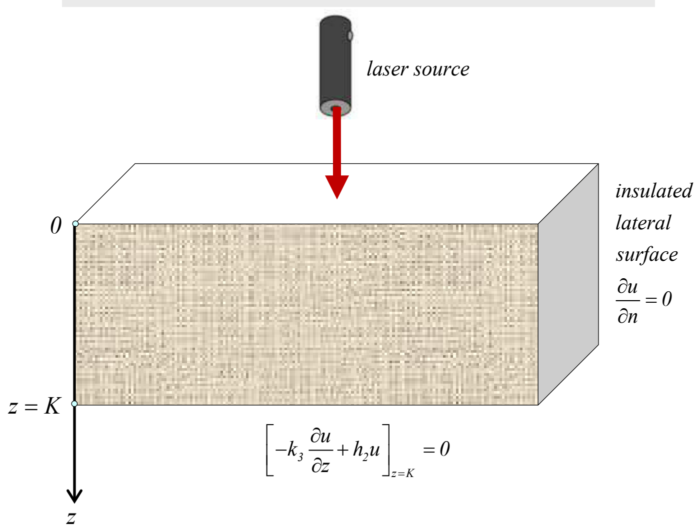
Denote $u\left(x,y,z,t\right) = T\left(x,y,z,t\right) - T_\infty$

Thermal diffusivity: $\alpha = \dfrac{k}{\rho c_p}$

incident impulse point source

$$q_0''\left(t\right) = \phi\left(x,y,t\right), \quad \left[\frac{W}{m^2}\right]$$

$$\left[-k_3\frac{\partial u}{\partial z} + h_1 u\right]_{z=0} = \phi\left(x,y,t\right) = \phi_0 \cdot \delta\left(x - \frac{L}{2}\right)\cdot\delta\left(y - \frac{M}{2}\right)\cdot\delta\left(t\right)$$



Heat Equation

$$k_1\frac{\partial^2 u}{\partial x^2} + k_2\frac{\partial^2 u}{\partial y^2} + k_3\frac{\partial^2 u}{\partial z^2} = \rho c_p\frac{\partial u}{\partial t}$$

$$\alpha_1\frac{\partial^2 u}{\partial x^2} + \alpha_2\frac{\partial^2 u}{\partial y^2} + \alpha_3\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}$$

Boundary conditions:

$$\left[\frac{\partial u}{\partial x}\right]_{x=0} = 0, \qquad \left[\frac{\partial u}{\partial x}\right]_{x=L} = 0$$

$$\left[\frac{\partial u}{\partial y}\right]_{y=0} = 0, \qquad \left[\frac{\partial u}{\partial y}\right]_{y=M} = 0$$

$$\left[-k_3\frac{\partial u}{\partial z} + h_1 u\right]_{z=0} = \phi_0 \cdot \delta\left(x - \frac{L}{2}\right)\cdot\delta\left(y - \frac{M}{2}\right)\cdot\delta\left(t\right)$$

$$\left[k_3\frac{\partial u}{\partial z} + h_2 u\right]_{z=L} = 0$$

Initial condition:

$$u\big|_{t=0} = 0$$

Determine what type of integral transform to use:

$z \in [0, N]$ $\Rightarrow$ Finite Fourier transform III-III (R-R)
$x \in [0, L]$ $\Rightarrow$ Finite Fourier transform II-II (N-N)
$y \in [0, M]$ $\Rightarrow$ Finite Fourier transform II-II (N-N)

$t > 0$ $\Rightarrow$ Laplace transform

The Finite Fourier Transforms (*Section IX.3 Finite Fourier Transform NEW VERSION, p.758)*

Finite Fourier Transform	$\overline{u}_n$	$= \int_0^L u(x) X_n(x)\,dx$	*integral transform*
	$u(x)$	$= \sum_n^\infty \overline{u}_n \dfrac{X_n(x)}{\lVert X_n(x) \rVert^2}$	*inverse transform*

Boundary conditions	**Eigenvalues** $\gamma_n,\ \beta_m,\ \lambda_k$	**Eigenfunctions**	**Norm**	**Operational property** $\int_0^L \left(\dfrac{\partial^2 u}{\partial x^2}\right) X_n(x)\,dx$
N $u'(0) = f_0$	$\gamma_0 = 0$	$X_0 = 1$	$L,\quad n = 0$	$-\mu_n^2 \overline{u}_n - f_0 X_n(0) + f_L X_n(L)$
N $u'(L) = f_L$	$\dfrac{n\pi}{L},\ n = 1,2,...$	$cos(\gamma_n x)$	$\dfrac{L}{2},\quad n = 1,2,...$	$X_n(0) = 1$
				$X_n(L) = cos(\mu_n L)$
N $u'(0) = g_0$	$\beta_0 = 0$	$Y_0 = 1$	$M,\quad m = 0$	$-\beta_m^2 \overline{u}_m - g_0 Y_m(0) + g_L Y_m(L)$
N $u'(M) = g_M$	$\dfrac{m\pi}{M},\ n = 1,2,...$	$cos(\beta_m y)$	$\dfrac{M}{2},\quad m = 1,2,...$	$Y_m(0) = 1$
				$Y_M(M) = cos(\beta_m M)$
R $-k_1 u'(0) + h_1 u(0) = f_0$	λ_k *are positive roots of*			$-\lambda_k^2 \overline{u}_k + \dfrac{f_0}{k_1} Z_k(0) + \dfrac{f_L}{k_2} Z_k(K)$
R $k_2 u'(K) + h_2 u(K) = f_K$	$(H_1 H_2 - \lambda^2) sin\,\lambda K + (H_1 + H_2)\lambda\, cos\,\lambda K = 0$			$Z_k(0) = \lambda_k$
		$Z_k = \lambda_k\, cos\,\lambda_k z + H_1\, sin\,\lambda_k z$		$Z_k(K) = \lambda_k\, cos\,\lambda_k K + H_1\, sin\,\lambda_k K$
		$\lVert Z_k \rVert^2 = \dfrac{(\lambda_k^2 + H_1^2)}{2}\left(L + \dfrac{H_2}{\lambda_k^2 + H_2^2}\right) + \dfrac{H_1}{2}$		

The Laplace transform $L\{u(t)\} = s\hat{u}(s) - u(0)$

Transformation of the Heat Equation.

1) Apply the Fourier transform in z (additional term appears because of operational property):

$$\alpha_1 \frac{\partial^2 \overline{u}_k(x,y,t)}{\partial x^2} + \alpha_2 \frac{\partial^2 \overline{u}_k(x,y,t)}{\partial y^2} + \alpha_3 \left[-\lambda_k^2 \overline{u}_k(x,y,t) + \frac{\phi(x,y,t)}{k_3} Z_k(0) \right] = \frac{\partial \overline{u}_k(x,y,t)}{\partial t}$$

2) Apply the Laplace transform in *t*

$$\alpha_1 \frac{\partial^2 \hat{\overline{u}}_k(x,y,s)}{\partial x^2} + \alpha_2 \frac{\partial^2 \hat{\overline{u}}_k(x,y,s)}{\partial y^2} - \alpha_3 \lambda_k^2 \hat{\overline{u}}_k(x,y,s) + \frac{\hat{\phi}(x,y,s)}{\rho c_p} Z_k(0) = s\hat{\overline{u}}_k(x,y,s)$$

3) Apply transform in *x*:

$$-\alpha_1 \gamma_n^2 \overline{\hat{\overline{u}}}_{n,k}(y,s) + \alpha_2 \frac{\partial^2 \overline{\hat{\overline{u}}}_{n,k}(y,s)}{\partial y^2} - \alpha_3 \lambda_k^2 \overline{\hat{\overline{u}}}_{n,k}(y,s) + \frac{\overline{\hat{\phi}}_n(y,s)}{\rho c_p} Z_k(0) = s\overline{\hat{\overline{u}}}_{n,k}(y,s)$$

4) Apply transform in *y*:

$$-\alpha_1 \gamma_n^2 \overline{\overline{\hat{\overline{u}}}}_{n,m,k}(s) - \alpha_2 \beta_m^2 \overline{\overline{\hat{\overline{u}}}}_{n,m,k}(s) - \alpha_3 \lambda_k^2 \overline{\overline{\hat{\overline{u}}}}_{n,m,k}(s) + \frac{\overline{\overline{\hat{\phi}}}_{n,m}(s)}{\rho c_p} Z_k(0) = s\overline{\overline{\hat{\overline{u}}}}_{n,m,k}(s)$$

5) Solution of the transformed equation:

$$\overline{\overline{\hat{\overline{u}}}}_{n,m,k}(s) = \frac{\overline{\overline{\hat{\phi}}}_{n,m}(s)}{\rho c_p} \cdot Z_k(0) \cdot \frac{1}{s - \left(-\alpha_1 \gamma_n^2 - \alpha_2 \beta_m^2 - \alpha_3 \lambda_k^2\right)}$$

For the case of the impulse point source, $\phi(x,y,t) = \phi_0 \cdot \delta\left(x - \frac{L}{2}\right) \cdot \delta\left(y - \frac{M}{2}\right) \cdot \delta(t)$, the transformation of ϕ is

$$\overline{\overline{\hat{\phi}}}_{n,m}(s) = \phi_0 \cdot cos\left(\frac{n\pi}{2}\right) \cdot cos\left(\frac{m\pi}{2}\right) = \phi_0 \cdot cos\left(\frac{n\pi}{2}\right) \cdot cos\left(\frac{m\pi}{2}\right)$$

Then solution of the transformed equation becomes: $n = 0,1,2,...$; $m = 0,1,2,...$; $k = 1,2,...$

$$\overline{\overline{\hat{\overline{u}}}}_{n,m,k}(s) = \frac{1}{\rho c_p} \cdot \phi_0 \cdot Z_k(0) \cdot cos\left(\frac{n\pi}{2}\right) \cdot cos\left(\frac{m\pi}{2}\right) \cdot \frac{1}{s - \left(-\alpha_1 \gamma_n^2 - \alpha_2 \beta_m^2 - \alpha_3 \lambda_k^2\right)}$$

6) Apply the inverse Laplace transform:

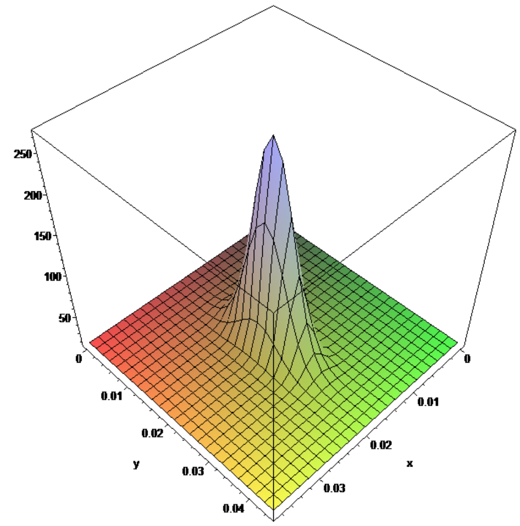
$$\overline{\overline{\overline{u}}}_{n,m,k}(t) = \frac{1}{\rho c_p} \cdot \phi_0 \cdot Z_k(0) \cdot cos\left(\frac{n\pi}{2}\right) \cdot cos\left(\frac{m\pi}{2}\right) \cdot e^{-\left(\alpha_1 \gamma_n^2 + \alpha_2 \beta_m^2 + \alpha_3 \lambda_k^2\right) \cdot t}$$

7) Apply the inverse Finite Fourier transforms to obtain the complete solution:

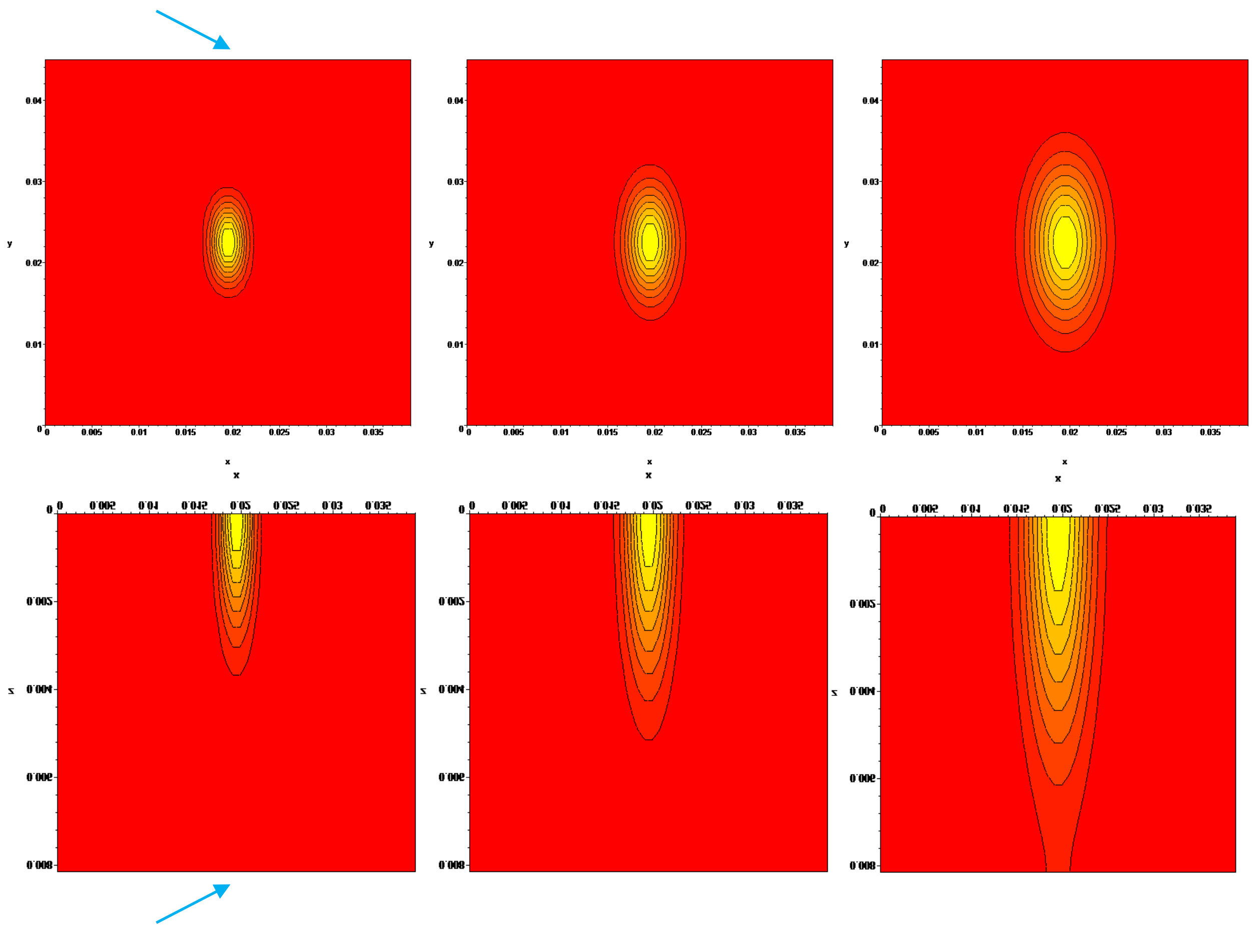
$$u(x,y,z,t) = \frac{\phi_0}{\rho c_p} \cdot \frac{4}{LM} \sum_{n=0} \sum_{m=0} \sum_{k=1} X_n(x) \cdot Y_m(y) \cdot \frac{Z_k(z)}{\left\| Z_k \right\|^2} \cdot \overbrace{Z_k(0)}^{\lambda_k} \cdot cos\left(\frac{n\pi}{2}\right) \cdot cos\left(\frac{m\pi}{2}\right) \cdot e^{-\left(\alpha_1 \gamma_n^2 + \alpha_2 \beta_m^2 + \alpha_3 \lambda_k^2\right) \cdot t}$$

Note, that in this summation for $n = 0$, $\gamma_0 = 0$, $X_0(x) = 1$; and for $m = 0$, $\beta_0 = 0$, $Y_0(y) = 1$.

Surface temperature at $z = 0$: $u(x, y, 0, t)$ at some fixed moment of time



Front surface temperature contour plots at different moments of time:

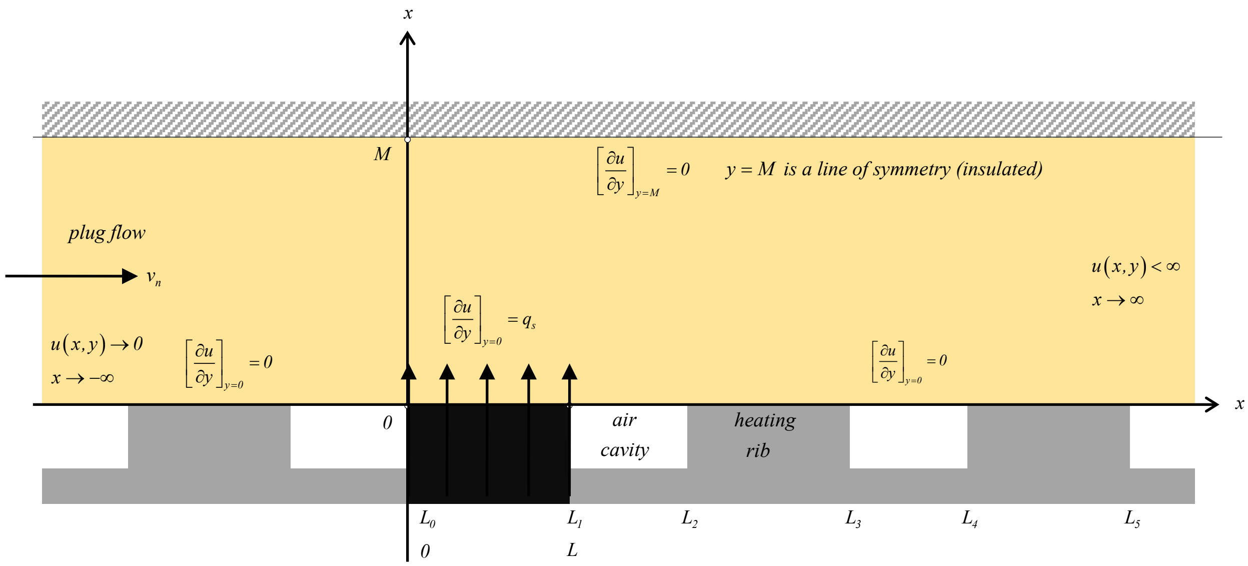


Contour plots of the cross-section in the vertical z-direction.

The shape of contours is affected by anisotropy of coefficient of conduction. *Maple code* 00 ELISSA-04.mws

Example 4 [Adam] **Conduction and advection (plug flow)** **old version of transform**

Details of solution can be found in IX.3 The Finite Fourier Transform (2018 version).



Equation: $$\frac{\partial^2 u}{\partial x^2} - \frac{\rho c_p v_n}{k}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (x,y) \in (-\infty,\infty) \times (0,M)$$

Boundary conditions: $$\frac{\partial}{\partial y}u(x,0) = -q_s \cdot \left[H(x) - H(x-L)\right]$$ (Attention: $-q_s < 0$ for heating)

$$\frac{\partial}{\partial y}u(x,M) = 0$$

Finite Fourier Transform in y : $\quad \bar{u}_n(x) = \int_0^M u(x,y)K_n(y)\,dy \quad$ *integral transform*

$\quad u(x,y) = \sum_{n=1}^{\infty} \bar{u}_n(x) K_n(y) \quad$ *inverse transform*

N $[u']_{y=0} = f_0 = -q_s$ *heating*

N $[u']_{y=M} = f_M = 0$ insulation

N $[u']_{y=0} = f_0$ **N** $[u']_{y=M} = f_M$	$\lambda_n = \frac{n\pi}{M} \quad n = 0,1,2,\ldots$ $Y_0 = 1, \quad Y_n = \cos\left(\frac{n\pi}{M}y\right)$ $K_0 = \frac{1}{\sqrt{M}}, \quad K_n = \sqrt{\frac{2}{M}}\cos\left(\frac{n\pi}{M}y\right) \quad n = 1,2,\ldots$	$-f_0K_n(0) + f_MK_n(M) - \lambda_n^2\bar{u}_n$

Transformed equation **Single Heating Rib**

$$\frac{\partial^2 \overline{u}_n}{\partial x^2} - \frac{\rho c_p v_n}{k}\frac{\partial \overline{u}_n}{\partial x} - \lambda_n^2 \overline{u}_n - f_0(x) K_n(0) = 0$$

Consider conjugate problems:

I $x < 0$

$$\frac{\partial^2 \overline{u}_n}{\partial x^2} - \frac{\rho c_p v_n}{k}\frac{\partial \overline{u}_n}{\partial x} - \lambda_n^2 \overline{u}_n = 0$$

$$\overline{u}_n^I\Big|_{x=0} = \overline{u}_n^{II}\Big|_{x=0}$$

$$\frac{\partial \overline{u}_n^I}{\partial x}\bigg|_{x=0} = \frac{\partial \overline{u}_n^{II}}{\partial x}\bigg|_{x=0}$$

II $0 < x < L$

$$\frac{\partial^2 \overline{u}_n}{\partial x^2} - \frac{\rho c_p v_n}{k}\frac{\partial \overline{u}_n}{\partial x} - \lambda_n^2 \overline{u}_n + q_s K_n(0) = 0$$

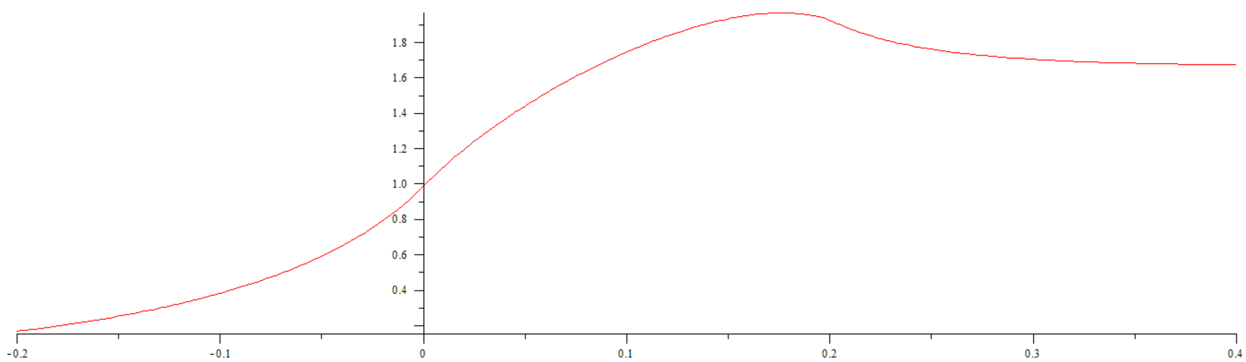
III $x > L$

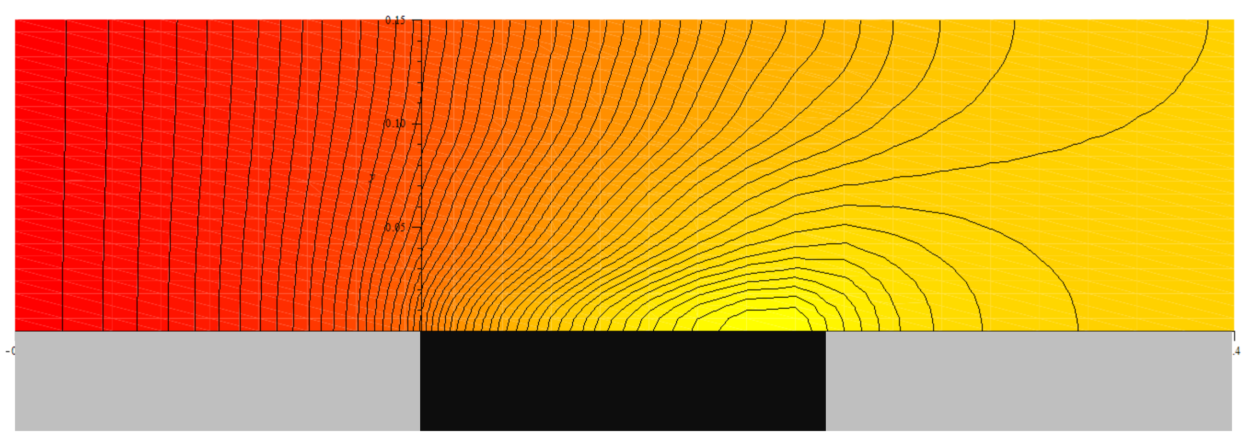
$$\frac{\partial^2 \overline{u}_n}{\partial x^2} - \frac{\rho c_p v_n}{k}\frac{\partial \overline{u}_n}{\partial x} - \lambda_n^2 \overline{u}_n = 0$$

$$\overline{u}_n^{II}\Big|_{x=L} = \overline{u}_n^{III}\Big|_{x=L}$$

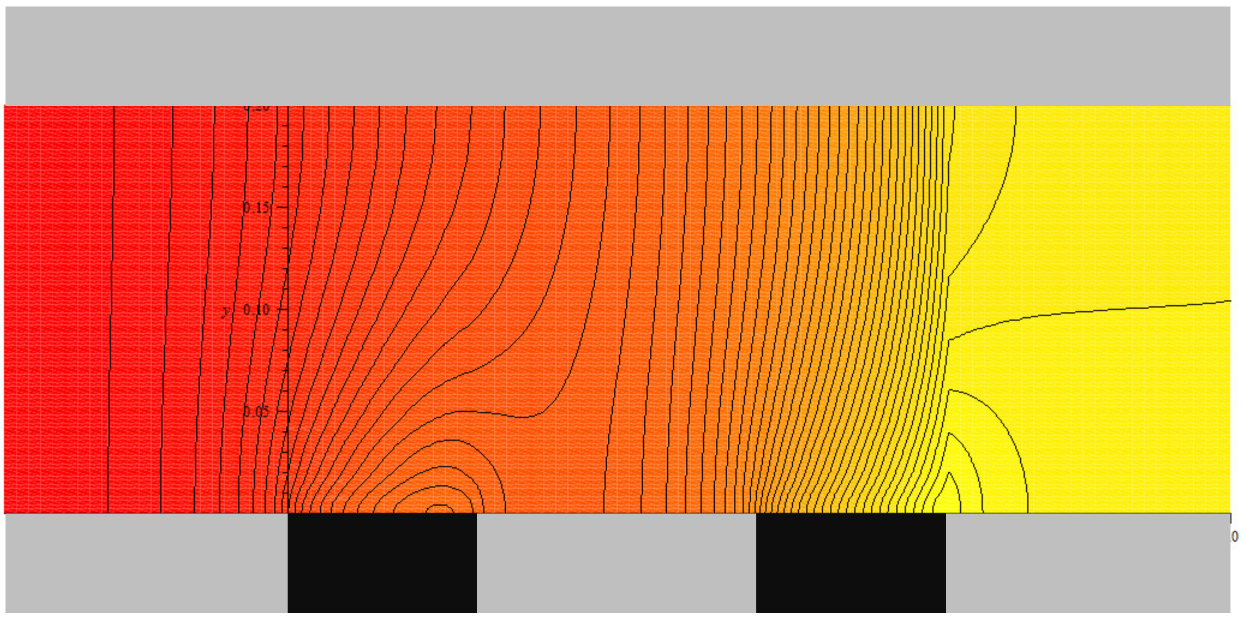
$$\frac{\partial \overline{u}_n^{II}}{\partial x}\bigg|_{x=L} = \frac{\partial \overline{u}_n^{III}}{\partial x}\bigg|_{x=L}$$

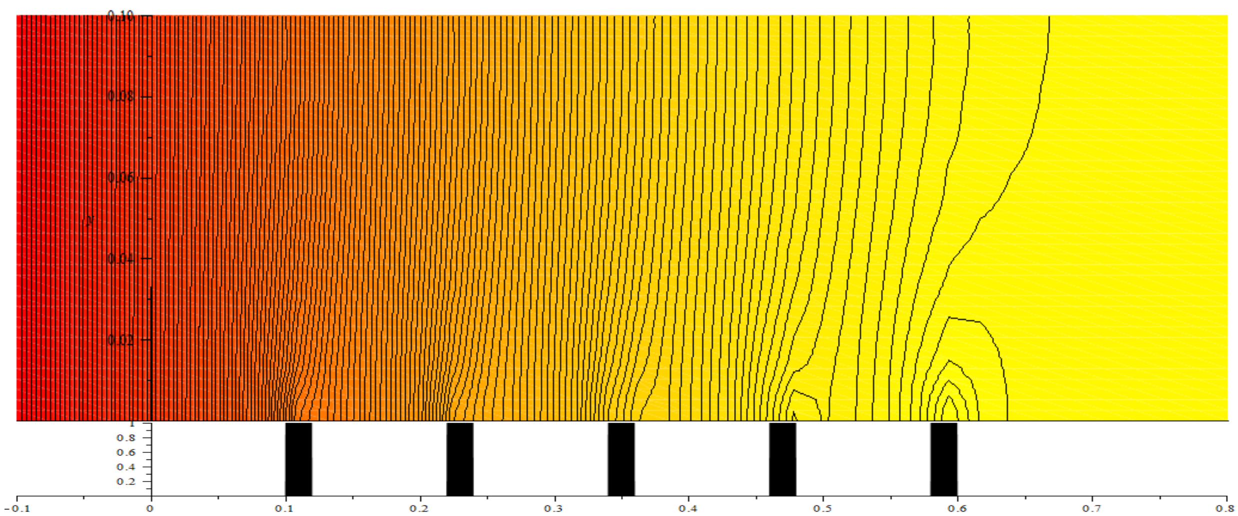
Plot of $u\left(x, \frac{M}{10}\right)$



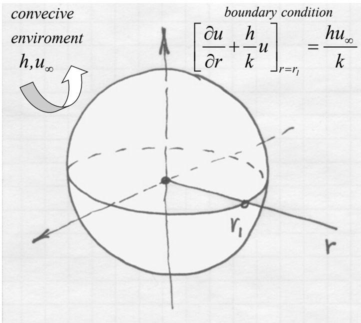


General Case **Arbitrary number *N* of heating ribs**





IX.3.6. Heat Equation in the Spherical Coordinates with angular symmetry – Reduction to Cartesian coords.



Consider heat conduction in the solid sphere with angular symmetry. The non-stationary temperature field $u(r,t)$ which depends both the temporal variable *t* and the radial variable *r*, is governed by the Heat Equation:

$$\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 u\right)+\frac{\dot{q}(r,t)}{k}=\frac{1}{a^2}\frac{\partial u}{\partial t} \qquad 0\le r<r_1 \qquad t>0\,,$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{\dot{q}(r,t)}{k}=\frac{1}{a^2}\frac{\partial u}{\partial t}$$

With the initial condition

$$u(r,0)=u_0(r)$$

and with the convective boundary condition:

$$\left[k\frac{\partial u}{\partial r}+hu\right]_{r=r_1}=h\,u_\infty(t)$$

where $u_\infty(t)$ is a temperature of the surroundings (generally, a function of time). We can rewrite the boundary condition in the standard form

$$\left[\frac{\partial u}{\partial r}+\frac{h}{k}u\right]_{r=r_1}=\frac{hu_\infty}{k}$$

1) Introduce the new dependent variable as

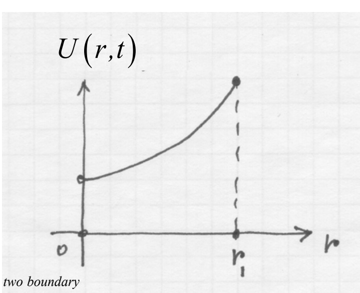
$$U(r,t)=r\,u(r,t)$$

reduction to 1-d Cartesian formulation by the change of variable $U(r,t)=r\,u(r,t)$

Differentiate twice

$$\frac{\partial}{\partial r}U=u+r\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 U}{\partial r^2}=2\frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2}=r\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}\right)$$



two boundary conditions

$U\big|_{r=0}=0 \qquad \left[\frac{\partial U}{\partial r}+HU\right]_{r=r_1}=f_{r_1}(t)$

Then the Heat Equation becomes

$$\frac{1}{r}\frac{\partial^2 U}{\partial r^2}+\frac{\dot{q}(r,t)}{k}=\frac{1}{a^2}\frac{\partial u}{\partial t}$$

$$\frac{\partial^2 U}{\partial r^2}+S(r,t)=\frac{1}{a^2}\frac{\partial U}{\partial t}\,, \qquad 0\le r<r_1\,,\ t>0\,,\ S(r,t)=r\frac{\dot{q}(r,t)}{k}$$

which formally is the 1-D Heat Equation in Cartesian coordinates.

Initial condition becomes:

$$U(r,0)=r\,u(r,0)=ru_0(r)$$

The first boundary condition at $r = 0$ is obtained directly from the equation used for a change of variable:

$$U\big|_{r=0} = ru\big|_{r=0} = 0 \qquad \textit{Dirichlet}$$

Consider the second boundary condition at $r = r_1$:

$$\left[\frac{\partial u}{\partial r} + \frac{h}{k}u\right]_{r=r_1} = \frac{hu_\infty}{k}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{U}{r}\right) + \frac{h}{k}\frac{U}{r}\right]_{r=r_1} = \frac{hu_\infty}{k}$$

$$\left[\frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2} + \frac{h}{k}\frac{U}{r}\right]_{r=r_1} = \frac{hu_\infty}{k}$$

$$\left[\frac{\partial U}{\partial r} + HU\right]_{r=r_1} = f_{r_1}(t), \quad H = \frac{h}{k} - \frac{1}{r_1}, \; f_{r_1}(t) = \frac{hu_\infty r_1}{k} \qquad \textit{Robin}$$

2) Finite Fourier Transform		$\bar{U}_n(t) = \int_0^{r_1} U(r,t)\,X_n(r)\,dr$		*integral transform*
		$U(r,t) = \sum_{n=1}^{\infty} \bar{U}_n \dfrac{X_n(r)}{\left\Vert X_n(r)\right\Vert^2}$		*inverse transform*
Boundary conditions	Eigenvalues μ_n	Eigenfunctions $X_n(r)$	Norm $\left\Vert X_n(r)\right\Vert^2$	Operational property $\int_0^{r_1}\left(\dfrac{\partial^2 U}{\partial r^2}\right)X_n(r)\,dr$
	μ_n *are positive root of*			
D $u(0) = f_0$	$\mu \cos\mu r_1 + H_2 \sin\mu r_1 = 0$	$\sin(\mu_n r)$	$\dfrac{r_1}{2} - \dfrac{\sin(2\mu_n r_1)}{4\mu_n}$	$-\mu_n^2\bar{U}_n + f_0 X_n'(0) + \dfrac{hu_\infty r_1}{k}X_n(r_1)$
R $\left[\dfrac{\partial U}{\partial r} + HU\right]_{r=r_1} = f_{r_1}(t), \quad H = \dfrac{h}{k} - \dfrac{1}{r_1}, \; f_{r_1}(t) = \dfrac{hu_\infty r_1}{k}$				$X_n'(0) = \mu_n, \; X_n(r_1) = \sin(\mu_n r_1)$

3) Apply the Finite Fourier transform

$$-\mu_n^2\bar{U}_n(t) + \not{f_0}X_n'(0) + \frac{hu_\infty(t)r_1}{k}X_n(r_1) + \bar{S}_n(t) = \frac{1}{a^2}\frac{\partial \bar{U}_n(t)}{\partial t}$$

$$\bar{U}_n(0) = F\left\{r\,u_0(r)\right\}$$

Apply the Laplace Transform

$$-\mu_n^2\bar{\bar{U}}_n(s) + \frac{hr_1}{k}X_n(r_1)\bar{u}_\infty(s) + L\left\{\bar{S}_n(t)\right\} = \frac{s}{a^2}\bar{\bar{U}}_n(s) - \frac{1}{a^2}\bar{U}_n(0)$$

$$\bar{\bar{U}}_n(s) = \frac{a^2hr_1}{k}X_n(r_1)\frac{1}{\left(s + a^2\mu_n^2\right)}L\left\{u_\infty(t)\right\} + a^2\frac{1}{\left(s + a^2\mu_n^2\right)}L\left\{\bar{S}_n(t)\right\} + \frac{\bar{U}_n(0)}{\left(s + a^2\mu_n^2\right)}$$

Transformed solution:

$$\bar{\bar{U}}_n(s)=\frac{a^2hr_1}{k}X_n(r_1)L\left\{e^{-a^2\mu_n^2t}\right\}L\left\{u_\infty(t)\right\}+a^2L\left\{e^{-a^2\mu_n^2t}\right\}L\left\{\bar{S}_n(t)\right\}+L\left\{e^{-a^2\mu_n^2t}\right\}\bar{U}_n(0)$$

4) Solution:

Inverse Laplace by Convolution

$$\bar{U}_n(t)=\frac{a^2hr_1}{k}X_n(r_1)\int_0^t e^{-a^2\mu_n^2\tau}u_\infty(t-\tau)d\tau+a^2L\int_0^t e^{-a^2\mu_n^2\tau}\bar{S}_n(t-\tau)d\tau+\bar{U}_n(0)e^{-a^2\mu_n^2t}$$

Inverse Finite Fourier transform

$$U(r,t)=\sum_{n=1}^{\infty}\bar{U}_n(t)\frac{X_n(r)}{\left\|X_n(r)\right\|^2}$$

Solution

$$u(r,t)=\frac{U(r,t)}{r}=\frac{1}{r}\sum_{n=1}^{\infty}\bar{U}_n(t)\frac{X_n(r)}{\left\|X_n(r)\right\|^2}$$

5) Particular Case $u_\infty=const,\ S=0$

$$\bar{U}_n(t)=\frac{hr_1u_\infty}{k\mu_n^2}X_n(r_1)\left(1-e^{-a^2\mu_n^2t}\right)+\bar{U}_n(0)e^{-a^2\mu_n^2t}$$

$$\bar{U}_n(t)=\frac{hr_1}{k\mu_n^2}X_n(r_1)u_\infty+\left[\bar{U}_n(0)-\frac{hr_1u_\infty}{k\mu_n^2}X_n(r_1)\right]e^{-a^2\mu_n^2t}$$

Solution

$$u(r,t)=\frac{U(r,t)}{r}$$

$$u(r,t)=\frac{1}{r}\sum_{n=1}^{\infty}\bar{U}_n(t)\frac{X_n(r)}{\left\|X_n(r)\right\|^2}$$

where

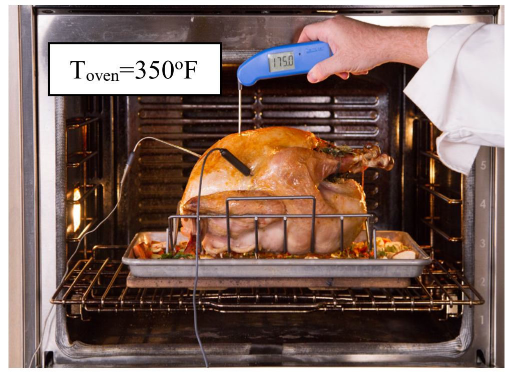
$$U(r,t)=u_\infty+\frac{1}{r}\sum_{n=1}^{\infty}\left[\bar{U}_n(0)-\frac{hr_1u_\infty}{k\mu_n^2}X_n(r_1)\right]\frac{X_n(r)}{\left\|X_n(r)\right\|^2}e^{-a^2\mu_n^2t}$$

In the absence of heat sources, the obvious limit as $t\to\infty$ is u_∞.

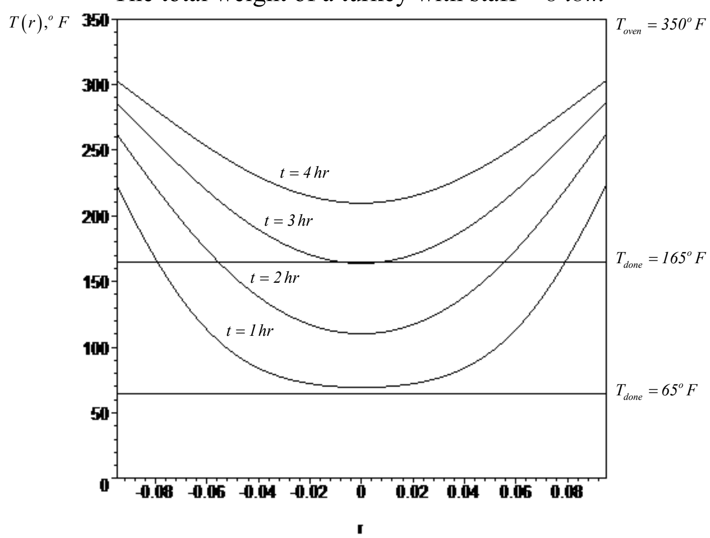
6) Maple solution **Roasting a turkey** (*05 TURKEY.mws*)

A turkey is considered done when its minimum temperature reaches $T_{done} = 165^o F$.

Thermophysical properties of turkey meat can be taken from the table (p.580), or they can be approximate by the properties of water.



The total weight of a turkey with staff = *8 lbm*



Examples with application of the Finite Fourier Transform

Firefly in the fog

Star wars

Raindrops

The Fokker-Planck Equation

Hanging Cable

Jumping board

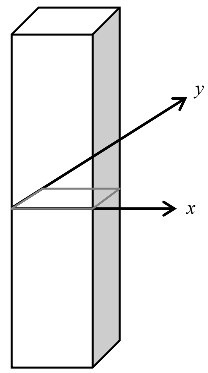
Friction welding

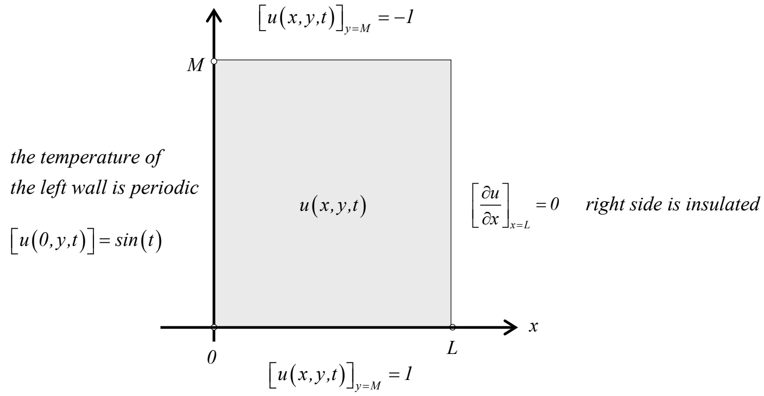
The Running on the waves

Jet Ski

November 17, 2023

Example 2 Consider heat conduction in the 2-dimensional cross-section of the long column

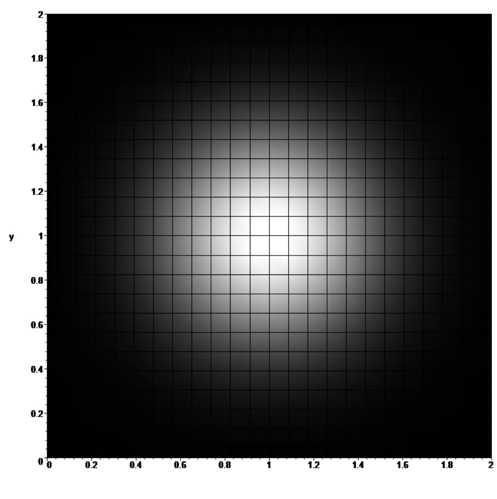


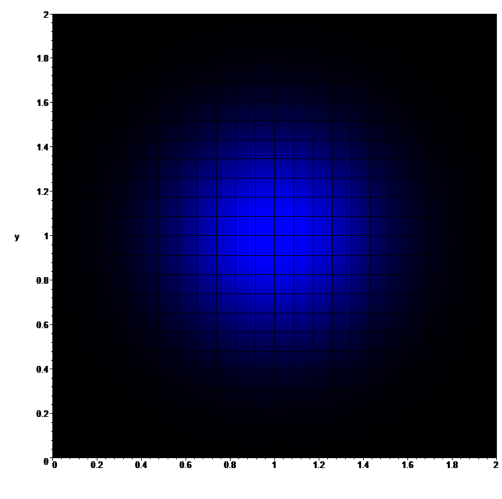


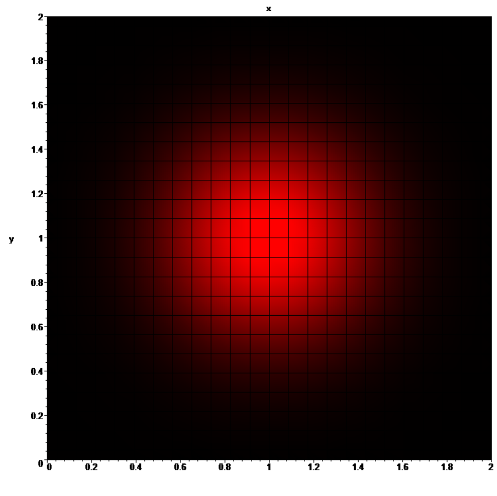
Equation: $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad (x,y)\in(0,L)\times(0,M) \qquad t>0$$

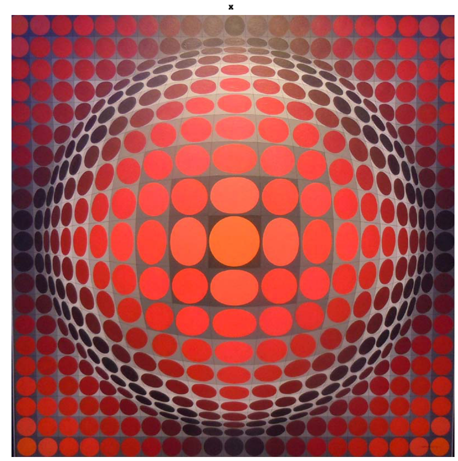
Initial condition: $$u(x,0)=u_0(x)=0$$

Example 3 Point heat source – moving or stationary. Impulse point source. *06-03 not moving source - Copy.mws*









November 2019