IX.4 The Hankel Transform

IX.4.1 HANKEL TRANSFORM

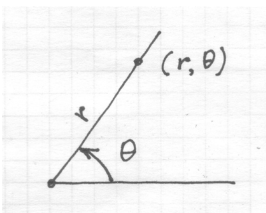
Hermann Hankel
(1839-1873)

The physical domain where we are supposed to apply the Hankel transform is the infinite 2-dimensional plane with polar coordinates (r,θ).

The Hankel transform which we consider in this section will be applied for transformation of the differential operator

$$Lu \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u \tag{1}$$

where $\nu \in \mathbb{R}$ is a parameter and domain is $0 \le r < \infty$.
The particular case of this operator with $\nu = 0$ is the radial term of Laplacian in cylindrical coordinates.



We assume that the function $u(r)$ is bounded for all r and that the function u and its derivative have radial decaying at infinity:

$$u(r) < \infty, \quad ru\big|_{r\to\infty} = 0, \quad r\frac{\partial u}{\partial r}\bigg|_{r\to\infty} = 0 \tag{2}$$

The bounded solution of the singular Sturm-Liouville problem (eigenvalue problem):

$Lu = \mu u$

has non-trivial solution when the parameter is non-positive, let us rename it as $\mu = -\lambda^2$. That yields the equation

$J_\nu(\lambda r)$ is a Bessel function of the 1^{st} kind of order ν.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u = -\lambda^2 u \tag{3}$$

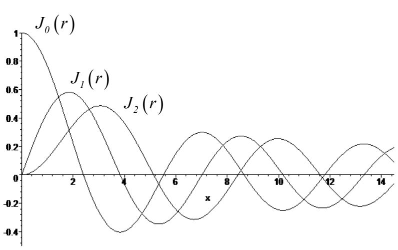
$J_\nu(\lambda r)$ is the bounded solution of the Bessel Equation
$r^2 y'' + ry' + \left(\lambda^2 r^2 - \nu^2\right) y = 0$
for $r \in [0,\infty)$

which is the Bessel equation of order ν, the bounded solutions of which are the Bessel functions of the 1st kind

$J_\nu(\lambda r)$

The self-adjoint form of the operator (1) is

$$Lu \equiv \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r}u\right] \tag{4}$$



Therefore, the weight function is $p(r) = r$, which is used for definition of the weighted inner product.

Formally, the Hankel transform can be constructed from the two-dimensional Fourier transform with transition to polar coordinates and application of the integral representation of the Bessel functions [Debnah]. Then the Hankel transform and the inverse Hankel transform are defined as

Hankel Transform of order ν

$$H_\nu\{f(r)\} = \quad \bar{f}_\nu(\lambda) = \int_0^\infty f(r) J_\nu(\lambda r)\, r\, dr \tag{5}$$

Inverse Hankel transform of order ν

$$H_\nu^{-1}\{\bar{f}_\nu(\lambda)\} = \quad f(r) = \int_0^\infty \bar{f}_\nu(\lambda) J_\nu(\lambda r)\, \lambda\, d\lambda \tag{6}$$

Apply the Hankel transform (5) of order ν to $\delta(r-\mu)$, $\mu > 0$

$$
\begin{aligned}
H_\nu\left\{\delta(r-\mu)\right\} &= \int_0^\infty \delta(r-\mu) J_\nu(\lambda r)\, r\, d\, r \\
&= J_\nu(\lambda\mu)\,\mu
\end{aligned}
$$

Then $\delta(r-\mu)$ the inverse Hankel transform (6) of $J_\nu(\lambda\mu)\,\mu$

$$
\begin{aligned}
\delta(r-\mu) &= H_\nu^{-1}\left\{ J_\nu(\lambda\mu)\,\mu \right\} = \\
&= \int_0^\infty \left[J_\nu(\lambda\mu)\,\mu \right] J_\nu(\lambda r)\, \lambda\, d\lambda \\
&= \mu \int_0^\infty J_\nu(\lambda r) J_\nu(\lambda\mu)\, \lambda\, d\lambda
\end{aligned}
$$

from which follows the orthogonality of Bessel functions:

Orthogonality of Bessel functions

$$
\int_0^\infty J_\nu(\lambda\mu) J_\nu(\lambda r)\, \lambda\, d\lambda = \frac{\delta(r-\mu)}{\mu}, \quad \mu > 0 \tag{7}
$$

Then we can formulate the Hankel Integral Theorem:

The Hankel Integral Theorem

$$
f(r) = \int_{\lambda=0}^\infty \left[\int_{\mu=0}^\infty f(\mu) J_\nu(\lambda\mu)\, \mu\, d\,\mu \right] J_\nu(\lambda r)\, \lambda\, d\lambda \tag{8}
$$

Proof using the orthogonality relationship (7):

$$
\int_{\lambda=0}^\infty \left[\int_{\mu=0}^\infty f(\mu) J_\nu(\lambda\mu)\, \mu\, d\mu \right] J_\nu(\lambda r)\, \lambda\, d\lambda
$$

$$
= \int_{\mu=0}^\infty f(\mu) \left[\int_{\lambda=0}^\infty J_\nu(\lambda\mu) J_\nu(\lambda r)\, \lambda\, d\lambda \right] \mu\, d\mu
$$

$$
= \int_{p=0}^\infty f(\mu) \frac{\lambda(\mu-r)}{\not{\mu}} \not{\mu}\, d\mu
$$

$$
= f(r) \qquad \blacksquare
$$

The Hankel integral theorem equation (8) can be used for definition of the Hankel transform pair (5,6). However, it should be noted, that equation (8) in this notes is derived by application of equations (5,6) assuming that they are true. We can state there is no circular reasoning (derivation of the result starting with assumption that it is true), if equations (5,6) are originated from some external derivation, for example, by the change of variables in Fourier transform to polar coordinates.

Properties of Hankel transform

Both, the Hankel transform and its inverse are linear operators.

$$H_\nu\left\{f\left(ar\right)\right\} = \frac{1}{a^2}\bar{f}_\nu\left(\frac{\lambda}{a}\right) \qquad \textit{scaling}$$

$$H_0\left\{1\right\} = \frac{\delta\left(\lambda\right)}{\lambda}$$

$$H_0\left\{\frac{e^{-ar}}{r}\right\}$$

$$H_0\left\{\frac{\delta\left(r-c\right)}{r}\right\}$$

$$H_0\left\{H\left(a-r\right)\right\}$$

$$H_0\left\{e^{-cr^2}\right\} = \frac{1}{2c}e^{\frac{-\lambda^2}{4c}}$$

$$H_1\left\{e^{-ar}\right\} = \frac{\lambda}{\sqrt{\left(a^2+\lambda^2\right)^3}}$$

$$H_1\left\{\frac{e^{-ar}}{r}\right\}$$

$$H_0\left\{r\,f'\left(r\right)\right\} = -2\bar{f}_0\left(\lambda\right)-\lambda\bar{f}_0'\left(\lambda\right)$$

$$H_0\left\{\bar{f}_0\left(r\right)\right\} = f\left(\lambda\right) \qquad \textit{reciprocity}$$

Have to be checked!

Operational Property

Consider application of the Hankel trnsform of order ν to differential operator (1), in which the additional term defines the order of the applied Hankel transform:

$$H_\nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right\}$$

$$=\int_0^\infty rJ_\nu(\lambda r)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right]dr$$

$$=\int_0^\infty J_\nu(\lambda r)\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)dr \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=\int_0^\infty J_\nu(\lambda r)d\left(r\frac{\partial u}{\partial r}\right) \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)J_\nu(\lambda r)\right]_0^\infty-\int_0^\infty r\frac{\partial u}{\partial r}d[J_\nu(\lambda r)] \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=-\int_0^\infty r\frac{\partial u}{\partial r}[J_\nu(\lambda r)]'dr \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=-\int_0^\infty r[J_\nu(\lambda r)]'du \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=\left\{ru\left[J_\nu(\lambda r)\right]'\right\}_0^\infty+\int_0^\infty ud\left\{r\left[J_\nu(\lambda r)\right]'\right\} \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

From self-adjoint form of the Bessel Equation:

$$(ry')'+\left(-\frac{\nu^2}{r}+\lambda^2 r\right)y=0$$

write

$$(ry')'=r\left(-\frac{\nu^2}{r^2}+\lambda^2\right)y$$

Then for $y=J_\nu(\lambda r)$

$$\left[rJ_\nu'(\lambda r)\right]'=r\left(\lambda^2-\frac{\nu^2}{r^2}\right)J_\nu(\lambda r)$$

$$=\int_0^\infty u\left\{r\left[J_\nu(\lambda r)\right]'\right\}'dr \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=\int_0^\infty ur\left[\frac{\nu^2}{r^2}-\lambda^2\right]J_\nu(\lambda r)dr \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=-\lambda^2\int_0^\infty rJ_\nu(\lambda r)udr+\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr \qquad -\int_0^\infty rJ_\nu(\lambda r)\left[\frac{\nu^2}{r^2}u\right]dr$$

$$=-\lambda^2\int_0^\infty rJ_\nu(\lambda r)udr$$

$$=-\lambda^2\,\overline{u}_\nu(\lambda)$$

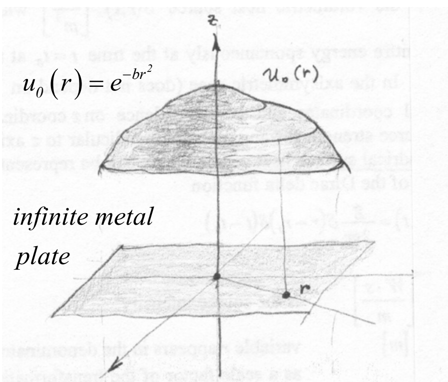
$$H_\nu\left\{\left[\nabla^2-\frac{\nu^2}{r^2}\right]u(r)\right\} = \int_0^\infty rJ_\nu(\lambda r)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right]dr = -\lambda^2\,\overline{u}_\nu(\lambda) \qquad (9)$$

0th order and 1st order Hankel Transforms

$H_0\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right\} = \int_0^\infty rJ_0(\lambda r)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]dr = -\lambda^2\,\overline{u}_0(\lambda)$	$\overline{u}_0(\lambda)=\int_0^\infty rJ_0(\lambda r)u(r)dr$ **0th order**
$H_1\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{u}{r^2}\right\} = \int_0^\infty rJ_1(\lambda r)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{u}{r^2}\right]dr = -\lambda^2\,\overline{u}_1(\lambda)$	$\overline{u}_1(\lambda)=\int_0^\infty rJ_1(\lambda r)u(r)dr$ **1st order**

We will use mostly the 0th order Hankel transform (in this case, index 0 in the transformed function is usually omitted)

EXAMPLE TRANSIENT HEAT TRANSFER Cooling of the bell-shaped temperature profile



Consider the axisymmetric case of the Heat Equation in cylindrical coordinates

$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r\in[0,\infty) \quad t>0$$

$$u(r,0)=u_0(r)=e^{-br^2} \qquad (\textit{initial condition})$$

There are no boundary conditions, but we assume that the function *u* and its derivative has radial decaying at infinity:

$$ru\big|_{r\to\infty}=0 \qquad r\frac{\partial u}{\partial r}\bigg|_{r\to\infty}=0$$

Physically it can be treated as a heated to bell shaped temperature profile infinite flat plate thermoinsulated from both sides.
Heat is conducted along the plate.

<u>1) Transformed equation</u> Apply 0th order Hankel transform

$$\bar{u}(\lambda,t) = \int_0^\infty rJ_0(\lambda r)u(r,t)\,dr$$

Then equation is transformed to

$$-\lambda^2\bar{u}=\frac{1}{\alpha}\frac{\partial\bar{u}}{\partial t}$$

With transformed initial condition (see p.806)

$$\bar{u}(\lambda,0)=\int_0^\infty rJ_0(\lambda r)u_0\,dr=\int_0^\infty rJ_0(\lambda r)e^{-br^2}\,dr=\frac{e^{-\frac{\lambda^2}{4b}}}{2b}$$

That is the 1st order linear ODE

$$\frac{\partial\bar{u}}{\partial t}+\alpha\lambda^2\bar{u}=0$$

The solution is

$$\bar{u}(\lambda,t) \quad =\bar{u}(\lambda,0)e^{-\alpha\lambda^2 t} \quad =\frac{1}{2b}e^{-\left(\frac{1}{4b}+\alpha t\right)\lambda^2}$$

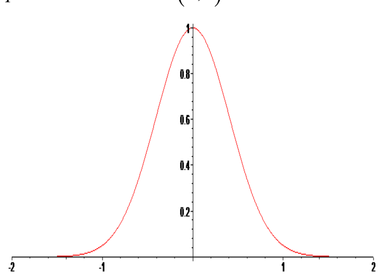
<u>2) Inverse transform</u> Solution of the problem (use table p.806)

$$u(r,t) \quad = H_0^{-1}\left\{\frac{1}{2b}e^{-\left(\frac{1}{4b}+\alpha t\right)\lambda^2}\right\}$$

$$=\frac{1}{2b}H_0^{-1}\left\{e^{-\frac{\frac{\lambda^2}{4}}{\frac{1}{b}+4\alpha t}}\right\} \qquad \text{denote } c=\frac{1}{\frac{1}{b}+4\alpha t}$$

$$=\frac{2c}{2b}H_0^{-1}\left\{\frac{1}{2c}e^{-\frac{\lambda^2}{4c}}\right\}`=\frac{c}{b}e^{-cr^2}`=\frac{1}{(1+4\alpha bt)}e^{-\frac{r^2}{\left(\frac{1}{b}+4\alpha t\right)}}$$

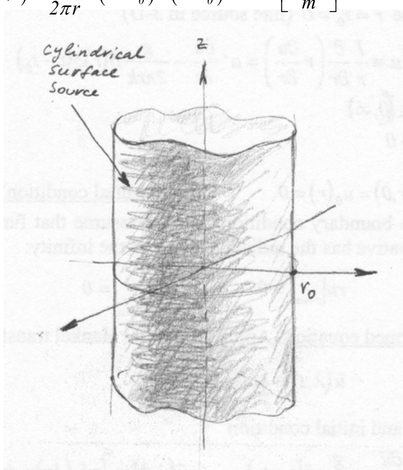
Hankel example p.808.mws $u(r,t)$



$$u(r,t) \quad =\frac{1}{(1+4\alpha bt)}e^{-\frac{r^2}{\left(\frac{1}{b}+4\alpha t\right)}}$$

IX.4.2 INSTANTENEOUS ENERGY SOURCE

$S(r,t)=\frac{g}{2\pi r}\delta(r-r_0)\delta(t-t_0)$ $\quad g\left[\frac{W\cdot s}{m}\right]$



We consider the volumetric heat source $S(r,t)$ which releases its entire energy spontaneously at the time $t=t_0$ at the points $r=r_0$ [Ozisik, HC, p.220].

In the axisymmetric case (does not depend on θ) of cylindrical coordinates and no dependence on the z coordinate (the same source strength for any plane perpendicular to the z axis), this is a ***cylindrical surface heat source*** which can be represented with the help of the Dirac delta function

$$S(r,t)=\frac{g}{2\pi r}\delta(r-r_0)\delta(t-t_0) \qquad \left[\frac{W}{m^3}\right]$$

where g $\left[\frac{W\cdot s}{m}\right]$ is the source strength per unit length

$\frac{g}{2\pi r}$ $\left[\frac{W\cdot s}{m^2}\right]$ is the source strength per unit area

r $[m]$ the variable r appears in the denominator as a scale factor of the transformation to polar coordinates

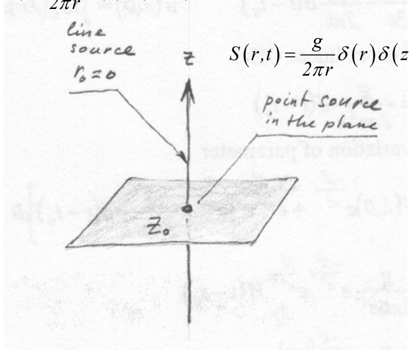
$\delta(r-r_0)$ $\left[\frac{1}{m}\right]$ defines the radius of the source $r=r_0$

$\delta(t-t_0)$ $\left[\frac{1}{s}\right]$ defines the moment of time $t=t_0$

$g\left[\frac{W\cdot s}{m}\right]$

$S(r,t)=\frac{g}{2\pi r}\delta(r)\delta(t-t_0)$ $\quad g[W\cdot s]$

$S(r,t)=\frac{g}{2\pi r}\delta(r)\delta(z-z_0)\delta(t-t_0)$

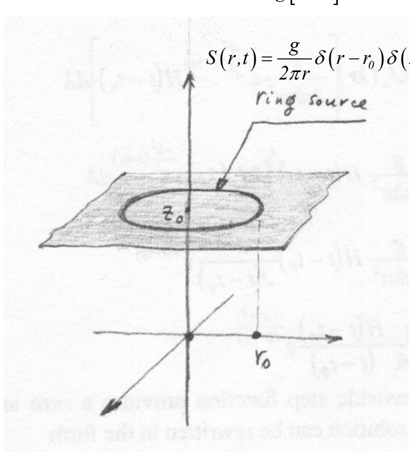


Integrated over the entire time-space domain (in the plane perpendicular to the z coordinate), it should yield the strength of the heat source (conservation of energy):

$$\begin{aligned}\int_0^\infty\int_0^{2\pi}\int_0^\infty S(r,t)rdrd\theta dt &= \int_0^\infty\int_0^{2\pi}\int_0^\infty\left[\frac{g}{2\pi r}\delta(r-r_0)\delta(t-t_0)\right]rdrd\theta dt\\ &=\frac{g}{2\pi}\int_0^\infty\int_0^{2\pi}\int_0^\infty\delta(r-r_0)\delta(t-t_0)drd\theta dt\\ &=\frac{g}{2\pi}\int_0^\infty\int_0^{2\pi}\delta(t-t_0)d\theta dt\\ &=\frac{g}{2\pi}\int_0^{2\pi}d\theta\\ &=\frac{g}{2\pi}[\theta]_0^{2\pi}\\ &=g\end{aligned}$$

$g[W\cdot s]$

$S(r,t)=\frac{g}{2\pi r}\delta(r-r_0)\delta(z-z_0)\delta(t-t_0)$



In the 3-dimensional case, the heat source is a ***circular ring*** in the plane $z=z_0$ defined by

$$S(r,t)=\frac{g}{2\pi r}\delta(r-r_0)\delta(z-z_0)\delta(t-t_0)$$

An instantaneous ***source*** is used for finding the Green function for the non-homogeneous PDE. The heat source for HE equation is defined by

$$\frac{S(r,t)}{k}$$

where k $\left[\frac{W}{m\cdot K}\right]$ is a thermal conductivity of the medium and

S $\left[\frac{W}{m^3}\right]$ is a volumetric heat generation rate.

IX.4.3 HEAT EQUATION WITH INSTANTANEOUS LINE SOURCE

$r = r_0 = 0$

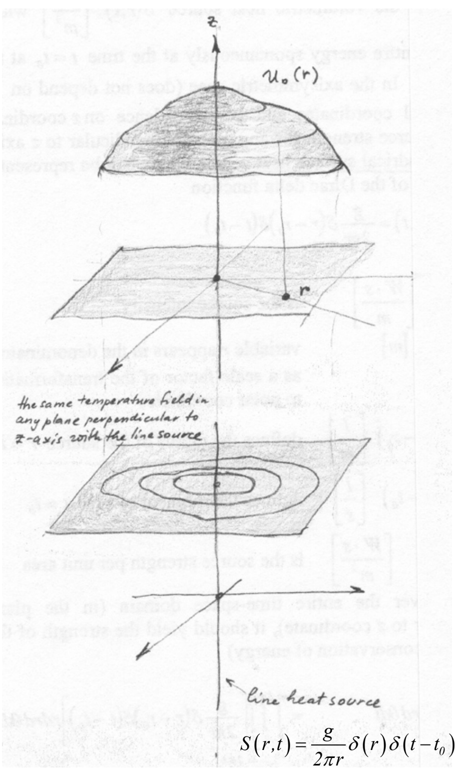
Consider the axisymmetric case of the Heat Equation in cylindrical coordinates with the instantaneous ***line source*** located at $r = r_0 = 0$ (line source in 3-D)

$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = a^2\frac{\partial u}{\partial t} - \frac{g}{2\pi r k}\delta(r)\delta(t-t_0) \quad r\in[0,\infty) \quad t>0$$

$$u(r,0) = u_0(r) = 0 \qquad (\textit{initial condition})$$

There are no boundary conditions but we assume that the function u and its derivative has radial decaying at infinity:

$$ru\big|_{r\to\infty} = 0 \qquad r\frac{\partial u}{\partial r}\bigg|_{r\to\infty} = 0$$



1) Transformed equation Apply zero order Hankel transform

$$\bar{u}(\lambda,t) = \int_0^\infty rJ_0(\lambda r)u\,dr$$

to the equation and initial condition

$$-\lambda^2\bar{u} = a^2\frac{\partial\bar{u}}{\partial t} - \frac{g}{2\pi k}\delta(t-t_0) \quad \bar{u}(\lambda,0) = \int_0^\infty rJ_0(\lambda r)u_0\,dr = 0$$

this is the 1st order linear ODE

$$\frac{\partial\bar{u}}{\partial t} + \frac{\lambda^2}{a^2}\bar{u} = \frac{g}{2\pi a^2 k}\delta(t-t_0)$$

The solution by variation of parameter

$$\begin{aligned}
\bar{u} &= \bar{u}(\lambda,0)e^{\frac{-\lambda^2}{a^2}t} + e^{\frac{-\lambda^2}{a^2}t}\int_0^t e^{\frac{\lambda^2}{a^2}t}\left[\frac{g}{2\pi a^2 k}\delta(t-t_0)\right]dt \\
&= \frac{g}{2\pi k a^2}e^{\frac{-\lambda^2}{a^2}t}e^{\frac{\lambda^2}{a^2}t_0}H(t-t_0) \\
&= \frac{g}{2\pi k a^2}e^{\frac{-\lambda^2}{a^2}(t-t_0)}H(t-t_0)
\end{aligned}$$

2) Inverse transform

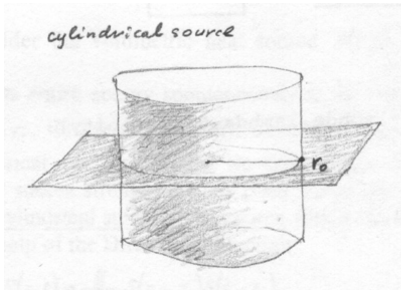
$$\begin{aligned}
u(r,t) &= \int_0^\infty \lambda J_0(\lambda r)\bar{u}(\lambda,t)\,d\lambda \\
&= \int_0^\infty \lambda J_0(\lambda r)\left[\frac{g}{2\pi k a^2}e^{\frac{-\lambda^2}{a^2}(t-t_0)}H(t-t_0)\right]d\lambda \\
&= \frac{g}{2\pi k a^2}H(t-t_0)\int_0^\infty \lambda J_0(\lambda r)e^{\frac{-\lambda^2}{a^2}(t-t_0)}d\lambda \\
&= \frac{g}{2\pi k a^2}H(t-t_0)\frac{a^2}{2(t-t_0)}e^{\frac{-a^2r^2}{4(t-t_0)}} \\
&= \frac{g}{4\pi k}\frac{H(t-t_0)}{(t-t_0)}e^{\frac{-a^2r^2}{4(t-t_0)}}
\end{aligned}$$

The solution can be rewritten also in the form

$$u(r,t) = \begin{cases} 0 & t \le t_0 \\ \dfrac{g}{4\pi k(t-t_0)}e^{\frac{-a^2r^2}{4(t-t_0)}} & t > t_0 \end{cases}$$

Solution for $u_0 = 0$ and $r_0 = 0$ (H-1.mws)

Cylindrical heat source



3) Case $r = r_0 \neq 0$

$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = a^2\frac{\partial u}{\partial t} - \frac{g}{2\pi r k}\delta(r - r_0)\delta(t - t_0)$$

$$-\lambda^2\overline{u} = a^2\frac{\partial \overline{u}}{\partial t} - \frac{g}{2\pi k}\delta(t - t_0)J_0(\lambda r_0)$$

$$\begin{aligned}
\overline{u} &= \overline{u}(\lambda,0)e^{\frac{-\lambda^2}{a^2}t} + e^{\frac{-\lambda^2}{a^2}t}\int_0^t e^{\frac{\lambda^2}{a^2}t}\left[\frac{g}{2\pi a^2 k}\delta(t - t_0)J_0(\lambda r_0)\right]dt \\
&= \frac{g}{2\pi k a^2}e^{\frac{-\lambda^2}{a^2}t}e^{\frac{\lambda^2}{a^2}t_0}J_0(\lambda r_0)H(t - t_0) \\
&= \frac{g}{2\pi k a^2}e^{\frac{-\lambda^2}{a^2}(t - t_0)}J_0(\lambda r_0)H(t - t_0)
\end{aligned}$$

$$\begin{aligned}
u(r,t) &= \int_0^\infty \lambda J_\nu(\lambda r)\overline{u}(\lambda,t)d\lambda \\
&= \int_0^\infty \lambda J_\nu(\lambda r)\left[\frac{g}{2\pi k a^2}e^{\frac{-\lambda^2}{a^2}(t - t_0)}J_0(\lambda r_0)H(t - t_0)\right]d\lambda \\
&= \frac{g}{2\pi k a^2}H(t - t_0)\int_0^\infty \lambda J_\nu(\lambda r)J_0(\lambda r_0)e^{\frac{-\lambda^2}{a^2}(t - t_0)}d\lambda
\end{aligned}$$

integration in [Ozisik, p.161, eqn(3-151)]

$$\begin{aligned}
&= \frac{g}{2\pi k a^2}H(t - t_0)\frac{a^2}{2(t - t_0)}I_0\left[\frac{a^2 r r_0}{2(t - t_0)}\right]e^{\frac{-a^2\left(r^2 + r_0^2\right)}{4(t - t_0)}} \\
&= \frac{g}{4\pi k(t - t_0)}I_0\left[\frac{a^2 r r_0}{2(t - t_0)}\right]e^{\frac{-a^2\left(r^2 + r_0^2\right)}{4(t - t_0)}}H(t - t_0)
\end{aligned}$$

<u>Solution for</u> $u_0 \neq 0$ **Green's Function**

$$u(r,t) = \frac{g}{4\pi k(t - t_0)}I_0\left[\frac{a^2 r r_0}{2(t - t_0)}\right]e^{\frac{-a^2\left(r^2 + r_0^2\right)}{4(t - t_0)}}H(t - t_0)$$

This is the Green function for the Heat Equation in cylindrical coordinates.

H-1.mws **Hankel transform solution of Heat Equation with instanteneous point source** $r_0 = 0$

```
> restart;
> with(plots):
> it:=int(exp(-lambda^2*(t-t0)/a^2)*
BesselJ(0,lambda*r)*lambda,lambda=0..infinity):
> u(r,t):=g*it/2/Pi/k;
```

$$u(r,t) := \frac{1}{4}\,\frac{g\,a^2\,e^{\left(-\frac{a^2 r^2}{4\,(t-t0)}\right)}}{(t-t0)\,\pi\,k}$$

```
> g:=1;a:=2;t0:=1;k:=1;
```

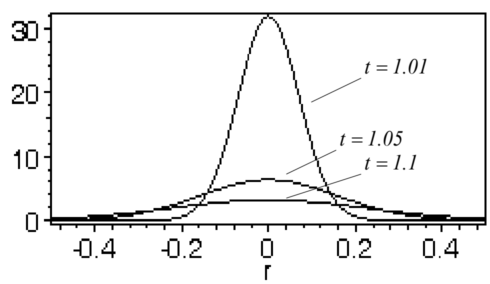
$$g := 1$$

$$a := 2$$

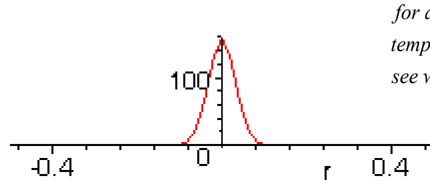
$$t0 := 1$$

$$k := 1$$

```
> u2(r,t):=subs(r=-r,u(r,t)):
> u1(r):=subs(t=1.01,u2(r,t)):u2(r):=subs(t=1.05,u2(r,t)):
u3(r):=subs(t=1.1,u2(r,t)):
> plot({u1(r),u2(r),u3(r)},r=-0.5..0.5,axes=boxed,color=black);
```

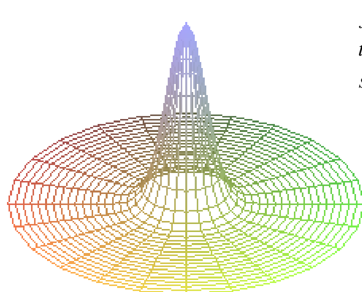



```
> animate(u(r,t),r=-0.5..0.5,t=1.002..1.2,frames=300);
```



for animation of temperature profile see web site

```
> animate3d([r,theta,u(r,t)],r=0..0.4,theta=0..2*Pi,t=1.004..1.2,
frames=200,style=wireframe,coords=cylindrical);
```



for animation of temperature profile see web site

Heat Equation with the instantantenous cylindrical source with $r_0 = 1$

```
> restart;
> g:=1;a:=2;t0:=0;r0:=1;k:=1;
```

$$g := 1$$

$$a := 2$$

$$t0 := 0$$

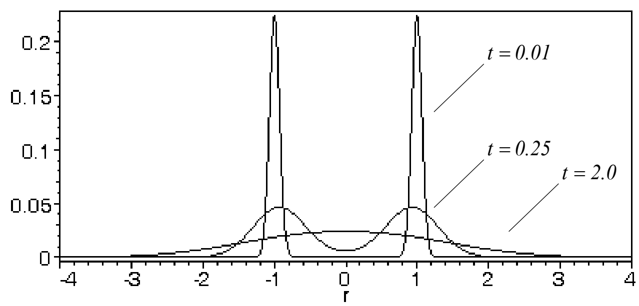
$$r0 := 1$$

$$k := 1$$

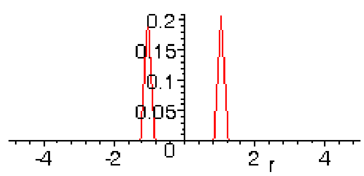
```
> u(r,t):=g/4/Pi/(t-t0)*BesselI(0,a^2*r0*r/2/(t-t0))*
exp(-(r^2+r0^2)/4/(t-t0)*a^2);
```

$$\mathrm{u}(r, t) := \frac{1}{4}\,\frac{\mathrm{BesselI}\left(0, \frac{2\,r}{t}\right)\,\mathbf{e}^{\left(-\frac{r^2+1}{t}\right)}}{\pi\,t}$$

```
> u2(r,t):=subs(r=-r,u(r,t)):
> u1(r):=subs(t=0.01,u2(r,t)):u2(r):=subs(t=0.25,u2(r,t)):
u3(r):=subs(t=2.0,u2(r,t)):
> plot({u1(r),u2(r),u3(r)},r=-4..4,axes=boxed,color=black);
```

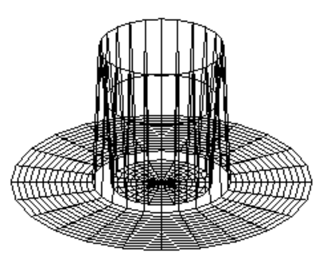



```
> with(plots):
> animate(u(r,t),r=-5..5,t=0.002..3,frames=300);
```



for animation of temperature profile see web site

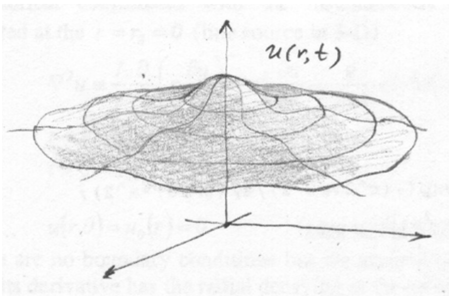
```
> animate3d([r,theta,u(r,t)],r=0..2.5,theta=0..2*Pi,t=0.004..3,
frames=200,style=wireframe,coords=cylindrical,color=black);
```



for animation of temperature profile see web site

H-1-3.gif

IX.4.5 THE WAVE EQUATION (*vibration of axisymmetric infinite membrane*)



$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad r \in [0,\infty) \quad t > 0$$

i.c. $u(r,0) = u_0(r)$

$$\frac{\partial u(r,0)}{\partial t} = u_1(r)$$

1) <u>Transformed equation</u> Apply zero order Hankel transform

$$\overline{u}(\lambda,t) = \int_0^\infty rJ_0(\lambda r)u(r,t)dr$$

to the equation and initial condition

$$-\lambda^2\overline{u} = \frac{1}{a^2}\frac{\partial^2\overline{u}}{\partial t^2} \qquad \overline{u}(\lambda,0) = \int_0^\infty rJ_0(\lambda r)u_0 dr = \overline{u}_0$$

$$\frac{\partial\overline{u}}{\partial t}(\lambda,0) = \int_0^\infty rJ_0(\lambda r)u_1 dr = \overline{u}_1$$

this is the 2nd order linear ODE

$$\frac{\partial^2\overline{u}}{\partial t^2} + a^2\lambda^2\overline{u} = 0$$

General solution

$$\overline{u} = c_1\cos(a\lambda t) + c_2\sin(a\lambda t)$$

$$\overline{u}' = -a\lambda c_1\sin(a\lambda t) + a\lambda c_2\cos(a\lambda t)$$

Initial conditions:

$$\overline{u}(\lambda,0) = \overline{u}_0 = c_1$$

$$\frac{\partial\overline{u}}{\partial t}(\lambda,0) = \overline{u}_1 = a\lambda c_2 \quad\Rightarrow\quad c_2 = \frac{\overline{u}_1}{a\lambda}$$

Solution:

$$\overline{u} = \overline{u}_0\cos(a\lambda t) + \frac{\overline{u}_1}{a\lambda}\sin(a\lambda t)$$

2) <u>Inverse transform</u>:

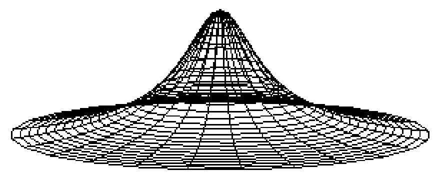
Solution of the problem is given by the inverse Hankel transform

$$u(r,t) = \int_0^\infty \lambda J_0(\lambda r)\overline{u}_0(\lambda)\cos(a\lambda t)d\lambda + \frac{1}{a}\int_0^\infty \lambda J_0(\lambda r)\overline{u}_1(\lambda)\sin(a\lambda t)d\lambda$$

3) Maple example *(Hankel-1.mws)*:

Let the membrane have initially the bell shape and $u_1(r) = 0$:

$$u_0(r) = \frac{1}{\sqrt{r^2+1}} \quad \text{or} \quad u_0(r) = \frac{A}{\sqrt{r^2+s}}$$



The Hankel transform of this function is

$$\overline{u}_0(\lambda) = \frac{e^{-\lambda}}{\lambda} \qquad \overline{u}_0(\lambda) = A\frac{e^{-s\lambda}}{\lambda}$$

Assume also zero initial velocity of the membrane

$$u_1(r) = 0$$

Then the solution of this problem is given by the following integral

$$u(r,t) = \int_0^\infty J_0(\lambda r) e^{-\lambda} \cos(a\lambda t) d\lambda$$

[I.S.Gradstein, I.M.Ryzhik, Table of Integrals #6.751.3]

$$\int_0^\infty J_0(c\lambda) e^{-b\lambda} \cos(a\lambda) d\lambda = \frac{\left[\sqrt{\left(b^2+c^2-a^2\right)^2+4a^2b^2}+b^2+c^2-a^2\right]^{\frac{1}{2}}}{\sqrt{2}\sqrt{\left(b^2+c^2-a^2\right)^2+4a^2b^2}}$$

identify: $a = at \quad b = -1 \quad c = r$

Then the solution becomes:

$$u(r,t) = = \frac{\left[\sqrt{\left(1+r^2-a^2t^2\right)^2+4a^2t^2}+1+r^2-a^2t^2\right]^{\frac{1}{2}}}{\sqrt{2}\sqrt{\left(1+r^2-a^2t^2\right)^2+4a^2t^2}}$$

Hankel-1.mws

Wave equation - vibration of the infinite membrane

```
> restart;
> with(plots):
```

Initial condition:

```
> u0(r):=1/sqrt(r^2+1);
```

$$u0(r) := \frac{1}{\sqrt{r^2+1}}$$

```
> cylinderplot([r,theta,u0(r)],r=0..6,theta=0..2*Pi);
```

Hankel transform of initial condition:

```
> ub0(lambda):=simplify(factor(int(r*BesselJ(0,lambda*r)*u0(r),
r=0..infinity)));
```

$$ub0(\lambda) := \frac{e^{(-\lambda)}}{\lambda}$$

```
> v:=2;
```

$$v := 2$$

Solution - inverse Hankel transform:

```
> u(r,t):=int(lambda*BesselJ(0,lambda*r)*ub0(lambda)*
cos(v*lambda*t),lambda=0..5);
```

$$u(r, t) := \int_0^5 \text{BesselJ}(0, \lambda\, r)\, e^{(-\lambda)} \cos(2\,\lambda\, t)\, d\lambda$$

Table of integrals [Gradshtein, Ryzhik #6.751.3]

```
> b:=-1;a:=-v*t;c:=r;
```

$$b := -1$$

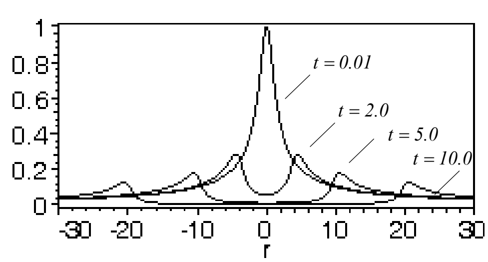
$$a := -2\,t$$

$$c := r$$

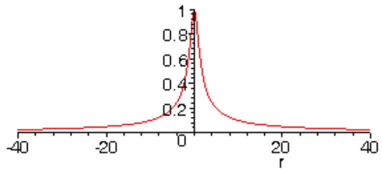
```
> u(r,t):=sqrt(sqrt((b^2+c^2-a^2)^2+4*a^2*b^2)+b^2+c^2-a^2)/
sqrt((b^2+c^2-a^2)^2+4*a^2*b^2)/sqrt(2);
```

$$u(r, t) := \frac{\sqrt{\sqrt{1+2\,r^2+8\,t^2+r^4-8\,r^2\,t^2+16\,t^4}+1+r^2-4\,t^2}\,\sqrt{2}}{2\sqrt{1+2\,r^2+8\,t^2+r^4-8\,r^2\,t^2+16\,t^4}}$$

```
> u2(r,t):=subs(r=-r,u(r,t)):
> u1(r):=subs(t=0.01,u2(r,t)):u2(r):=subs(t=2.0,u2(r,t)):
u3(r):=subs(t=5.0,u2(r,t)):u4(r):=subs(t=10.0,u2(r,t)):
> plot({u1(r),u2(r),u3(r),u4(r)},r=-30..30,axes=boxed,color=black);
```

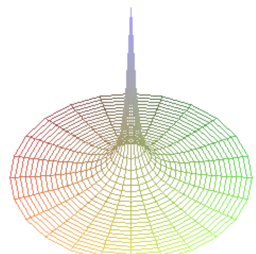


```
> animate({u2(r,t),u(r,t)},r=-40..40,t=0..20,frames=200);
```



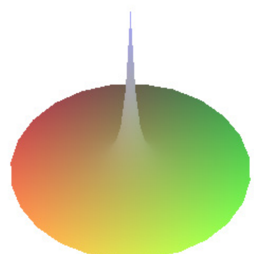
for animation see web site

```
> animate3d([r,theta,u(r,t)],r=0..40,theta=0..2*Pi,t=0..20,
frames=100,style=wireframe,coords=cylindrical);
```



for animation see web site

```
> animate3d([r,theta,u(r,t)],r=0..40,theta=0..2*Pi,t=0..20,
frames=100,style=patchnogrid,coords=cylindrical);
```



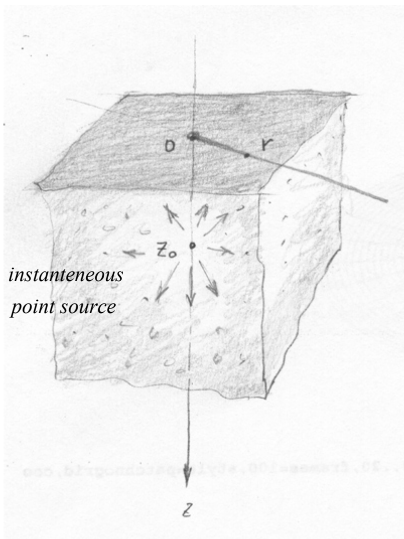
for animation see web site

IX.4.6 Solution of PDE with application of two integral transforms (Fourier and Hankel transforms)

Diffusion of light (heat) in semi-infinite space

Consider a problem of diffusion of light in the half-space $z \geq 0$ with instantaneous point source

$$S(r,t) = \frac{Q}{2\pi r}\delta(r)\delta(z - z_0)\delta(t)$$



Since the problem has the angular symmetry in the *xy*-plane, it is reasonable to consider it in the cylindrical coordinates. In the purely scattering medium, the diffusion approximation of light propagation is described by the following equation:

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} - \frac{1}{d}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) - \frac{1}{d}\frac{\partial^2 \varphi}{\partial z^2} = \frac{Q}{2\pi}\delta(z)\delta(t)\frac{\delta(r)}{r}$$

The boundary condition describes partial reflection of light from the surface and partial escape of the light through the surface

$$\left[-d\frac{\partial \varphi}{\partial z} + h\varphi\right]_{z=0} = 0$$

Initial condition:

$$\varphi\big|_{t=0} = 0$$

where	$\varphi(r,z,t)$	fluency describes the radiation field
	$d = 3(1-g)\mu$	diffusion coefficient
	μ	scattering coefficient
	g	anisotropy factor
	c	speed of light
	Q	source power
	$h = \frac{1}{2}\frac{1-R}{1+R}$	
	R	surface reflectivity (optical property)
	$H = \frac{h}{d}$	

Determine what type of integral transform to use:
The Laplacian includes derivatives w.r.t. two variables:
$z \in [0,\infty)$ (*Robin b.c*) $\Rightarrow$ Fourier standard transform
$r \in [0,\infty)$ $\Rightarrow$ Hankel transform

Fourier standard transform

$$\overline{\varphi}(r,\omega,t) = \int_0^\infty \varphi(r,z,t)K(\omega,z)dz$$

$$\varphi(r,z,t) = \int_0^\infty \overline{\varphi}(z,\omega,t)K(\omega,z)d\omega$$

$$K(\omega,z) = \frac{\omega\cos\omega z + H\sin\omega z}{N(\omega)}$$

$$N(\omega) = \sqrt{\frac{\pi}{2}\left(\omega^2 + H^2\right)} \qquad \frac{1}{N^2} = \frac{2}{\pi}\frac{1}{\omega^2 + H^2}$$

Hankel transform

$$\overline{\overline{\varphi}}(\lambda,\omega,t) = \int_0^\infty rJ_0(\lambda r)\overline{\varphi}(r,\omega,t)dr$$

$$\overline{\varphi}(r,\omega,t) = \int_0^\infty \lambda J_0(\lambda r)\overline{\varphi}(\lambda,\omega,t)d\lambda$$

Apply integral transforms to the equation and initial condition:

1) Fourier Transform
$$\int_0^\infty \delta(z)K(\omega,z)dz = \frac{\omega}{N(\omega)}$$

$$\frac{1}{c}\frac{\partial\overline{\varphi}}{\partial t}-\frac{1}{d}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{\varphi}}{\partial r}\right)+\frac{\omega^2}{d}\overline{\varphi}=\frac{Q}{2\pi}\frac{\omega}{N(\omega)}\delta(t)\frac{\delta(r)}{r}$$

2) Hankel Transform
$$\int_0^\infty rJ_0(\lambda r)\frac{\delta(r)}{r}d\rho = 1$$

transformed equation
$$\frac{1}{c}\frac{\partial\overline{\overline{\varphi}}}{\partial t}+\frac{\lambda^2+\omega^2}{d}\overline{\overline{\varphi}}=\frac{Q}{2\pi}\frac{\omega}{N(\omega)}\delta(t) \qquad \left.\overline{\overline{\varphi}}\right|_{t=0}=0$$

solution by variation of parameter
$$\overline{\overline{\varphi}} = \frac{cQ}{2\pi}\frac{\omega}{N(\omega)}e^{-(\lambda^2+\omega^2)\frac{c}{d}t}\int_0^\tau e^{-(\lambda^2+\omega^2)\frac{c}{d}\tau}\delta(\tau)d\tau$$
$$= \frac{cQ}{2\pi}\frac{\omega}{N(\omega)}e^{-(\lambda^2+\omega^2)\frac{c}{d}t}$$

3) Inverse Hankel Transform
$$\overline{\varphi} = \frac{cQ}{2\pi}\frac{\omega}{N(\omega)}e^{-\omega^2\frac{c}{d}t}\int_0^\infty \lambda J_0(\lambda r)e^{-\lambda^2\frac{c}{d}t}d\lambda$$
$$= \frac{cQ}{2\pi}\frac{\omega}{N(\omega)}e^{-\omega^2\frac{c}{d}t}\frac{d}{2ct}e^{\frac{-d}{4ct}r^2}$$
$$= \frac{Qd}{4\pi t}e^{\frac{-d}{4ct}r^2}\frac{\omega}{N(\omega)}e^{-\omega^2\frac{c}{d}t}$$

4) Inverse Fourier Transform
$$\varphi = \frac{Qd}{4\pi t}e^{\frac{-d}{4ct}r^2}\int_0^\infty \frac{\omega}{N(\omega)}e^{-\omega^2\frac{c}{d}t}K(\omega,z)d\omega$$
$$= \frac{Qd}{4\pi t}e^{\frac{-d}{4ct}r^2}\int_0^\infty \frac{\omega}{N(\omega)}\frac{\omega\cos\omega z+H\sin\omega z}{N(\omega)}e^{-\omega^2\frac{c}{d}t}d\omega$$
$$= \frac{Qd}{2\pi^2 t}e^{\frac{-d}{4ct}r^2}\int_0^\infty \frac{\omega^2\cos\omega z+H\omega\sin\omega z}{\omega^2+H^2}e^{-\omega^2\frac{c}{d}t}d\omega$$

Solution
$$\varphi(r,z,t) = \frac{Qd}{2\pi^2 t}e^{\frac{-d}{4ct}r^2}\int_0^\infty \frac{\omega^2\cos\omega z+H\omega\sin\omega z}{\omega^2+H^2}e^{-\omega^2\frac{c}{d}t}d\omega$$

$$= \frac{Qd}{2\pi^2 t}e^{\frac{-d}{4ct}r^2}\int_0^\infty \frac{(\omega^2+H^2-H^2)\cos\omega z+H\omega\sin\omega z}{\omega^2+H^2}e^{-\omega^2\frac{c}{d}t}d\omega$$
$$= \frac{Qd}{2\pi^2 t}e^{\frac{-d}{4ct}r^2}\int_0^\infty\left[\cos\omega z-\frac{H^2\cos\omega z}{\omega^2+H^2}+\frac{H\omega\sin\omega z}{\omega^2+H^2}\right]e^{-\omega^2\frac{c}{d}t}d\omega$$

$$= \frac{Qd}{2\pi^2 t}e^{\frac{-d}{4ct}r^2}\left\{\int_0^\infty[\cos\omega z]e^{-\omega^2\frac{c}{d}t}d\omega - H^2\int_0^\infty\left[\frac{\cos\omega z}{\omega^2+H^2}\right]e^{-\omega^2\frac{c}{d}t}d\omega + H\int_0^\infty\left[\frac{\omega\sin\omega z}{\omega^2+H^2}\right]e^{-\omega^2\frac{c}{d}t}d\omega\right\}$$
$$= \frac{Qd}{2\pi^2 t}e^{\frac{-d}{4ct}r^2}\left\{\frac{1}{2}\sqrt{\frac{\pi d}{ct}}e^{\frac{-z^2 d}{4ct}} - H^2\int_0^\infty\left[\frac{\cos\omega z}{\omega^2+H^2}\right]e^{-\omega^2\frac{c}{d}t}d\omega + H\int_0^\infty\left[\frac{\omega\sin\omega z}{\omega^2+H^2}\right]e^{-\omega^2\frac{c}{d}t}d\omega\right\}$$

SI-7b.mws **Diffusion of Radiation in the semi-infinite space from instantaneous point source**

```
> restart;with(plots):
```

Fluency:

```
> phi:=Q*d/2/Pi^2/t*exp(-d*r^2/4/c/t)*(1/2*sqrt(Pi*d/c/t)*exp(-z^2*d/4/c/t)-H^2*int((cos(omega*z))/(omega^2+H^2)*exp(-omega^2*c*t/d),omega=0..25)+int((H*omega*sin(omega*z))/(omega^2+H^2)*exp(omega^2*c*t/d),omega=0..25));
```

$$\phi := \frac{1}{2} Q\, d\, e^{\left(-1/4 \frac{d r^2}{c t}\right)} \left(\frac{1}{2}\sqrt{\frac{\pi d}{c t}}\, e^{\left(-1/4 \frac{z^2 d}{c t}\right)} - H^2 \int_0^{25} \frac{\cos(\omega z)\, e^{\left(-\frac{\omega^2 c t}{d}\right)}}{\omega^2 + H^2} d\omega + \int_0^{25} \frac{H \omega \sin(\omega z)\, e^{\left(-\frac{\omega^2 c t}{d}\right)}}{\omega^2 + H^2} d\omega \right) \Big/ (\pi^2 t)$$

```
> d:=1;c:=300000000;Q:=1;H:=10;G:=10000000;
```

$$d := 1$$

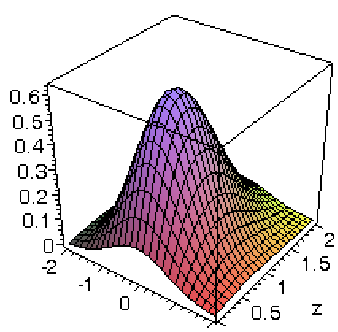
$$c := 300000000$$

$$Q := 1$$

$$H := 10$$

$$G := 10000000$$

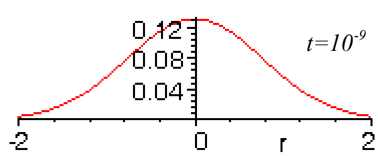
```
> f1:=subs(t=0.000000001,phi)/G:
> f2:=subs(r=-r,f1):
> plot3d({f1,f2},r=-2..2,z=0..2,axes=boxed);
```



distribution of fluency in the medium at t=10^{-9}

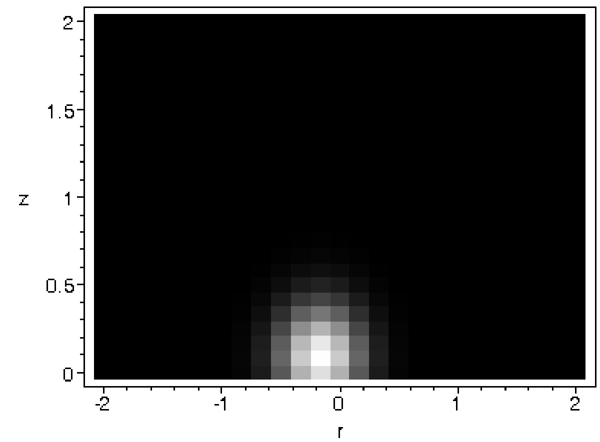
Reflection – fluency at the surface z=0 - light exiting the layer:

```
> R(r):=subs(z=0,f1):
> RR(r):=subs(r=-r,R(r)):
> plot({RR(r),R(r)},r=-2..2);
```



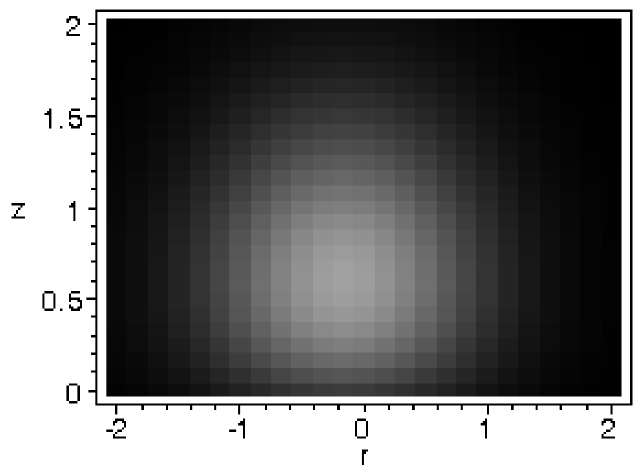
t=10^{-9}

```
> densityplot({f1,f2},r=-2..2,z=0..2,axes=boxed,style=patchnogrid);
```



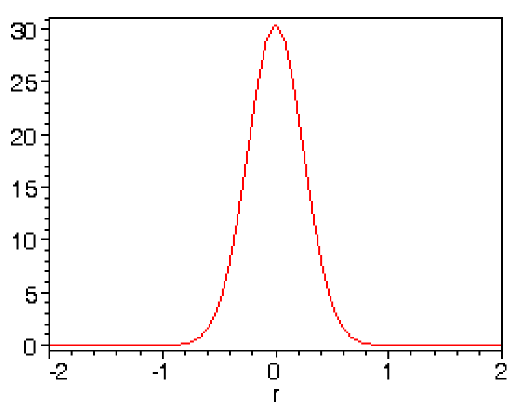
time = 10^{-10}

```
> densityplot({f1,f2},r=-2..2,z=0..2,style=patchnogrid);
```



time = 10^{-9}

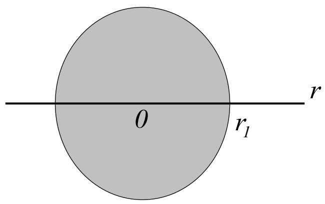
```
> R1:=subs(z=0,phi)/G:
> R2:=subs(r=-r,R1):
> animate({R1,R2},r=-2..2,t=0.0000000001..0.000000001,frames=100);
```



for animation of reflection profile see web site

IX.4.7 FINITE HANKEL TRANSFORM CIRCULAR DOMAIN – SOLID CYLINDER

IX.4.7.1 Circular Domain



The Hankel transform which we consider in this section will be applied for transformation of the differential operator

$$Lu \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u \tag{1}$$

where $\nu \in \mathbb{R}$ is a parameter and domain is $0 \le r < r_1$.

The particular case of this operator with $\nu = 0$ is the radial term of Laplacian in cylindrical coordinates.

We assume that the function $u(r) < \infty$ is bounded for all r and it satisfies a boundary condition at $r = r_1$ (which can be one of the the three types):

$$u(r_1) = f \tag{2}$$

$$\frac{\partial}{\partial r}u(r_1) = f$$

$$\left[k\frac{\partial u}{\partial r}+hu\right]_{r=r_1} = f$$

The self-adjoint form of operator is

$$Lu \equiv \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\left(-\frac{\nu^2}{r}\right)u\right]$$

Supplemental Sturm-Liouville problem.

The singular eigenvalue problem for considered operator applied to function $R(r)$ is

$$LR = \lambda R$$

subject to one of homogeneous boundary conditions

$$R(r_1) = 0, \quad R'(r_1) = 0, \quad \text{or} \quad \left[k\frac{\partial R}{\partial r}+hR\right]_{r=r_1} = 0$$

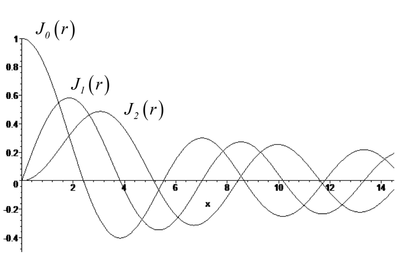
and it should be bounded $R(0) < \infty$ at the center.

Equation can be rewritten in Sturm-Liouville form as

$$(rR')' + \left[-\frac{\nu^2}{r}+(-\lambda)r\right]R = 0$$

has non-trivial solution when the parameter is non-positive, let us rename it as $-\lambda^2$, and the weight function is $p(r) = r$.

That yields the equation



$$r^2R'' + rR' + \left(\lambda^2 r^2 - \nu^2\right)R = 0 \tag{3}$$

which is the Bessel equation of order ν, the bounded solutions of which are the Bessel functions of the 1st kind

$$R(r) = J_\nu(\lambda r)$$

Application of boundary conditions (see Chapter VII, p.502) yields the characteristic equations for eigenvalues λ_n, and the corresponding eigenfunctions

$$R_n(r) = J_\nu(\lambda_n r)$$

Inner product Vector Space

Inner product and the norm are defined as

$$(u,v)_p = \int_0^{r_1} u(r)\,v(r)\,r\,dr$$

$$\|u\|_p^2 = (u,u)_p = \int_0^{r_1} u^2(r)\,r\,dr$$

According to SLT, the set of eigenfunctions $\{J_\nu(\lambda_n r)\}$ is orthogonal on $r \in [0, r_1]$ with the weight function r and can be used for Fourier-Bessel expansion of the function u:

$$u(r) = \sum_{n=1}^{\infty} c_n J_\nu(\lambda_n r)$$

where

$$c_n = \frac{\int_0^{r_1} u(r) J_\nu(\lambda_n r) r dr}{\int_0^{r_1} J_\nu^2(\lambda_n r) r dr} = \frac{\int_0^{r_1} u(r) J_\nu(\lambda_n r) r dr}{\|J_\nu(\lambda_n r)\|^2}$$

If we rearrange the coefficients in the expansion

$$u(r) = \sum_{n=1}^{\infty} \left[\frac{\int_0^{r_1} u(r) J_\nu(\lambda_n r) r dr}{\|J_\nu(\lambda_n r)\|^2} \right] J_\nu(\lambda_n r) = \sum_{n=1}^{\infty} \left[\int_0^{r_1} u(r) J_\nu(\lambda_n r) r dr \right] \frac{J_\nu(\lambda_n r)}{\|J_\nu(\lambda_n r)\|^2}$$

then the integral transform pair of order ν can be defined as

Finite Hankel Transform (FHT)

$$H_\nu\{u\} = \quad \bar{u}_n = \int_0^{r_1} u(r) J_\nu(\lambda_n r) r dr$$

Inverse Finite Hankel Transform

$$H_\nu^{-1}\{\bar{u}_n\} = u(r) = \sum_{n=1}^{\infty} \bar{u}_n \frac{J_\nu(\lambda_n r)}{\|J_\nu(\lambda_n r)\|^2}$$

with the inverse transform in the from of an infinite series.

Because functions $J_\nu(\lambda_n r)$ are solutions of SLP, they are solutions of BE

$$\left[r J_\nu'(\lambda_n r) \right]' + \left(-\frac{\nu^2}{r} + \lambda_n^2 r \right) J_\nu(\lambda_n r) = 0$$

or it can be rewritten as

$$\left[r J_\nu'(\lambda_n r) \right]' = \left(\frac{\nu^2}{r} - \lambda_n^2 r \right) J_\nu(\lambda_n r)$$

and satisfy one of the homogeneous boundary conditions:

I) *Dirichlet* $J_\nu(\lambda_n r_1) = 0$

II) *Neumann* $J_\nu'(\lambda_n r_1) = 0$

III) *Robin* $k J_\nu'(\lambda_n r_1) + h J_\nu(\lambda_n r_1) = 0$ or $J_\nu'(\lambda_n r_1) = -\frac{h}{k} J_\nu(\lambda_n r_1)$

$k\left[-J_{\nu+1}(\lambda_n r_1)\lambda_n + \frac{\nu}{r} J_\nu(\lambda_n r_1) \right] + h$?

Operational property of Finite Hankel Transform

Consider the differential operator with respect to radial variable r

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u$$

which is the objects for elimination with the help of FHT. Apply FHT of order ν

$$H_\nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right\}$$

$$=\int_0^{r_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

$$=\int_0^{r_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]J_\nu(\lambda_n r)rdr-\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

$$=\int_0^{r_1}J_\nu(\lambda_n r)d\left(r\frac{\partial u}{\partial r}\right)-\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)J_\nu(\lambda_n r)\right]_0^{r_1}-\int_0^{r_1}r\frac{\partial u}{\partial r}J'_\nu(\lambda_n r)dr-\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)J_\nu(\lambda_n r)\right]_{r=r_1}-\int_0^{r_1}rJ'_\nu(\lambda_n r)du-\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

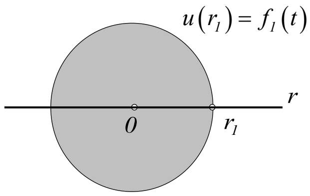
$$=\left[\left(r\frac{\partial u}{\partial r}\right)J_\nu(\lambda_n r)\right]_{r=r_1}-\left[rJ'_\nu(\lambda_n r)u\right]_0^{r_1}+\int_0^{r_1}u\left[rJ'_\nu(\lambda_n r)\right]'dr-\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)J_\nu(\lambda_n r)\right]_{r=r_1}-\left[rJ'_\nu(\lambda_n r)u\right]_{r=r_1}+\int_0^{r_1}u\left[\left(\frac{\nu^2}{r}-\lambda_n^2 r\right)J_\nu(\lambda_n r)\right]dr-\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)J_\nu(\lambda_n r)\right]_{r=r_1}-\left[rJ'_\nu(\lambda_n r)u\right]_{r=r_1}-\lambda_n^2\int_0^{r_1}uJ_\nu(\lambda_n r)rdr+\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr-\int_0^{r_1}\left[\frac{\nu^2}{r^2}u\right]J_\nu(\lambda_n r)rdr$$

$$=r_1\left.\frac{\partial u}{\partial r}\right|_{r=r_1}J_\nu(\lambda_n r_1)-r_1J'_\nu(\lambda_n r_1)u(r_1)-\lambda_n^2\bar{u}_n$$ for any of I, II, or III type of b.c.

For the particular type of boundary conditions, the result can be specified as following:



I **Dirichlet** $u(r_1) = f_1(t)$

$J_\nu(\lambda_n r_1) = 0$

Eigenvalues λ_n are the positive roots of $J_\nu(\lambda r_1) = 0$

Eigenfunctions $R_n(r) = J_0(\lambda_n r)$

Norm $\|R_n(r)\|^2 = \dfrac{r_1^2}{2} J_{\nu+1}^2(\lambda_n r_1)$

Operational property

$$H_\nu^{(1)}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u\right\} = r_1\lambda_n J_{\nu+1}(\lambda_n r_1) f_1(t) - \lambda_n^2 \overline{u}_n$$

$$H_\nu^{(1)}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u\right\} = -r_1\lambda_n J_{\nu-1}(\lambda_n r_1) f_1(t) - \lambda_n^2 \overline{u}_n$$

Fourier-Bessel series

$$\overline{u}_n = \int_0^{r_1} u(r) J_\nu(\lambda_n r)\, r\, dr \qquad \text{Finite Hankel Transform}$$

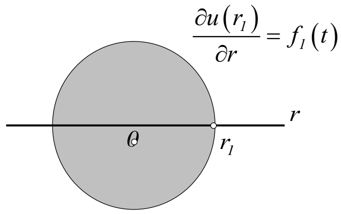
$$u(r) = \sum_{n=1}^{\infty} \overline{u}_n \frac{J_\nu(\lambda_n r)}{\|J_\nu(\lambda_n r)\|^2} \qquad \text{Inverse Hankel Transform}$$

Derivation of operational property

$$\begin{aligned}
H_\nu^{(1)}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u\right\} &= r_1 \left.\frac{\partial u}{\partial r}\right|_{r=r_1} J_\nu(\lambda_n r_1) - r_1 J_\nu'(\lambda_n r_1) u(r_1) - \lambda_n^2 \overline{u}_n \\
&= -r_1 J_\nu'(\lambda_n r_1) f_1(t) - \lambda_n^2 \overline{u}_n \\
&= -r_1\lambda_n\left[-J_{\nu+1}(\lambda_n r_1) + \frac{\nu}{\lambda_n r_1} J_\nu(\lambda_n r_1)\right] f_1(t) - \lambda_n^2 \overline{u}_n \\
&= r_1\lambda_n J_{\nu+1}(\lambda_n r_1) f_1(t) - \lambda_n^2 \overline{u}_n
\end{aligned}$$

or if the other recurrence formula is used, then

$$\begin{aligned}
&= -r_1\lambda_n\left[J_{\nu-1}(\lambda_n r_1) - \frac{\nu}{\lambda_n r_1} J_\nu(\lambda_n r_1)\right] f_1(t) - \lambda_n^2 \overline{u}_n \\
&= -r_1\lambda_n J_{\nu-1}(\lambda_n r_1) f_1(t) - \lambda_n^2 \overline{u}_n
\end{aligned}$$



II Neumann $\frac{\partial u(r_1)}{\partial r} = f_1(t)$

$J'_\nu(\lambda_n r_1) = 0$ or

$-\lambda J_{\nu+1}(\lambda L) + \frac{\nu}{L} J_\nu(\lambda L) = 0$ (Ch.VII, p.507)

$\nu > 0$

Eigenvalues λ_n are the $\lambda_0 = 0$

and the positive roots of $-\lambda J_{\nu+1}(\lambda L) + \frac{\nu}{L} J_\nu(\lambda L) = 0$

Eigenfunctions $R_0 = 1$

$R_n(r) = J_0(\lambda_n r)$

Norm: $\|R_0\|^2 = \int_0^{r_1} r dr = \frac{r_1^2}{2}$

$$\|R_n(r)\|^2 = \frac{r_1^2}{2}\left(1 - \frac{\nu^2}{\lambda_n^2 r_1^2}\right) J_\nu^2(\lambda_n r_1)$$

$\nu = 0$

Eigenvalues λ_n are the $\lambda_0 = 0$

and the positive roots of $J_1(\lambda L) = 0$,

Eigenfunctions: $R_0 = 1$

$R_n(r) = J_0(\lambda_n r)$

Squared Norm: $\|R_0\|^2 = \int_0^{r_1} r dr = \frac{r_1^2}{2}$

$$\|R_n(r)\|^2 = \int_0^{r_1} J_0^2(\lambda_n r) r dr = \frac{r_1^2}{2} J_1^2(\lambda_n r_1)$$

Operational Property

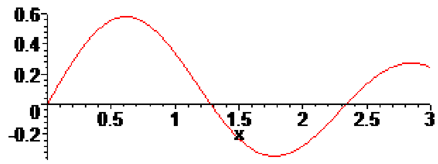
$$H_{\nu>0}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u\right\} = 0, \qquad n = 0$$

$$H_{\nu>0}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u\right\} = r_1 J_\nu(\lambda_n r_1) f_1(t) - \lambda_n^2 \bar{u}_n, \qquad n = 1,2,\ldots$$

$$H_{\nu=0}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{0^2}{r^2}u\right\} = r_1 f_1(t) \qquad n = 0$$

$$H_{\nu=0}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{0^2}{r^2}u\right\} = r_1 J_0(\lambda_n r_1) f_1(t) - \lambda_n^2 \bar{u}_n \qquad n = 1,2,\ldots$$

The case $\nu = 0$, $\lambda_0 = 0,\ \lambda_n > 0,\ n = 1,2,...$ λ_n *are the roots of the equation* $J_1(\lambda r_1) = 0$



Characteristic equation $J_1(\lambda_n r_1) = 0$

Eigenfunctions $y_0(r) = 1$

$$y_n(r) = J_0(\lambda_n r)$$

Squared norm $\|y_0(r)\|^2 = \frac{r_1^2}{2}$

$$\|y_n(r)\|^2 = \frac{r_1^2}{2} J_0^2(\lambda_n r_1)$$

FHT $\bar{u}_0 = \int_0^{r_1} u(r) r dr$ For $u = const$, $\bar{u}_0 = u r_1$

$$\bar{u}_n = \int_0^{r_1} u(r) J_0(\lambda_n r) r dr \qquad \bar{u}_n = u\frac{J_1(\lambda_n r_1)}{\lambda_n} = 0$$

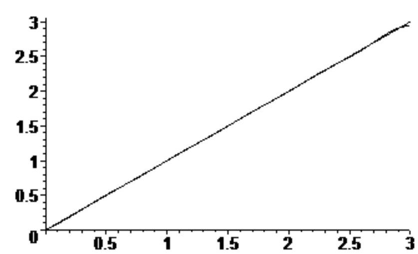
Inverse FHT $u(r) = \frac{\bar{u}_0}{\|J_0(\lambda_0 r)\|^2} + \sum_{n=1}^{\infty} \bar{u}_n \frac{J_\nu(\lambda_n r)}{\|J_\nu(\lambda_n r)\|^2}$

$$u(r) = \frac{2}{r_1^2}\bar{u}_0 + \frac{2}{r_1^2}\sum_{n=1}^{\infty} \bar{u}_n \frac{J_0(\lambda_n r)}{J_0^2(\lambda_n r_1)}$$

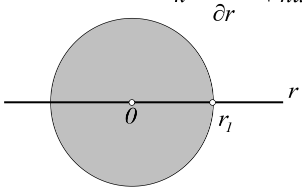
Operational Property $H_{\nu=0}^{(II)}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{0^2}{r^2}u\right\} = r_1 J_0(\lambda_n r_1) f_1(t) - \lambda_n^2 \bar{u}_n \quad n = 1,2,...$

$$H_{\nu=0}^{(II)}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{0^2}{r^2}u\right\} = r_1 f_1(t) \qquad n = 0$$

DIDIER / 00 Hajduk Finite Hankel example



$$k\frac{\partial u(r_1)}{\partial r}+hu(r_1)=f_1(t)$$



III **Robin**

$$k\frac{\partial u(r_1)}{\partial r}+hu(r_1)=f_1(t)$$

$$kJ'_\nu(\lambda_n r_1)+hJ_\nu(\lambda_n r_1)=0 \quad \text{or}$$

$$J'_\nu(\lambda_n r_1)=-\frac{h}{k}J_\nu(\lambda_n r_1)$$

Eigenvalues λ_n are the positive roots of

$$-\lambda\, J_{\nu+1}(\lambda r_1)+\left(H+\frac{\nu}{L}\right)J_\nu(\lambda r_1)=0 \qquad \nu>0$$

$$-\lambda\, J_1(\lambda r_1)+H\, J_0(\lambda r_1)=0 \qquad \nu=0$$

Eigenfunctions

$$R_n(r)=J_0(\lambda_n r)$$

Norm

$$\left\|R_n(r)\right\|^2=\frac{r_1^2}{2}\left[\frac{H^2}{\lambda_n^2}+\left(1-\frac{\nu^2}{\lambda_n^2 r_1^2}\right)\right]J_\nu^2(\lambda_n r_1) \qquad \nu>0$$

$$\left\|R_n(r)\right\|^2=\frac{r_1^2}{2}\left(\frac{H^2}{\lambda_n^2}+1\right)J_0^2(\lambda_n r_1) \qquad \nu=0$$

Fourier-Bessel series

$$\overline{u}_n=\int_0^{r_1}u(r)J_\nu(\lambda_n r)\,r\,dr \qquad \text{Finite Hankel Transform}$$

$$u(r)=\sum_{n=1}^{\infty}\overline{u}_n\frac{J_\nu(\lambda_n r)}{\left\|J_\nu(\lambda_n r)\right\|^2} \qquad \text{Inverse Hankel Transform}$$

Operational property

$$H_\nu^{(1)}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right\}=r_1 J_\nu(\lambda_n r_1)\frac{f_1(t)}{k}-\lambda_n^2\overline{u}_n$$

Derivation of operational property

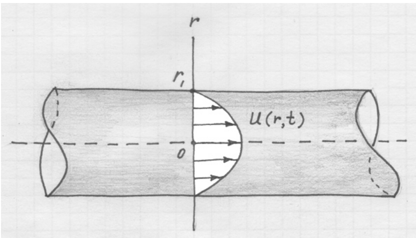
$$\begin{aligned}
H_\nu^{(1)}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right\}&=r_1\left.\frac{\partial u}{\partial r}\right|_{r=r_1}J_\nu(\lambda_n r_1)-r_1 J'_\nu(\lambda_n r_1)u(r_1)-\lambda_n^2\overline{u}_n\\
&=r_1\left.\frac{\partial u}{\partial r}\right|_{r=r_1}J_\nu(\lambda_n r_1)+r_1\frac{h}{k}J_\nu(\lambda_n r_1)u(r_1)-\lambda_n^2\overline{u}_n\\
&=r_1\left[\left.\frac{\partial u}{\partial r}\right|_{r=r_1}+\frac{h}{k}u(r_1)\right]J_\nu(\lambda_n r_1)-\lambda_n^2\overline{u}_n\\
&=r_1 J_\nu(\lambda_n r_1)\frac{f_1(t)}{k}-\lambda_n^2\overline{u}_n
\end{aligned}$$

Properties

$$H_0\left\{ 1 \right\} = \qquad = \frac{r_1}{\lambda_n} J_1\left(\lambda_n r_1\right)$$

January, 2019

IX.4.7.2 Development of velocity profile in a pipe under a pressure gradient



Consider the viscous fluid which is initially at rest inside the long circular cylinder, and then set in motion by a spontaneously imposed difference between the pressures at the ends of the pipe. Let G be the axial pressure gradient, then axial velocity $u(r,t)$ is described by the equation [Batchelor]:

$$\frac{\partial u}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{G}{\rho} \qquad r \in [0, r_1]$$

with the initial condition (initially fluid is at rest)
$u(r,0) = 0$
and the boundary condition
$u(r,t) = 0 \quad t > 0$ Dirichlet

1) Integral transform The form of integral transform is determined by the domain (finite) and the boundary condition (Dirichlet), the order of FHT is determined by the first term in the r.h.s. of the equation - order 0. According to (Bessel Functions, section 12, example 3):

Eigenvalues λ_n are positive roots of $J_0(\lambda r_1) = 0$

Eigenfunctions $X_n(r) = J_0(\lambda_n r) \qquad \left\| J_0(\lambda_n r) \right\|^2 = \frac{r_1^2}{2} J_1^2(\lambda_n r_1)$

$$\overline{u}_n(t) = \int_0^{r_1} u(r,t) J_0(\lambda_n r) r dr$$

$$\overline{S}_n = \int_0^{r_1} \frac{G}{\rho} J_0(\lambda_n r) r dr = \frac{G}{\rho}\left[\frac{rJ_1(\lambda_n r)}{\lambda_n}\right]_0^{r_1} = \frac{G}{\rho}\frac{r_1 J_1(\lambda_n r_1)}{\lambda_n}$$

2) Transformed equation Operational property I of FHT-I (p.825) yields

$$\frac{\partial \overline{u}_n}{\partial t} = -\nu\lambda_n^2 \overline{u}_n + \frac{G}{\rho}\frac{r_1 J_1(\lambda_n r_1)}{\lambda_n}$$

$$\frac{\partial \overline{u}_n}{\partial t} + \nu\lambda_n^2 \overline{u}_n = \frac{r_1 G}{\rho}\frac{J_1(\lambda_n r_1)}{\lambda_n}$$

Solution by variation of parameter

$$\overline{u}_n = \frac{r_1 G}{\rho}\frac{J_1(\lambda_n r_1)}{\lambda_n}\frac{1}{\nu\lambda_n^2}\left(1 - e^{-\nu\lambda_n^2 t}\right)$$

3) Inverse transform Solution of IBVP:

$$u(r,t) = \sum_{n=1}^{\infty} \overline{u}_n \frac{J_0(\lambda_n r)}{\left\| J_0(\lambda_n r) \right\|^2} = \frac{2G}{\nu\rho r_1}\sum_{n=1}^{\infty}\frac{\left(1 - e^{-\nu\lambda_n^2 t}\right)}{\lambda_n^3}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_1)}$$

4) Steady state

$$u_s(r) = \frac{G}{4\nu\rho}\left(r_1^2 - r^2\right) = \frac{2G}{\nu\rho r_1}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_1)}$$

6) Maple Example (*pipe-01.mws)*

pipe-01.mws

Starting Flow in a Pipe (Batchelor, p.193) Solution with the Finite Hankel Transform **FHT-1**

```
> restart;with(plots):
> nu:=0.1;G:=3;rho:=2;r1:=0.5;
```

$$\nu := 0.1$$

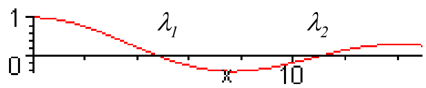
$$G := 3$$

$$\rho := 2$$

$$r1 := 0.5$$

```
> w(x):=BesselJ(0,x*r1);plot(w(x),x=0..15);
```

$$w(x) := \mathrm{BesselJ}(0, 0.5\,x)$$



Eigenvalues:

```
> lambda:=array(1..50);
```

$$\lambda := \mathrm{array}(1 .. 50, [\ ])$$

```
> n:=1: for m from 1 to 100 do z:=fsolve(w(x)=0,x=m*1..(m+1)*1): if type(z,float) then lambda[n]:=z: n:=n+1 fi od:
> for i to 2 do lambda[i] od;
```

$$4.809651115$$

$$11.04015622$$

```
> N:=n-1;
```

$$N := 5$$

```
> n:='n':i:='i':x:='x':x:='z':
```

Eigenfunctions:

```
> u[n](r,t):=2*G/r1/nu/rho*BesselJ(0,lambda[n]*r)/lambda[n]^3/BesselJ(1,lambda[n]*r1)*(1-exp(-nu*lambda[n]^2*t));
```

$$u_n(r,t) := \frac{60.00000000\,\mathrm{BesselJ}(0, \lambda_n\, r)\left(1 - \mathbf{e}^{\left(-0.1\,\lambda_n^{\,2}\, t\right)}\right)}{\lambda_n^{\,3}\,\mathrm{BesselJ}(1, 0.5\,\lambda_n)}$$

Solution:

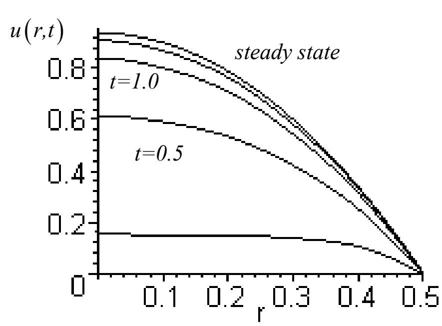
```
> u(r,t):=sum(u[n](r,t),n=1..N):
> u0:=subs(t=0.1,u(r,t)):
> u1:=subs(t=0.5,u(r,t)):
> u2:=subs(t=1,u(r,t)):
> u3:=subs(t=1.5,u(r,t)):
```

Steady state solution (from Batchelor)**:**

```
> f:=G*(r1^2-r^2)/4/nu/rho;
```

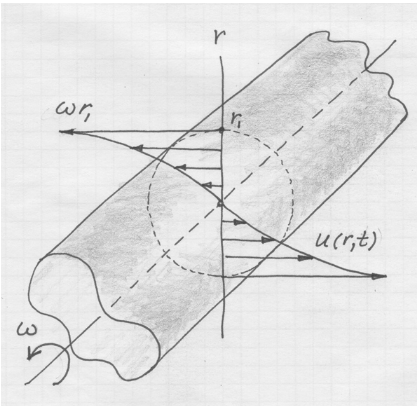
$$f := 0.9375000000 - 3.750000000\, r^2$$

```
> plot({f,u0,u1,u2,u3},r=0..r1,color=black);
```



for animation of the velocity profile development see web site

IX.4.7.3 Viscous flow inside of the rotating cylinder – development of the velocity profile



Consider the motion of viscous fluid inside of the circular cylinder generated by rotation of cylinder around its axis. If the motion of fluid remains purely rotational then the axial component of velocity is equal to zero and the development of the angular velocity is described by the equation

$$\frac{1}{a}\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{1}{r^2}u\,, \qquad r\in[0,r_1]$$

with initial condition (initially fluid is at rest)

$$u(r,0)=0$$

and boundary condition

$$u(r,t)=\omega r_1\,,\ t>0 \qquad \text{Dirichlet} \qquad \omega=const$$

1) **Steady state solution** has to satisfiy the boundary value problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_s}{\partial r}\right)-\frac{1}{r^2}u_s=0\,,\quad u_s(r_1)=\omega r_1\,,\qquad u_s(0)<\infty$$

That yields the Euler equation

$$r^2u''+ru'-u=0$$

Then bounded solution satisfying the boundary condition is

$$u_s(r)=\omega r$$

2) Integral transform The form of integral transform is determined by the domain (finite) and the boundary condition (Dirichlet), the order of FHT is determined by the second term in the r.h.s. of equation - order 1. According to (Bessel Functions, section 12, example 3):

Eigenvalues λ_n are positive roots of $J_1(\lambda r_1)=0$

Eigenfunctions $X_n(r)=J_1(\lambda_n r)$ $\qquad \|J_1(\lambda_n r)\|^2=\frac{r_1^2}{2}J_0^2(\lambda_n r_1)$

FHT of $u(r,t)$ $\qquad \bar{u}_n(t)=\int_0^{r_1}u(r,t)J_1(\lambda_n r)rdr$

3) Transformed equation Operational property I of FHT (p.825) yields

$$\frac{1}{a}\frac{\partial\bar{u}_n(t)}{\partial t}=-\lambda_n^2\bar{u}_n(t)-\lambda_n r_1 J_0(\lambda_n r_1)\omega r_1\,,\qquad \bar{u}_n(0)=0$$

Apply the Laplace transform

$$s\bar{U}_n(s)=-a\lambda_n^2\bar{U}_n(s)-a\lambda_n r_1 J_0(\lambda_n r_1)\omega r_1\cdot\frac{1}{s}$$

Then

$$\bar{U}_n(s)=-a\lambda_n r_1 J_0(\lambda_n r_1)\omega r_1\cdot\frac{1}{s}\cdot\frac{1}{s+a\lambda_n^2}$$

Use convolution theorem for the inverse Laplace transform, then

$$\bar{u}_n(t)=\omega r_1^2\,\frac{J_0(\lambda_n r)}{\lambda_n}\left(e^{-\nu\lambda_n^2 t}-1\right)$$

4) Inverse Finite Hankel transform provides solution of the problem

$$u(r,t)=\sum_{n=1}^{\infty}\bar{u}_n\frac{J_1(\lambda_n r)}{\|J_1(\lambda_n r)\|^2}=2\omega\sum_{n=1}^{\infty}\frac{\left(e^{-\nu\lambda_n^2 t}-1\right)}{\lambda_n}\frac{J_1(\lambda_n r)}{J_0(\lambda_n r_1)} \qquad (1)$$

We can split solution into the steady state and the transient parts:

$$u(r,t) = \underbrace{-2\omega\sum_{n=1}^{\infty}\frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n r_1)}}_{u_s(r)} + 2\omega\sum_{n=1}^{\infty}\frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n r_1)}e^{-a\lambda_n^2 t}$$

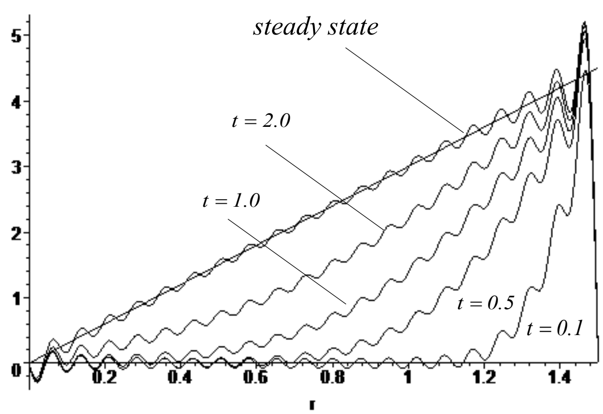
$$u(r,t) = \omega r + 2\omega\sum_{n=1}^{\infty}\frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n r_1)}e^{-a\lambda_n^2 t} \qquad (2)$$

That will allow eliminating oscillations in plotting the truncated solution

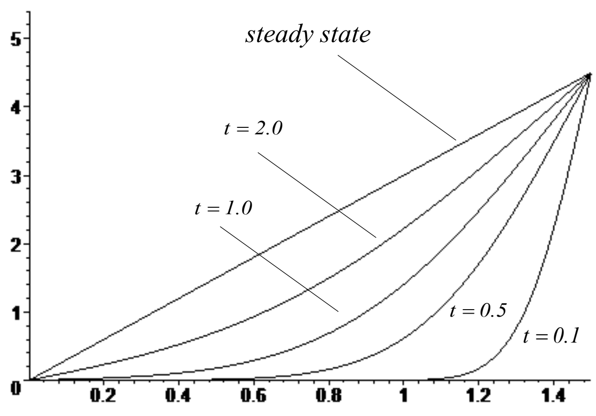
5) Example **Viscous flow inside of rotating cylinder**
Solution with the Finite Hankel Transform **FHT-1**

pipe-02.mws $a = 0.1$, $\omega = 3.0$, $r_1 = 1.5$.

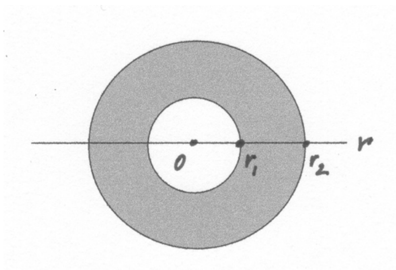
Non-splitted solution, Eqn. (1)



Solution splitted into the analytical steady state and the transient parts, Eqn.(2)



IX.4.7.4 ANNULAR DOMAIN HOLLOW CYLINDER



Auxiliary Sturm-Liouville Problem. Consider a two-parameter Bessel equation of order ν in the annular domain

$$r^2u'' + ru' + \left(\lambda^2 r^2 - \nu^2\right)u = 0, \qquad r \in (r_1, r_2) \tag{BE}$$

with the homogeneous boundary conditions:

$$\left[-k_1 \frac{\partial u}{\partial r} + h_1 u\right]_{r=r_1} = 0$$

$$\left[k_2 \frac{\partial u}{\partial r} + h_2 u\right]_{r=r_2} = 0$$

According to SLT, this regular boundary value problem has infinitely many solutions (eigenfunctions)

$$X_n(r) = c_{1,n} J_\nu(\lambda_n r) + c_{2,n} Y_\nu(\lambda_n r)$$

where eigenvalues λ_n are the roots of the corresponding characteristic equation (table Circular Domain).

The set of eigenfunctions $\{X_n(r)\}$ is orthogonal on $r \in [r_1, r_2]$ with the weight function r and can be used for Fourier-Bessel expansion of the function u:

$$u(r) = \sum_{n=1}^{\infty} a_n X_n(r)$$

where

$$a_n = \frac{\int_{r_1}^{r_2} u(r) X_n(r) r dr}{\int_{r_1}^{r_2} X_n^2(r) r dr} = \frac{\int_{r_1}^{r_2} u(r) X_n(r) r dr}{\left\| X_n(r) \right\|^2}$$

If we rearrange the coefficients in the expansion

$$u(r) = \sum_{n=1}^{\infty} \left[\frac{\int_{r_1}^{r_2} u(r) X_n(r) r dr}{\left\| X_n(r) \right\|^2} \right] X_n(r) = \sum_{n=1}^{\infty} \left[\int_{r_1}^{r_2} u(r) X_n(r) r dr \right] \frac{X_n(r)}{\left\| X_n(r) \right\|^2}$$

then the integral transform pair of order ν can be defined as

Finite Hankel Transform

$$\bar{u}_n = \int_{r_1}^{r_2} u(r) X_n(r) r\, d r$$

Inverse Finite Hankel Transform

$$u(r) = \sum_{n=1}^{\infty} \bar{u}_n \frac{X_n(r)}{\left\| X_n(r) \right\|_p^2}$$

with the inverse transform in the form of an infinite series.

Because the eigenfunctions $X_n(r)$ are solutions of SLP, they are solutions of BE

$$[rX_n'(r)]' + \left(-\frac{\nu^2}{r} + \lambda^2 r\right)X_n(r) = 0 \quad \text{or} \quad [rX_n'(r)]' = \left(\frac{\nu^2}{r} - \lambda^2 r\right)X_n(r)$$

and satisfy one of the combinations of the homogeneous boundary conditions:

I) Dirichlet $X_n(r_1) = 0$ $X_n(r_2) = 0$

II) Neumann $X_\nu'(r_1) = 0$ $X_\nu'(r_2) = 0$

III) Robin $-k_1 X_n'(r_1) + h_1 X_n(r_1) = 0$ $+k_2 X_n'(r_2) + h_2 X_n(r_2) = 0$

There are 9 possible combinations of these boundary conditions.

Operational property Consider the application of the Finite Hankel transform to the following operator

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u$$

Apply FHT of order ν

$$\int_{r_1}^{r_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\nu^2}{r^2}u\right]X_n(r)rdr =$$

$$= \int_{r_1}^{r_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]X_n(r)rdr - \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr$$

$$= \int_{r_1}^{r_2} X_n(r)d\left(r\frac{\partial u}{\partial r}\right) - \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr$$

$$= \left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}^{r_2} - \int_{r_1}^{r_2} r\frac{\partial u}{\partial r}X_n'(r)dr - \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr$$

$$= \left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}^{r_2} - \int_{r_1}^{r_2} rX_n'(r)du - \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr$$

$$= \left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}^{r_2} - [rX_n'(r)u]_{r_1}^{r_2} + \int_{r_1}^{r_2} u[rX_n'(r)]'dr - \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr$$

$$= \left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}^{r_2} - [rX_n'(r)u]_{r_1}^{r_2} + \int_{r_1}^{r_2} u\left[\left(\frac{\nu^2}{r} - \lambda^2 r\right)X_n(r)\right]dr - \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr$$

$$= \left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}^{r_2} - [rX_n'(r)u]_{r_1}^{r_2} - \lambda^2\int_{r_1}^{r_2} uX_n(r)rdr + \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr - \int_{r_1}^{r_2}\left[\frac{\nu^2}{r^2}u\right]X_n(r)rdr$$

$$= \left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}^{r_2} - [rX_n'(r)u]_{r_1}^{r_2} - \lambda^2\bar{u}_n$$

For the particular type of boundary conditions, this result can be specified. The complete solution of the Sturm-Liouville problem in the annular domain is presented in the table" SLP in annular domain". Here, consider derivation of operational properties of FHT-2 for three types of boundary conditions:

Dirichlet-Dirichlet

$$X_n(r_1)=0\,,\qquad u(r_1)=f_1(t)$$

$$X_n(r_2)=0\,,\qquad u(r_2)=f_2(t)$$

$$=r_1X_n'(r_1)f_1(t)-r_2X_n'(r_2)f_2(t)-\lambda_n^2\overline{u}_n \qquad (☼\text{ D-D})$$

[*Solution of Jacob Badger and Jared Butler, Final exam, 2017*]

Robin-Robin

$$-k_1X_n'(r_1)+h_1X_n(r_1)=0 \qquad \Rightarrow \qquad X_n'(r_1)=\frac{h_1}{k_1}X_n(r_1)$$

$$k_2X_n'(r_2)+h_2X_n(r_2)=0 \qquad \Rightarrow \qquad X_n'(r_2)=\frac{-h_2}{k_2}X_n(r_2)$$

$$-k_1\frac{\partial u(r_1)}{\partial r}+h_1u(r_1)=f_1(t)$$

$$k_2\frac{\partial u(r_2)}{\partial r}+h_2u(r_2)=f_2(t)$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_2}-\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}-\left[rX_n'(r)u\right]_{r_2}+\left[rX_n'(r)u\right]_{r_1}-\lambda^2\overline{u}_n$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_2}-\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}-\left[-r\frac{h_2}{k_2}X_n(r)u\right]_{r_2}+\left[r\frac{h_1}{k_1}X_n(r)u\right]_{r_1}-\lambda^2\overline{u}_n$$

$$=\left(r_2\frac{\partial u(r_2)}{\partial r}\right)X_n(r_2)-\left(r_1\frac{\partial u(r_1)}{\partial r}\right)X_n(r_1)+r_2\frac{h_2}{k_2}X_n(r_2)u(r_2)+r_1\frac{h_1}{k_1}X_n(r_1)u(r_1)-\lambda^2\overline{u}_n$$

$$=r_1X_n(r_1)\left[-\frac{\partial u(r_1)}{\partial r}+\frac{h_1}{k_1}u(r_1)\right]+r_2X_n(r_2)\left[\frac{\partial u(r_2)}{\partial r}+\frac{h_2}{k_2}u(r_2)\right]-\lambda^2\overline{u}_n$$

$$=\frac{r_1}{k_1}X_n(r_1)f_1(t)+\frac{r_2}{k_2}X_n(r_2)f_2(t)-\lambda_n^2\overline{u}_n \qquad (☼\text{ R-R})$$

Neumann-Dirichlet

$$X_n'(r_1)=0 \qquad \frac{\partial u(r_1)}{\partial r}=f_1(t)$$

$$X_n(r_2)=0 \qquad u(r_2)=f_2(t)$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_2}-\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}-\left[rX_n'(r)u\right]_{r_2}+\left[rX_n'(r)u\right]_{r_1}-\lambda^2\overline{u}_n$$

$$=-\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}-\left[rX_n'(r)u\right]_{r_2}-\lambda^2\overline{u}_n$$

$$=-\left(r_1\frac{\partial u(r_1)}{\partial r}\right)X_n(r_1)-r_2X_n'(r_2)u(r_2)-\lambda^2\overline{u}_n$$

$$=-r_1X_n(r_1)f_1(t)-r_2X_n'(r_2)f_2(t)-\lambda_n^2\overline{u}_n \qquad (☼\text{ N-D})$$

Dirichlet-Neumann

$$X_n(r_1)=0 \qquad u(r_1)=f_1(t)$$
$$X_n'(r_2)=0 \qquad \frac{\partial u(r_2)}{\partial r}=f_2(t)$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_2}+\left[rX_n'(r)u\right]_{r_1}-\lambda^2\overline{u}_n$$

$$=\left(r_2\frac{\partial u(r_2)}{\partial r}\right)X_n(r_2)+r_1X_n'(r_1)u(r_1)-\lambda^2\overline{u}_n$$

$$\boxed{=r_1X_n'(r_1)f_1(t)+r_2X_n(r_2)f_2(t)-\lambda_n^2\overline{u}_n} \qquad \text{(☼ D-N)}$$

Robin-Dirichlet

$$\boxed{=\frac{r_1}{h_1}X_n'(r_1)f_1(t)-r_2X_n(r_2)f_2(t)-\lambda_n^2\overline{u}_n} \qquad \text{(☼ R-D)}$$

Neumann-Robin

$$\frac{\partial u(r_1)}{\partial r}=f_1(t) \qquad\qquad \frac{\partial u(r_1)}{\partial r}=f_1(t)$$

$$k_2\frac{\partial u(r_2)}{\partial r}+h_2u(r_2)=f_2(t) \qquad \Rightarrow \qquad \frac{\partial u(r_2)}{\partial r}+\frac{h_2}{k_2}u(r_2)=\frac{f_2(t)}{k_2}$$

$$X_n'(r_1)=0$$
$$k_2X_n'(r_2)+h_2X_n(r_2)=0 \qquad \Rightarrow \qquad X_n'(r_2)=\frac{-h_2}{k_2}X_n(r_2)$$

Operational property:

$$\int_{r_1}^{r_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\nu^2}{r^2}u\right]X_n(r)\,rdr$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}^{r_2}-\left[rX_n'(r)u\right]_{r_1}^{r_2}-\lambda^2\overline{u}_n$$

$$=r_2\left.\frac{\partial u}{\partial r}\right|_{r_2}X_n(r_2)-r_1\left.\frac{\partial u}{\partial r}\right|_{r_1}X_n(r_1)-r_2X_n'(r_2)u(r_2)+r_1X_n'(r_1)u(r_1)-\lambda^2\overline{u}_n$$

$$=r_2\left.\frac{\partial u}{\partial r}\right|_{r_2}X_n(r_2)-r_1\left.\frac{\partial u}{\partial r}\right|_{r_1}X_n(r_1)+r_2\frac{h_2}{k_2}X_n(r_2)u(r_2)+r_1\cancel{X_n'(r_1)}u(r_1)-\lambda^2\overline{u}_n$$

$$=r_2\left[\left.\frac{\partial u}{\partial r}\right|_{r_2}+\frac{h_2}{k_2}u(r_2)\right]X_n(r_2)-r_1\left.\frac{\partial u}{\partial r}\right|_{r_1}X_n(r_1)-\lambda^2\overline{u}_n$$

$$=r_2\frac{f_2(t)}{k_2}X_n(r_2)-r_1f_1(t)X_n(r_1)-\lambda^2\overline{u}_n$$

$$\boxed{=r_2\frac{f_2(t)}{k_2}X_n(r_2)-r_1f_1(t)X_n(r_1)-\lambda_n^2\overline{u}_n} \qquad \text{(☺ N-R}$$

Neumann- Neumann

$X_n'(r_1)=0\ ,\qquad \frac{\partial u(r_1)}{\partial r}=f_1(t)$

$X_n'(r_2)=0\ ,\qquad \frac{\partial u(r_2)}{\partial r}=f_2(t)$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_2}-\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}-\left[rX_n'(r)u\right]_{r_2}+\left[rX_n'(r)u\right]_{r_1}-\lambda^2\overline{u}_n$$

$$=\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_2}-\left[\left(r\frac{\partial u}{\partial r}\right)X_n(r)\right]_{r_1}-\lambda^2\overline{u}_n$$

$$=\left(r_2\frac{\partial u(r_2)}{\partial r}\right)X_n(r_2)-\left(r_1\frac{\partial u(r_1)}{\partial r}\right)X_n(r_1)-\lambda^2\overline{u}_n$$

$$\boxed{=-r_1X_n(r_1)f_1(t)+r_2X_n(r_2)f_2(t)-\lambda_n^2\overline{u}_n}\qquad (☼\ \text{N-N})$$

Robin-Neumann **[Final exam, 2018]** *zz Final F18 SF-AD-8-1.mw*

$$-k_1\frac{\partial u(r_1)}{\partial r}+h_1u(r_1)=f_1(t)\qquad\Rightarrow\qquad -\frac{\partial u(r_1)}{\partial r}+\frac{h_1}{k_1}u(r_1)=\frac{f_1(t)}{k_1}$$

$$\frac{\partial u(r_2)}{\partial r}=f_2(t)$$

$$-k_1X_n'(r_1)+h_1X_n(r_1)=0\qquad\Rightarrow\qquad X_n'(r_1)=\frac{h_1}{k_1}X_n(r_1)$$

$$X_n'(r_1)=0$$

Operational property:

$$\boxed{=\frac{r_1}{k_1}X_n(r_1)f_1(t)+r_2X_n(r_2)f_2(t)-\lambda_n^2\overline{u}_n}\qquad (☺\ \text{R-N})$$

Dirichlet-Robin **[HW#10, 2018]** *SF-AD-5-0.mws – boiling water*

$$u(r_1)=f_1(t)$$

$$+k_2\frac{\partial u(r_2)}{\partial r}+h_2u(r_2)=f_2(t)\qquad\Rightarrow\qquad +\frac{\partial u(r_2)}{\partial r}+\frac{h_2}{k_2}u(r_2)=\frac{f_2(t)}{k_2}$$

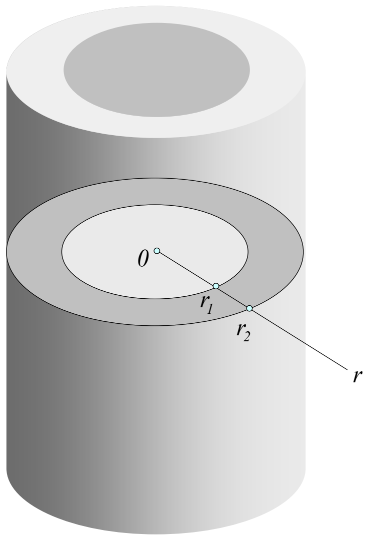
$$X_n(r_1)=0$$

$$+k_2X_n'(r_2)+h_2X_n(r_2)=0\qquad\Rightarrow\qquad X_n'(r_2)=-\frac{h_2}{k_2}X_n(r_2)$$

Operational property:

$$\boxed{=r_1X_n'(r_1)f_1(t)+\frac{r_2}{k_2}X_n(r_2)f_2(t)-\lambda_n^2\overline{u}_n}\qquad (☺\ \text{D-R})$$

Example **Heat Equation – Dirichlet problem**



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+S(r,t)=\frac{1}{a^2}\frac{\partial u}{\partial t} \qquad r_1<r<r_2,\ t>0$$

$$u(r,0)=u_0(r) \qquad r_1\le r\le r_2$$

$$u(r_1,t)=f_1(t) \qquad t>0$$

$$u(r_2,t)=f_2(t) \qquad t>0$$

1) Finite Hankel Transform of order $\nu=0$ (D-D, p.834, p.516)

$$-\lambda_n^2\,\overline{u}_n(t)+r_1X_n'(r_1)f_1(t)-r_2X_n'(r_2)f_2(t)+\overline{S}_n(t)=\frac{1}{a^2}\frac{\partial\overline{u}_n(t)}{\partial t}, \qquad \overline{u}_n(0)=\overline{u}_{0,n}$$

2) Apply Laplace transform to equation, $U_n(s)=\int_0^\infty \overline{u}_n(t)e^{-st}dt$, $\hat{\overline{S}}_n(s)=\int_0^\infty \overline{S}_n(t)e^{-st}dt$

$$\left(s+a^2\lambda_n^2\right)U_n(s)=-a^2r_1X_n'(r_1)\hat{f}_1(s)+a^2r_2X_n'(r_2)\hat{f}_2(s)-a^2\hat{\overline{S}}_n(s)+a^2\overline{u}_{0,n}$$

Solve for $U_n(s)$

$$U_n(s)=-a^2r_1X_n'(r_1)\hat{f}_1(s)\frac{1}{s+a^2\lambda_n^2}+a^2r_2X_n'(r_2)\hat{f}_2(s)\frac{1}{s+a^2\lambda_n^2}-a^2\hat{\overline{S}}_n(s)\frac{1}{s+a^2\lambda_n^2}+a^2\overline{u}_{0,n}\frac{1}{s+a^2\lambda_n^2}$$

3) Apply Inverse Laplace transform

$$\overline{u}_n(t)-a^2r_1X_n'(r_1)\int_{\tau=0}^{t}e^{-a^2\lambda_n^2(t-\tau)}f_1(\tau)d\tau+a^2r_2X_n'(r_2)\int_{\tau=0}^{t}e^{-a^2\lambda_n^2(t-\tau)}f_2(\tau)d\tau-a^2\int_{\tau=0}^{t}e^{-a^2\lambda_n^2(t-\tau)}\overline{S}_n(\tau)d\tau+a^2\overline{u}_{0,n}e^{-a^2\lambda_n^2t}$$

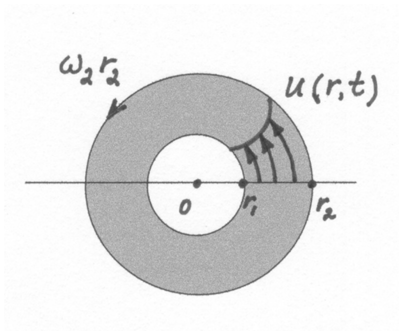
4) Apply Inverse Finite Hankel Transform to write the solution

$$u(r,t)=\sum_{n=1}^{\infty}\overline{u}_n(t)\frac{X_n(r)}{\left\|X_n(r)\right\|^2}$$

5) Maple example

November 30, 2019

IX.4.7.5 Viscous fluid flow between two cylinders



$$\frac{\partial u}{\partial t} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{1}{r^2}u\right] \quad = \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}u\right]$$

initial condition:

$$u(r,0) = u_0(r) = \omega_2 r_2\left(1 - \frac{r - r_2}{r_1 - r_2}\right)$$

boundary conditions:

$$\frac{\partial u(r_1)}{\partial r} = f_1(t) = c \qquad \text{Neumann}$$

$$u_2(r_2) = f_2(t) = \omega_2 r_2 \qquad \text{Dirichlet}$$

1) Integral transform FHT-2 of order $\nu = 1$

λ_n are positive roots of the equation:

$$\left[-J_2(\lambda r_1) + \frac{1}{\lambda r_1}J_1(\lambda r_1)\right]Y_1(\lambda r_1) - \left[-Y_2(\lambda r_1) + \frac{1}{\lambda r_1}Y_1(\lambda r_1)\right]J_1(\lambda r_1) = 0$$

Eigenfunctions:

$$X_n(r) = \frac{J_1(\lambda_n r)}{J_1(\lambda_n r_2)} - \frac{Y_1(\lambda_n r)}{Y_1(\lambda_n r_2)}$$

Transformed initial condition:

$$\overline{u}_{0,n} = \int_{r_1}^{r_2} u_0(r)X_n(r)rdr$$

2) Transformed equation Use equation (☼☼):

$$\frac{\partial \overline{u}_n}{\partial t} = \nu\left[-r_1 X_n(r_1)f_1(t) - r_2 X_n'(r_2)f_2(t) - \lambda^2\overline{u}_n\right]$$

Note: here, ν is a viscosity

$$\frac{\partial \overline{u}_n}{\partial t} + \nu\lambda^2\overline{u}_n = -\nu\left[r_1 X_n(r_1)f_1(t) + r_2 X_n'(r_2)f_2(t)\right] \quad = \nu Q_n$$

Notation: $Q_n(t) = \left[r_1 X_n(r_1)f_1(t) + r_2 X_n'(r_2)f_2(t)\right]$

Solution by variation of parameter

$$\begin{aligned}\overline{u}_n(t) &= \overline{u}_{0,n}e^{-\nu\lambda_n^2 t} - \nu e^{-\nu\lambda_n^2 t}\int_0^t Q_n(\tau)e^{\nu\lambda_n^2\tau}d\tau \\ &= \overline{u}_{0,n}e^{-\nu\lambda_n^2 t} - \nu Q_n e^{-\nu\lambda_n^2 t}\int_0^t e^{\nu\lambda_n^2\tau}d\tau \qquad (\text{assume } Q_n(t) = Q_n = const) \\ &= \overline{u}_{0,n}e^{-\nu\lambda_n^2 t} - \frac{\nu Q_n}{\nu\lambda_n^2}e^{-\nu\lambda_n^2 t}\left[e^{\nu\lambda_n^2\tau}\right]_0^t \\ &= \left(\overline{u}_{0,n} + \frac{Q_n}{\lambda_n^2}\right)e^{-\nu\lambda_n^2 t} - \frac{Q_n}{\lambda_n^2}\end{aligned}$$

3) Inverse transform

$$u(r) = \sum_{n=1}^{\infty}\left[\left(\overline{u}_{0,n} + \frac{Q_n}{\lambda_n^2}\right)e^{-\nu\lambda_n^2 t} - \frac{Q_n}{\lambda_n^2}\right]\frac{X_n(r)}{\|X_n(r)\|^2}$$

4) Steady state solution Euler-Cauchy equation

$$u_s(r) = \frac{1}{r_1^2 + r_1^2}\left[\frac{r_1^2 r_2^2(\omega_2 - f_1)}{x} + \left(r_1^2 f_1 + r_2^2\omega_2\right)x\right] \quad = \sum_{n=1}^{\infty}\frac{Q_n}{\lambda_n^2}\frac{X_n(r)}{\|X_n(r)\|^2}$$

FHT-2-01.mws

Annular domain : Neuman-Dirichlet boundary conditions

```
> restart;with(plots):
> u0:=omega2*L2*(1-(x-L2)/(L1-L2));
```

$$u0 := \omega 2\ L2\left(1-\frac{x-L2}{L1-L2}\right)$$

```
> f1:=1;f2:=omega2*L2;
```

$$f1 := 1$$

$$f2 := \omega 2\ L2$$

```
> omega1:=2;omega2:=5;v:=2;
```

$$\omega 1 := 2$$

$$\omega 2 := 5$$

$$v := 2$$

```
> L1:=2;L2:=3;
```

$$L1 := 2$$

$$L2 := 3$$

```
> nu:=1;
```

$$\nu := 1$$

Characteristic equation (Table SLP in annular domain):

```
> a11:=-x*BesselJ(nu+1,x*L1)+nu/L1*BesselJ(nu,x*L1);
```

$$a11 := -x\ \mathrm{BesselJ}(2, 2\,x) + \frac{1}{2}\mathrm{BesselJ}(1, 2\,x)$$

```
> a12:=-x*BesselY(nu+1,x*L1)+nu/L1*BesselY(nu,x*L1);
```

$$a12 := -x\ \mathrm{BesselY}(2, 2\,x) + \frac{1}{2}\mathrm{BesselY}(1, 2\,x)$$

```
> a21:=BesselJ(nu,x*L2);
```

$$a21 := \mathrm{BesselJ}(1, 3\,x)$$

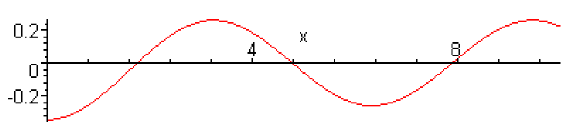
```
> a22:=BesselY(nu,x*L2);
```

$$a22 := \mathrm{BesselY}(1, 3\,x)$$

```
> w(x):=a11*a22-a12*a21;
```

$$\mathrm{w}(x) := \left(-x\ \mathrm{BesselJ}(2, 2\,x) + \frac{1}{2}\mathrm{BesselJ}(1, 2\,x)\right)\mathrm{BesselY}(1, 3\,x) - \left(-x\ \mathrm{BesselY}(2, 2\,x) + \frac{1}{2}\mathrm{BesselY}(1, 2\,x)\right)\mathrm{BesselJ}(1, 3\,x)$$

```
> plot(w(x),x=0..10);
```



Eigenvalues:

```
> lambda:=array(1..200);
```

$$\lambda := \mathrm{array}(1\,..\,200, [\ ])$$

```
> n:=1: for m from 1 to 150 do y:=fsolve(w(x)=0,x=m/2..(m+1)/2): if type(y,float) then lambda[n]:=y: n:=n+1 fi od:
> for i to 4 do lambda[i] od;
```

1.756990510

4.778245758

```
> N:=n-1;
```

$N := 24$

```
>n:='n':i:='i':m:='m':y:='y':x:='x':
```

Eigenfunctions:

```
> a11[n]:=subs(x=lambda[n],a11):
> a21[n]:=subs(x=lambda[n],a21):
> a12[n]:=subs(x=lambda[n],a12):
> a22[n]:=subs(x=lambda[n],a22):
> X[n](x):=BesselJ(nu,lambda[n]*x)/a21[n]-
BesselY(nu,lambda[n]*x)/a22[n];
```

$$X_n(x) := \frac{\mathrm{BesselJ}(1, \lambda_n x)}{\mathrm{BesselJ}(1, 3\,\lambda_n)} - \frac{\mathrm{BesselY}(1, \lambda_n x)}{\mathrm{BesselY}(1, 3\,\lambda_n)}$$

Derivative of eigenfunction, and the values of eigenfunction and its derivative at the boundaries:

```
> XP[n](x):=diff(X[n](x),x):
> X[n](L1):=subs(x=L1,X[n](x)):
> XP[n](L2):=subs(x=L2,XP[n](x)):
```

Norm:

```
> NN[n]:=int(x*X[n](x)^2,x=L1..L2):
```

Transformed initial condition:

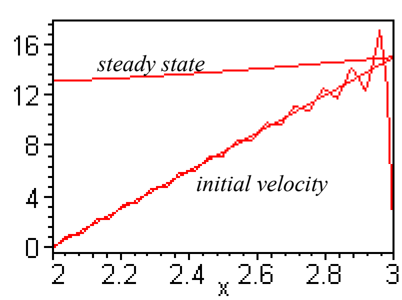
```
> u0t[n]:=int(u0*X[n](x)*x,x=L1..L2):
> Q[n]:=L1*X[n](L1)*f1+L2*XP[n](L2)*f2:
```

Fourier-Bessel series:

```
> u[n](x,t):=((u0t[n]+Q[n]/lambda[n]^2)*exp(-v*lambda[n]^2*t)-
Q[n]/lambda[n]^2)*X[n](x)/NN[n]:
> u(x,t):=sum(u[n](x,t),n=1..N):
```

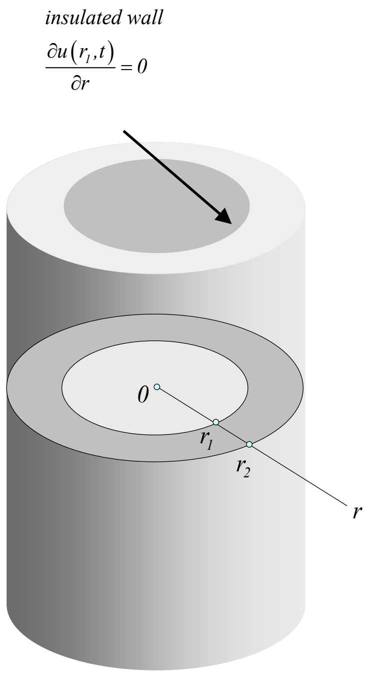
Steady State Solution:

```
> c1:=(f2-f1*L2)/(L2/L1^2+1/L2):c2:=(f1/L2+f2/L1^2)/(L2/L1^2+1/L2)
:us(x):=c1/x+c2*x:
> animate({u(x,t),us(x),u0},x=L1..L2,t=0..1,axes=boxed,frames=100);
```



for animation of the the velocity profile see web site

IX.4.7.7 Simple example of the Transient heat transfer through the cylindrical wall (modified IX.7.4.6) 2015



Heat equation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad \nu=0$$

Consider the following initial boundary value problem:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r_1<r<r_2,\ t>0$$

Uniform initial temperature:

$$u(r,0)=1 \qquad r_1\le r\le r_2$$

Boundary $r=r_1$ is thermoinsulated:

$$\frac{\partial u(r_1,t)}{\partial r}=\overbrace{0}^{f_1(t)}$$

Boundary $r=r_2$ is exposed to convective heat transfer with surroundings at 0 temperature:

$$k\frac{\partial u(r_2)}{\partial r}+h_2u(r_2)=0 \quad\Rightarrow\quad \frac{\partial u(r_2)}{\partial r}+H_2u(r_2)=\overbrace{0}^{f_2(t)},\ H_2=\frac{h_2}{k}$$

1) Finite Hankel Transform of order $\nu=0$ (see Neumann-Robin, Chapter 3, p.38)

$$\bar{u}_n(t)=\int_{r_1}^{r_2}u(r,t)X_n(r)rdr$$

Inverse Finite Hankel Transform

$$u(r,t)=\sum_{n=1}^{\infty}\bar{u}_n(t)\frac{X_n(r)}{\|X_n(r)\|^2}$$

where eigenfunctions (Finite Hankel of 0 order) are

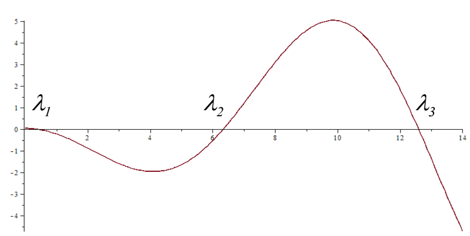
$$X_n^{\nu=0}(x)=\frac{J_0(\lambda_n x)}{-\lambda_nJ_1(\lambda_nr_2)+H_2J_\nu(\lambda_nr_2)}-\frac{Y_0(\lambda_n x)}{-\lambda_nY_1(\lambda_nr_2)+H_2Y_0(\lambda_nr_2)}$$

The squared norm of eigenfunctions is

$$\|X_n^0(r)\|^2=\int_{r_1}^{r_2}\left[X_n^0(r)\right]^2 rdr$$

Eigenvalues are positive roots of equation:

$$\lambda J_1(\lambda r_1)\left[-\lambda Y_1(\lambda r_2)+H_2Y_0(\lambda r_2)\right]-\lambda Y_1(\lambda r_1)\left[-\lambda J_1(\lambda r_2)+H_2J_\nu(\lambda r_2)\right]=0$$



2) Take Finite Hankel transform of equation (use operational property ☺☺, p.93):

$$r_2 \frac{f_2(t)}{k_2} X_n(r_2) - r_1 f_1(t) X_n(r_1) - \lambda_n^2 \overline{u}_n = \frac{1}{\alpha}\frac{\partial \overline{u}_n}{\partial t}$$

$$\frac{\partial \overline{u}_n(t)}{\partial t} + \alpha\lambda_n^2 \overline{u}_n(t) = 0$$

initial condition: $\overline{u}_n(0) = \int_{r_1}^{r_2} u_0(r,0) X_n(r) r dr = \int_{r_1}^{r_2} 1 \cdot X_n(r) r dr$

3) Apply Laplace transform to equation

$$sU_n(s) - \overline{u}_n(0) + \alpha\lambda^2 U_n(s) = 0$$

where Laplace transform is defined by $U_n(s) = \int_0^\infty \overline{u}_n(t) e^{-st} dt$

$$U_n(s) = \overline{u}_n(0)\frac{1}{s+\alpha\lambda_n^2}$$

4) Apply Inverse Laplace transform

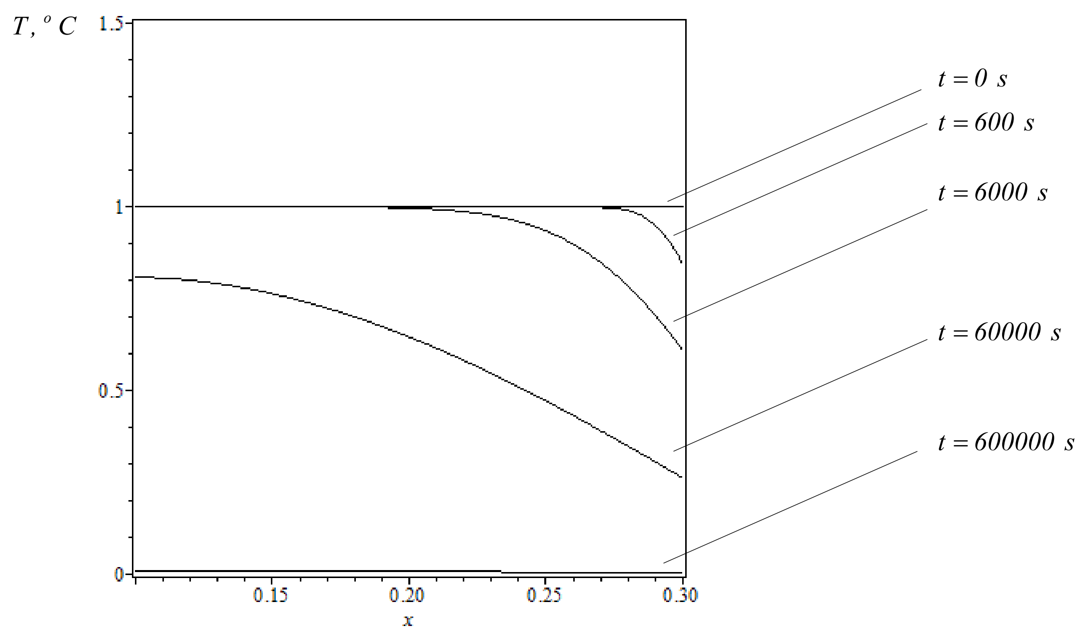
$$\overline{u}_n(t) = \overline{u}_n(0) e^{-\alpha\lambda_n^2}$$

5) Inverse Hankel Transform – solution of the IBVP:

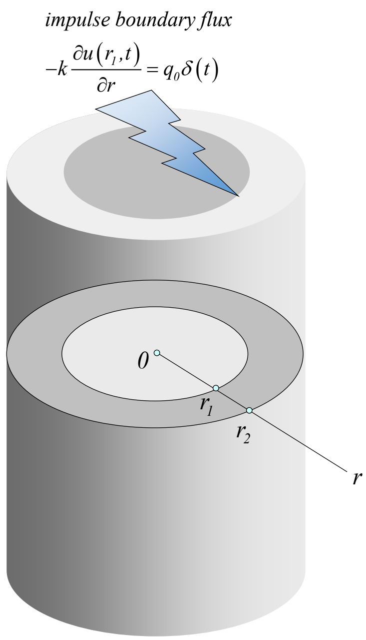
$$u(r,t) = \sum_{n=1}^{\infty} \overline{u}_n(0) \cdot \frac{X_n(r)}{\left\| X_n(r) \right\|^2} \cdot e^{-\alpha\lambda_n^2}$$

6) **Example** Cooling of shaurma (my favorite food on Taksim square)

Maple: SF-AD-6-0 Heat Transfer Problem N-R finite cylinder 02. Mws



IX.4.7.6 Transient heat transfer through the cylindrical wall (thermal shock) [for Vedat Suat Erturk 2015



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad \nu = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r_1 < r < r_2,\ t > 0$$

$$u(r,0) = 0 \qquad r_1 \le r \le r_2$$

$$-k\frac{\partial u(r_1,t)}{\partial r} = q_0\delta(t) \quad \Rightarrow \quad \frac{\partial u(r_1,t)}{\partial r} = \overbrace{-\frac{q_0}{k}\delta(t)}^{f_1(t)}$$

$$k\frac{\partial u(r_2)}{\partial r} + h_2 u(r_2) = 0 \quad \Rightarrow \quad \frac{\partial u(r_2)}{\partial r} + H_2 u(r_2) = \overbrace{0}^{f_2(t)},\ H_2 = \frac{h_2}{k}$$

1) Finite Hankel Transform of order $\nu = 0$ (see Neumann-Robin, Chapter 3, p.38)

$$\bar{u}_n(t) = \int_{r_1}^{r_2} u(r,t)X_n(r)\,rdr$$

Inverse Finite Hankel Transform

$$u(r,t) = \sum_{n=1}^{\infty}\bar{u}_n(t)\frac{X_n(r)}{\|X_n(r)\|^2}$$

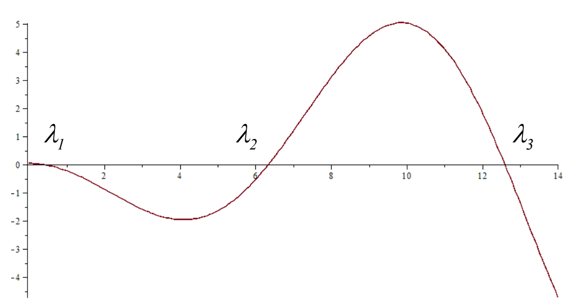
where eigenfunctions and squared norm are

$$X_n^{\nu=0}(x) = \frac{J_0(\lambda_n x)}{-\lambda_n J_1(\lambda_n r_2) + H_2 J_\nu(\lambda_n r_2)} - \frac{Y_0(\lambda_n x)}{-\lambda_n Y_1(\lambda_n r_2) + H_2 Y_0(\lambda_n r_2)}$$

$$\|X_n(r)\|^2 = \int_{r_1}^{r_2} X_n^2(r)\,rdr$$

and eigenvalues are positive roots of equation:

$$\lambda J_1(\lambda r_1)\left[-\lambda Y_1(\lambda r_2) + H_2 Y_0(\lambda r_2)\right] - \lambda Y_1(\lambda r_1)\left[-\lambda J_1(\lambda r_2) + H_2 J_\nu(\lambda r_2)\right] = 0$$



2) Hankel transform of equation (use operational property ☺ ☺, p.93):

$$r_2 \frac{f_2(t)}{k_2} X_n(r_2) - r_1 f_1(t) X_n(r_1) - \lambda_n^2 \overline{u}_n == \frac{1}{\alpha}\frac{\partial \overline{u}_n}{\partial t}$$

$$\frac{\partial \overline{u}_n(t)}{\partial t} + \alpha\lambda_n^2 \overline{u}_n(t) = \alpha r_1 \frac{q_0}{k}\delta(t) X_n(r_1)$$

initial condition: $\overline{u}_n(0) = 0$

3) Apply Laplace transform

$$sU_n(s) + \alpha\lambda^2 U_n(s) = \alpha r_1 \frac{q_0}{k} X_n(r_1)$$

where $U_n(s) = \int_0^\infty \overline{u}_n(t) e^{-st} dt$

$$U_n(s) = \alpha r_1 \frac{q_0}{k} X_n(r_1) \frac{1}{s + \alpha\lambda_n^2}$$

4) Apply Inverse Laplace tranform

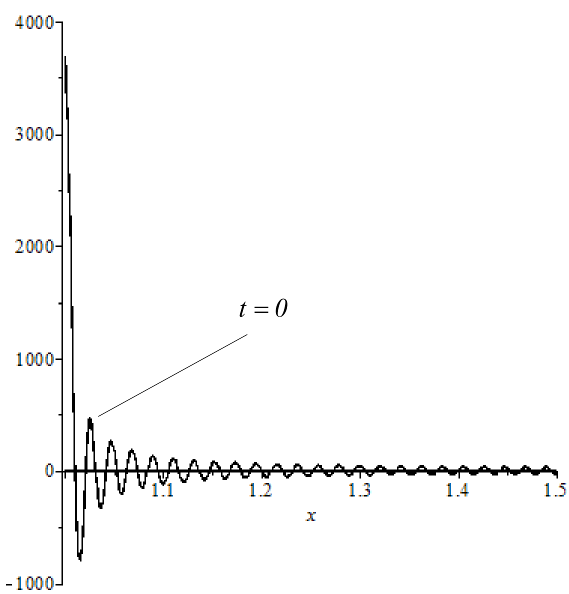
$$\overline{u}_n(t) = \alpha r_1 \frac{q_0}{k} X_n(r_1) e^{-\alpha\lambda_n^2}$$

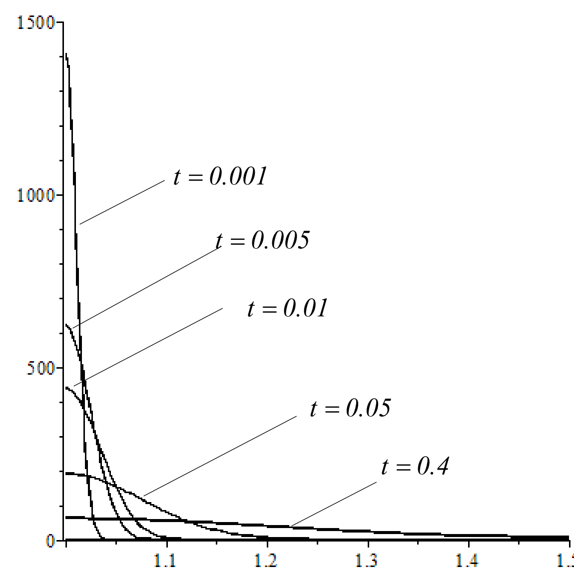
5) Inverse Hankel Transform – solution of the IBVP:

$$u(r,t) = \alpha r_1 \frac{q_0}{k} \sum_{n=1}^{\infty} X_n(r_1) e^{-\alpha\lambda_n^2} \frac{X_n(r)}{\left\| X_n(r) \right\|^2}$$

6) **Example** (paper of Yun et al, 2009 ASME JPVT)

$r_1 = 1.0,\ r_2 = 1.5,\ k = 0.111,\ \alpha = 0.06,\ h_2 = 0.00677988,\ q_0 = 36.0$





Simple way to obtain the other solution for verification.
Cylindrical wall can be approximated by a plane wall.
If the values of t are not big then the influence of other boundary is negligible.
Problem can be reduced to following:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{k_d}\frac{\partial u(x,t)}{\partial t} \quad \text{for} \quad x>0\,, \quad \text{and} \quad 0<t< \textit{not too big}$$

$$u(x,0)=0, \quad x \ge 0$$

$$-K\frac{\partial u(0,t)}{\partial x} = q_0\delta(t) \qquad \textit{Neumann}$$

Apply Laplace Transform:

$$\frac{\partial^2 U(x,s)}{\partial x^2} = \frac{1}{k_d} sU(x,s)$$

$$-K\frac{\partial U(0,s)}{\partial x} = q_0 \qquad \textit{Neumann}$$

Solve:

$$U(x,s) = c_1 e^{-\sqrt{\frac{s}{k_d}}x} + c_2 e^{+\sqrt{\frac{s}{k_d}}x} \qquad c_2 = 0 \quad \text{for solution to be bounded}$$

$$U(x,s) = c_1 e^{-\sqrt{\frac{s}{k_d}}x}$$

$$\frac{\partial}{\partial x}U(x,s) = -c_1\sqrt{\frac{s}{k_d}}e^{-\sqrt{\frac{s}{k_d}}x}$$

$$-K\frac{\partial}{\partial x}U(0,s) = c_1 K\sqrt{\frac{s}{k_d}}e^{-\sqrt{\frac{s}{k_d}}\cdot 0} = q_0 \qquad \Rightarrow \quad c_1 = \frac{q_0}{K}\sqrt{\frac{k_d}{s}}$$

$$U(x,s) = \frac{q_0}{K}\sqrt{\frac{k_d}{s}}e^{-\sqrt{\frac{s}{k_d}}x}$$

Apply inverse Laplace transform to get a solution:

$$u(x,t) = L^{-1}\{U(x,s)\} = \frac{\sqrt{k_d}}{K} q_0 \frac{e^{-\frac{x^2}{4k_d t}}}{\sqrt{\pi t}}$$

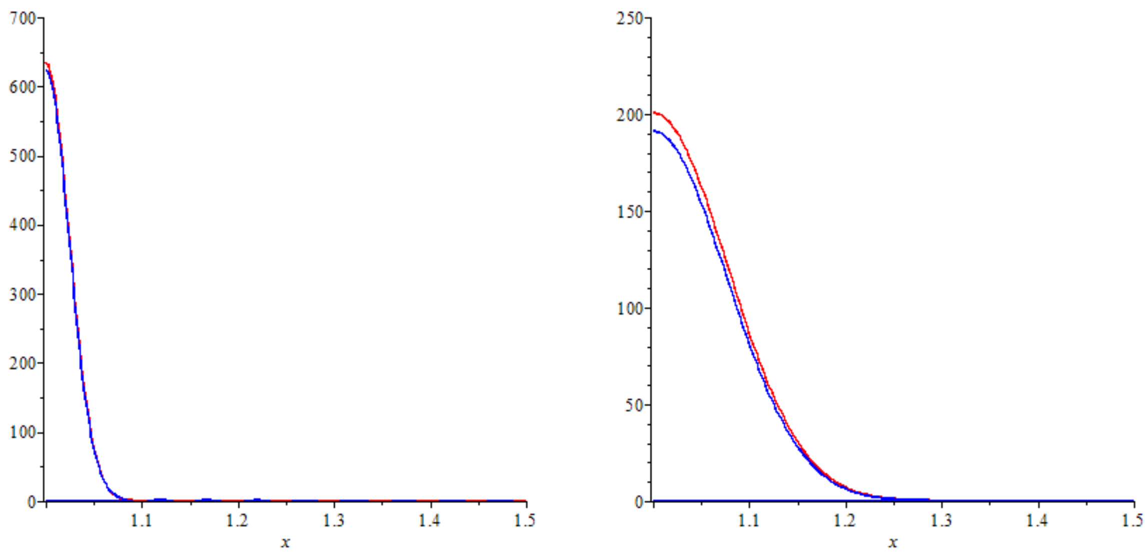
Shift solution to $x = a$

Temperature profile for near future prediction:

$$v(x,t) = \frac{\sqrt{k_d}}{K} q_0 \frac{e^{-\frac{(x-a)^2}{4k_d t}}}{\sqrt{\pi t}} \quad \text{for } x \ge a\,, \quad \text{and} \quad 0<t< \textit{not too big}$$

This solution yields nice smooth curves, solution is exact, it provides very good prediction for time for which temperature profiles do not reach the second boundary, and the layer can be considered semi-infinite.
This solution is good for comparison with the solution in the finite layer.
Comparison of this solution with the solution by Hankel transform (48 terms in summation):

For $t=0.005$ $t=0.05$



Solution are almost identical.
It makes me absolutely confident that both solutions are correct.
For shorter time $t<0.005$, oscillations in Hankel solution are noticible.

!!! This solution at $t_0=0.005$ (or some other, for example at $t_0=0.0001$) can be used as an initial condition for Numerical (finite-difference) solution of simpler problem:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r_1<r<r_2\,,\ t>0$$

$$u(r,0)=\frac{\sqrt{\alpha}}{k}q_0\frac{e^{-\frac{(r-r_1)^2}{4\alpha t_0}}}{\sqrt{\pi t}} \qquad r_1\le r\le r_2$$

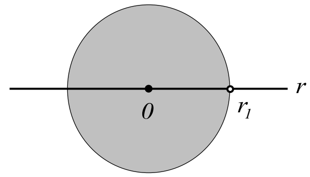
$$\frac{\partial u(r_1,t)}{\partial r}=0$$

$$k\frac{\partial u(r_2)}{\partial r}+h_2u(r_2)=0 \qquad \text{no impulse!}$$

Hotdog Problem

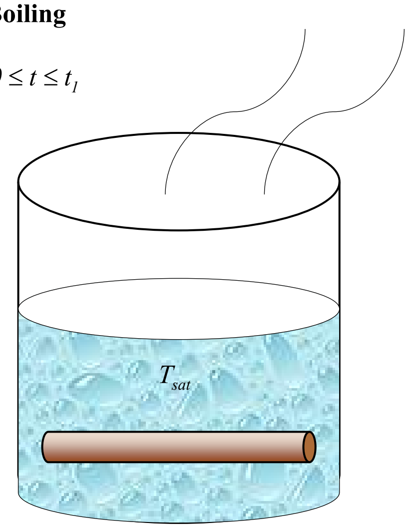
The Heat Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial u}{\partial t}, \qquad 0 \le r < r_1$$



I Boiling

$0 \le t \le t_1$



Initial condition:

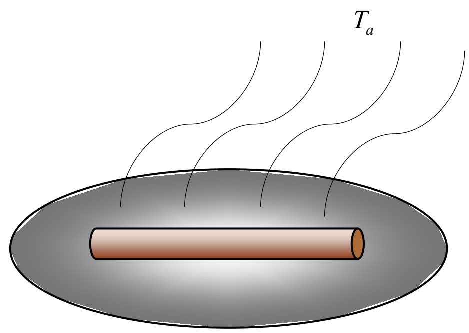
$$u^{I}(r,0)=u_0(r)=T_0$$

Boundary conditions:

$$u(r_1)=T_{sat}$$

II Cooling

$t > t_1$



$$u^{II}(r,0)=u_0^{II}(r)=u^{I}(r,t_1)$$

$$k\frac{\partial u(r_1)}{\partial r}+hu(r_1)=hT_a$$

1) What time is needed for the hotdog to be completely cooked?
2) What time is needed for the hotdog to be cool enough not to be burned?
3) Sketch the temperature field of the middle cross-section as a function of time.
4) Sketch the temperature of the center point of the middle cross-section as a function of time.

IX.4.7.8 Some additional notes on HANKEL TRANSFORM Feb, 2015

The Hankel transform of order ν of the function $f(r)$ of the polar variable $r \in [0,\infty)$ is defined as

$$H_\nu\{f(r)\} = \bar{f}_\nu(\lambda) = \int_0^\infty r J_\nu(\lambda r) f(r)\, dr$$

The **inverse Hankel transform** of order ν is defined as

$$H_\nu^{-1}\{\bar{f}_\nu(\lambda)\} = f(r) = \int_0^\infty \lambda J_\nu(\lambda r)\bar{f}_\nu(\lambda)\, d\lambda$$

Calculation of the Hankel transforms is performed by direct application of the definition. The main difficulty will be integration. Therefore, the special tables of integrals involving Bessel functions should be applied. Integration can be performed with the help of Maple or any other computer software.
But the simplest is to have a table of Hankel transforms similar to 17.1 and 17.2.

Examples:

1) Calculate 0th order Hankel transform of $f(r) = \frac{1}{r}$

Apply definition:

$$\bar{f}_0(\lambda) \equiv H_0\left\{\frac{1}{r}\right\} = \int_0^\infty r J_0(\lambda r)\frac{1}{r}dr = \int_0^\infty J_0(\lambda r)dr = \frac{1}{\lambda}\underbrace{\int_0^\infty J_0(\lambda r)d(\lambda r)}_{1} = \frac{1}{\lambda}$$

2) Calculate inverse of the 0th order Hankel transform of $\bar{f}_0(\lambda) = \frac{1}{\lambda}$ to reconstruct function $f(r)$.

Apply definition of the inverse transform:

$$f(r) = H_0^{-1}\left\{\frac{1}{\lambda}\right\} = \int_0^\infty \lambda J_0(\lambda r)\frac{1}{\lambda}d\lambda = \int_0^\infty J_0(\lambda r)d\lambda = \frac{1}{r}\int_0^\infty J_0(\lambda r)d(r\lambda) = \frac{1}{r}$$

3) Calculate 0th order Hankel transform of $f(r) = 1 - H(r-R) = \begin{cases} 1 & 0 < r < R \\ 0 & r > R \end{cases}$ Heaviside function

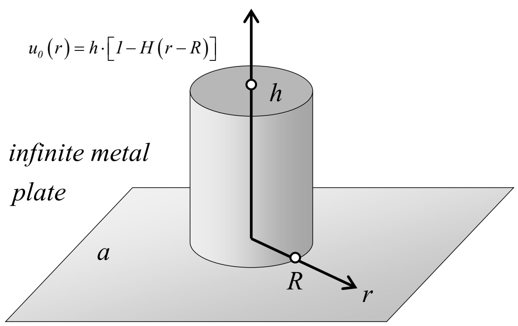
$$\bar{f}_0(\lambda) \equiv H_0\{1 - H(r-R)\} = \int_0^\infty r J_0(\lambda r)\left[1 - H(r-R)\right]dr$$

$$= \int_0^R r J_0(\lambda r)dr = \frac{1}{\lambda}\int_{r=0}^R d\left[r J_1(\lambda r)\right] = \frac{1}{\lambda}\left[r J_1(\lambda r)\right]_{r=0}^R = \frac{1}{\lambda} R J_1(\lambda R)$$

EXAMPLE TRANSIENT HEAT TRANSFER Cooling of the cylindrical-shaped temperature profile

initial temperature distrubution of the thin infinite metal plate

heated to temperature h i nside a circle of raius R and 0 outside of the circle



Consider the axisymmetric case of the Heat Equation in cylindrical coordinates

$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = a^2\frac{\partial u}{\partial t} \qquad r\in[0,\infty) \quad t>0$$

Initial condition (for r<R temperature is T=h, for r>R, T=0)

$$u(r,0)=u_0(r)=h\cdot[1-H(r-R)]=\begin{cases} h & 0<r<R \\ 0 & r>R \end{cases}$$

There are no boundary conditions but we assume that the function *u* and its derivative has radial decaying at infinity:

$$ru\big|_{r\to\infty}=0 \qquad r\frac{\partial u}{\partial r}\bigg|_{r\to\infty}=0$$

Interpretation: infinite plate thermoinsulated on both sides.
Heat is conducted along the plate.
Thermal diffusivity of the plate is a^2 .

<u>1) Transformed equation</u> Apply zero order Hankel transform

$$\overline{u}(\lambda,t)=\int_0^\infty rJ_0(\lambda r)u\,dr$$

to the equation

$$-\lambda^2\overline{u}=a^2\frac{\partial\overline{u}}{\partial t}$$

and to initial condition (see Table 17.1 Exponential functions)

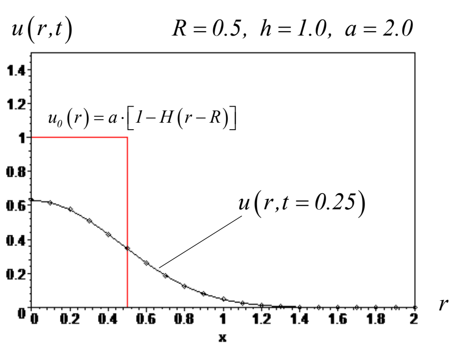
$$\overline{u}(\lambda,0)=\int_0^\infty rJ_0(\lambda r)u_0\,dr=h\int_0^R rJ_0(\lambda r)\,dr=\frac{hR}{\lambda}J_1(\lambda R)$$

That is the 1st order linear ODE

$$\frac{\partial\overline{u}}{\partial t}+\frac{\lambda^2}{a^2}\overline{u}=0$$

The solution is

$$\begin{aligned}\overline{u}(\lambda,t) &= \overline{u}(\lambda,0)e^{\frac{-\lambda^2}{a^2}t}\\ &= \frac{hR}{\lambda}J_1(\lambda R)e^{\frac{-\lambda^2}{a^2}t}\end{aligned}$$



<u>2) Inverse transform</u> Solution of the problem

$$\begin{aligned}u(r,t) &= \int_0^\infty \lambda J_0(\lambda r)\overline{u}(\lambda,t)\,d\lambda\\ &= \int_0^\infty \lambda J_0(\lambda r)\frac{hR}{\lambda}J_1(\lambda R)e^{\frac{-\lambda^2}{a^2}t}\,d\lambda\\ &= hR\int_0^\infty J_0(\lambda r)J_1(\lambda R)e^{\frac{-\lambda^2}{a^2}t}\,d\lambda\end{aligned}$$

$$u(r,t) = hR\int_0^\infty J_0(\lambda r)J_1(\lambda R)e^{\frac{-\lambda^2}{a^2}t}\,d\lambda$$

HANKEL.mws Maple numerical integration:

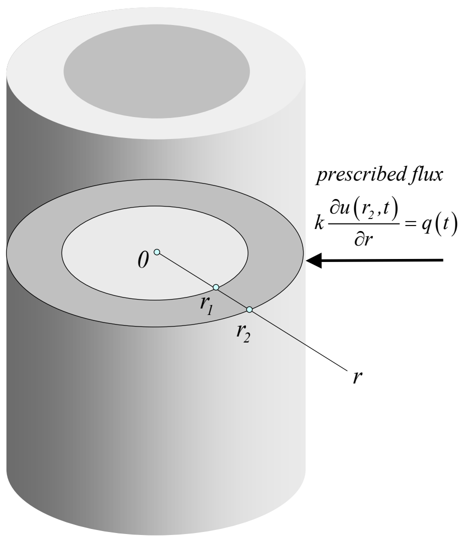
```
> u:=h*R*evalf(Int(BesselJ(0,r*x)*BesselJ(1,R*x)*exp(-
x^2/a^2*t),x=0..infinity));
```

Example Transient heat transfer through the cylindrical wall [problem in Guzii] March, 2015 - not finished

Heat equation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad \nu=0$$

Consider the following initial boundary value problem:



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r_1<r<r_2,\ t>0$$

Uniform initial temperature:

$$u(r,0)=0 \qquad r_1\le r\le r_2$$

Boundary $r=r_1$ is thermoinsulated:

$$\frac{\partial u(r_1,t)}{\partial r}=\overbrace{0}^{f_1(t)}$$

Boundary $r=r_2$, prescribed flux:

$$k\frac{\partial u(r_2)}{\partial r}=q(t) \qquad \Rightarrow \qquad \frac{\partial u(r_2)}{\partial r}=\overbrace{\frac{1}{k}q(t)}^{f_2(t)}$$

1) Finite Hankel Transform of order $\nu=0$ (see Chapter 3, p.38?) Neumann-Neumann 0 term?

$$\overline{u}_n(t)=\int_{r_1}^{r_2}u(r,t)X_n(r)rdr$$

Inverse Finite Hankel Transform

??? 0 term

$$u(r,t)=\sum_{n=1}^{\infty}\overline{u}_n(t)\frac{X_n(r)}{\left\|X_n(r)\right\|^2}$$

where eigenfunctions (Finite Hankel of 0 order) are

???

$$X_n^{\nu=0}(x)=\frac{J_0(\lambda_n x)}{-\lambda_n J_1(\lambda_n r_2)+H_2J_\nu(\lambda_n r_2)}-\frac{Y_0(\lambda_n x)}{-\lambda_n Y_1(\lambda_n r_2)+H_2Y_0(\lambda_n r_2)}$$

The squared norm of eigenfunctions is

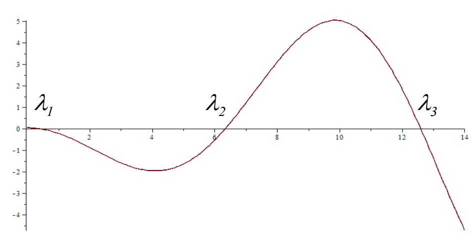
???

$$\left\|X_n^0(r)\right\|^2=\int_{r_1}^{r_2}\left[X_n^0(r)\right]^2 rdr$$

Eigenvalues are positive roots of equation:

???

$$\lambda J_1(\lambda r_1)\left[-\lambda Y_1(\lambda r_2)+H_2Y_0(\lambda r_2)\right]-\lambda Y_1(\lambda r_1)\left[-\lambda J_1(\lambda r_2)+H_2J_\nu(\lambda r_2)\right]=0$$



2) Take Finite Hankel transform of equation (use operational property ☺☺, p.93):

$$= -r_1 X_n(r_1) f_1(t) + r_2 X_n(r_2) f_2(t) - \lambda_n^2 \bar{u}_n$$ (☼ N-N)

$$-r_1 X_n(r_1) \not{f_1(t)} + r_2 X_n(r_2)\frac{q(t)}{k} - \lambda_n^2 \bar{u}_n = \frac{1}{\alpha}\frac{\partial \bar{u}_n}{\partial t}$$

$$\frac{\partial \bar{u}_n(t)}{\partial t} + \alpha\lambda_n^2 \bar{u}_n(t) = \alpha r_2 X_n(r_2)\frac{q(t)}{k}$$

initial condition: $\bar{u}_n(0) = \int_{r_1}^{r_2} u_0(r,0) X_n(r)\, r dr = 0$

3) Apply Laplace transform to equation

$$sU_n(s) - \not{\bar{u}_n(0)} + \alpha\lambda^2 U_n(s) = \alpha r_2 X_n(r_2)\frac{Q(s)}{k}$$

where Laplace transform is defined by

$$U_n(s) = \int_0^\infty \bar{u}_n(t) e^{-st} dt$$

$$U_n(s) = \frac{\alpha}{k} r_2 X_n(r_2)\frac{1}{s+\alpha\lambda_n^2} Q(s)$$

4) Apply Inverse Laplace transform (use convolution theorem):

$$\bar{u}_n(t) = \frac{\alpha}{k} r_2 X_n(r_2)\int_0^t e^{-\alpha\lambda_n^2\tau} q(t-\tau) d\tau \qquad = \frac{\alpha}{k} r_2 X_n(r_2)\int_0^t e^{-\alpha\lambda_n^2(t-\tau)} q(\tau) d\tau$$

Example: $q(t) = 1 - H(t-b)$

$$\begin{aligned}
\bar{u}_n(t) &= \frac{\alpha}{k} r_2 X_n(r_2)\int_0^t e^{-\alpha\lambda_n^2(t-\tau)} q(\tau) d\tau \\
&= \frac{\alpha}{k} r_2 X_n(r_2)\int_0^b e^{-\alpha\lambda_n^2(t-\tau)} d\tau \qquad = -\frac{\alpha}{\alpha\lambda_n^2 k} r_2 X_n(r_2)\int_{\tau=0}^{\tau=b} e^{\alpha\lambda_n^2(\tau-t)} d\left(\alpha\lambda_n^2(\tau-t)\right) \\
&= \frac{1}{\lambda_n^2 k} r_2 X_n(r_2)\left[e^{\alpha\lambda_n^2(\tau-t)}\right]_0^b \\
&= \frac{1}{\lambda_n^2 k} r_2 X_n(r_2)\left[e^{\alpha\lambda_n^2(b-t)} - e^{-\alpha\lambda_n^2 t}\right] \\
&= \frac{1}{\lambda_n^2 k} r_2 X_n(r_2)\left[e^{\alpha\lambda_n^2 b} - 1\right]\cdot e^{-\alpha\lambda_n^2 t}
\end{aligned}$$

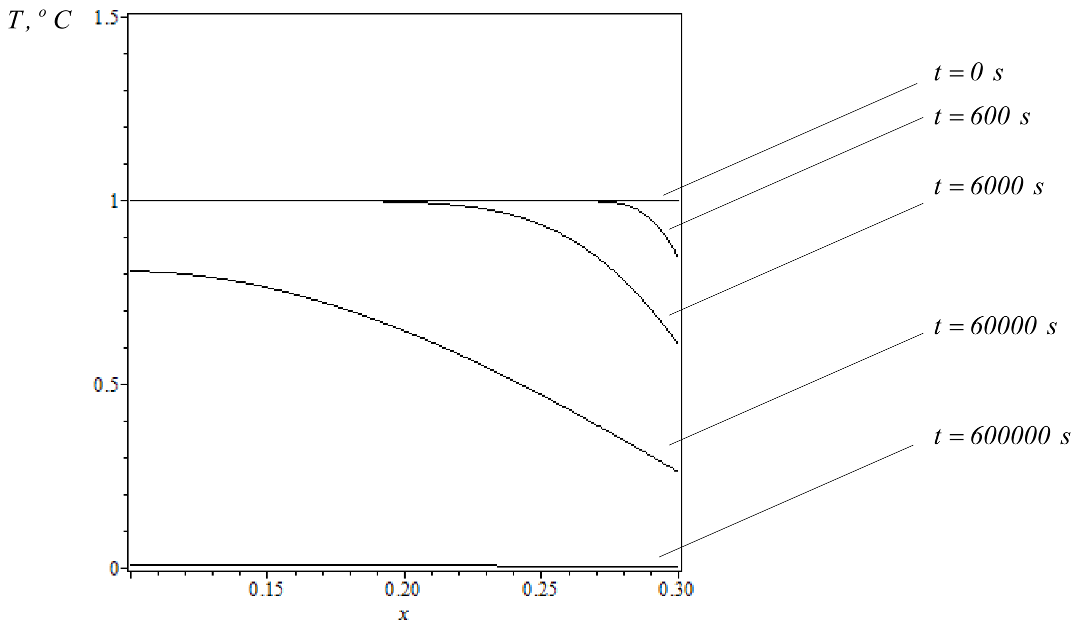
5) Inverse Hankel Transform – solution of the IBVP:

$$u(r,t) = \sum_{n=1}^\infty \bar{u}_n(t)\cdot\frac{X_n(r)}{\|X_n(r)\|^2}$$

0 term ???

6) **Example**

Maple: SF-AD-6-0 Heat Transfer Problem N-R finite cylinder 02. Mws



IX.4.7.9 Comments about approximation of the Dirac function for Vedat *May 30, 2015*

Consider the boundary condition appeared in the Section 6.4.5.5

$$-k\frac{\partial u(r_1,t)}{\partial r}=q_0\delta(t) \qquad \Rightarrow \qquad \frac{\partial u(r_1,t)}{\partial r}=\overbrace{-\frac{q_0}{k}\delta(t)}^{f_1(t)}$$

Can this condition be numerically approximated?

In my opinion, the answer is NOT!

The Dirac function $\delta(t)$ is not a classical function, because it is not defined at $t=0$.

The rigorous definition of the Dirac function can be done only in terms of the Theory of distributions.
In applied mathematics, we use a simplified version of this definition:

Definition of the Dirac function $\delta(t)$:
$\delta(t)$ has a distribution property

$\int_{-h}^{h} f(t)\delta(t)\,dt = f(0),\ \forall h>0$ for any continuous on $(-\infty,\infty)$ function $f(t)$ with a finite support

Only in terms of this definition we can make any conclusions about approximation of $\delta(t)$.

Other definitions used in practice, for example,

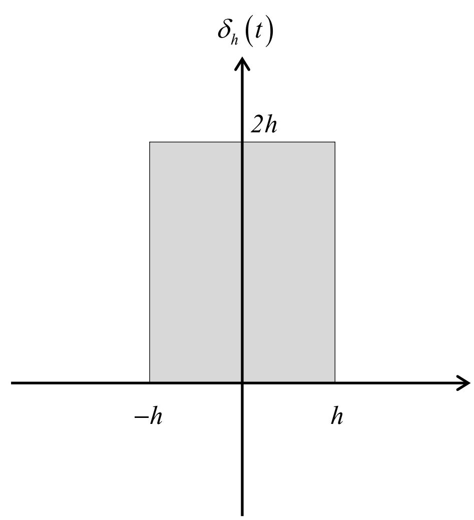
$$\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty}\delta(t)\,dt=1$$

are NOT mathematically correct – they only help to have some idea about this function.

There are many “approximations” of $\delta(t)$ which also help to understand the physical meaning of the Dirac function, but they cannot be used as a numerical approximation.

For example: a step function with the area equal to unity

$$\delta_h(t)=\begin{cases}0 & t<-h \text{ and } t>h\\ \dfrac{1}{2h} & -h<t<h\end{cases}$$



Then usually you can see that it is written that $\delta(t)=\lim_{h\to 0}\delta_h(t)$

But this convergence is not a point-wise because $\lim_{h\to 0}\delta_h(t)=\infty$, however $\delta_h(t)$ is defined for any $h\neq 0$.

It is correctly to say that $\delta_h(t) \to \delta(t)$ in terms of distributions:

$$\lim_{h\to 0} \int_{-h}^{h} f(t)\delta_h(t)\,dt = f(0)$$

If you use this "approximation" for boundary condition, it will yield some results. But they will not be more accurate if you decrease h, as it should be with a proper approximation (actually, I have never tried that).

If you want a correct setting of the problem for numerical (finite-difference) solution it can be properly done in the following way (already discussed before):

1) $u(r,0) = 0$

2) Choose any small value t_1 (for example, $t_1 = 0.001$)

3) Then the initial-boundary value problem for numerical solution does not involve the Dirac function:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r_1 < r < r_2,\ t > t_1$$

$$u(r,t_1) = \frac{\sqrt{k_d}}{K} q_0 \frac{e^{-\frac{(r-a)^2}{4k_d t_1}}}{\sqrt{\pi t_1}} \qquad r_1 \le r \le r_2$$ initial condition (EXACT!!! See derivation in p.)

$$\frac{\partial u(r_1,t)}{\partial r} = 0$$

$$k\frac{\partial u(r_2)}{\partial r} + h_2 u(r_2) = 0$$

This approach with small enough value of t_1 and small enough step in r to represent the initial profile should yield an accurate numerical solution.

IX.4.7.10 One-sided Dirac-delta which can be used in B.C. From review of the paper of Lappa [see reviews]

ii) Second, interpretation of the "delta Dirac function" is not consistent (even taken into account controversial mathematical nature of Dirac function):

"where … δ_s is well-known Dirac function (*which takes value 1 on the interface and is zero elsewhere*). Though such representation is correct from a mathematical point of view..."

In our opinion, it is not correct. Dirac function is not 1 at interface. Some definitions assign to Dirac function "∞" at the interface, but never 1. And more strictly speaking, Dirac function is not a function at all, but rather a distribution, which possesses distribution property given by equation

$$\int_{-\infty}^{\infty} \delta(s) ds = 1 .$$

Moreover, this property should be held over any open interval containing point $s = 0$:

$$\int_{-h}^{h} \delta(s) ds = 1$$ for any $h > 0$ (does not matter how small).

Therefore, insertion of standard Dirac function to the momentum equation is not appropriate, because Dirac function is "loosely" equal zero outside of the interface, but the interface $y = \frac{1}{2}$ is not included into the domain of momentum equation, which is $-\frac{1}{2} < y < \frac{1}{2}$.

Sometimes, Dirac function is interpreted as a "limit" (not point-by-by limit), which can be called a limit in distribution sense or a weak limit of the sequence of regular functions, for example, of a sequence of triangular functions

$$\tau_h(y) = \left(-y/h^2 + 1/h\right)\cdot\left[H(y+h) - H(y)\right] + \left(y/h^2 + 1/h\right)\cdot\left[H(y) - H(y+h)\right] ,$$

where $H(y)$ is a Heaviside unit-step function. As can be seen, when $h \to 0$, the value of function at zero $\tau_h(0) \to \infty$. The distribution property of this function is

$$\int_{-h}^{h} \tau_h(y) dy = 1$$ for any $h > 0$ (unit power).

Then, in the limit in distribution sense,

$$\delta(y) = \lim_{h \to 0} \tau_h(y)$$ (impulse of a unit power)

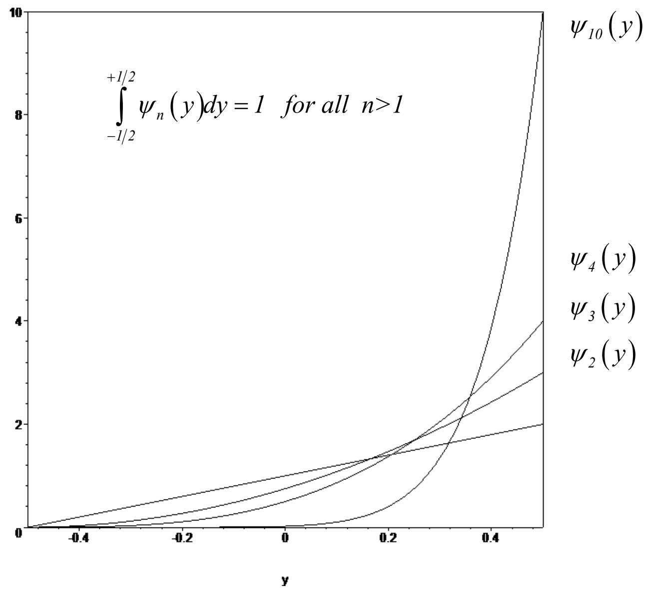
Therefore, for Dirac delta function, open neighborhood, which includes points from both sides of the interface is needed. That is, definitely, not the case for momentum equation, which is valid only for interior of the domain, and not valid at the boundary and outside of the boundary.

So, "spreading" of boundary condition at $y = +\frac{1}{2}$ into the domain with the help of Dirac delta is not possible. In this sense, equations (20),(23-25) is just a symbolic description of what the author is trying to accomplish.

However, author's approach, in fact, **is correct "spreading"** of boundary condition into the domain. But this "spreading" is performed not with the help of traditional Dirac function, but with the help of the function which can be defined as

$$\psi_n(y) = n\left(y + \frac{1}{2}\right)^{n-1}$$

which for any $n > 1$ is defined for all $-\frac{1}{2} \le y \le +\frac{1}{2}$.



The property of this function is similar to property of Dirac function,

$$\int_{-1/2}^{+1/2} \psi_n(y)dy = 1 \qquad \text{(area under any curve is equal to unity)}$$

however, integration is only over interior side of interface (i.e. over the domain of the governing equation).
In a limit (in distribution sense), one can introduce an impulse function of a unit power

$$\psi(y) = \lim_{n\to\infty} \psi_n(y)$$

which describes the spread of unit power only inside the domain.
This function formally can be obtained by differentiation of the function $\phi(y)$ defined by equation (26) in the paper:

$$\phi'(y) = \left[\left(y+\frac{1}{2}\right)^n\right]' = n\left(y+\frac{1}{2}\right)^{n-1} = \psi_n(y)$$

and equation (27) can be rewritten with the help of $\psi_n(y)$ as

$$\frac{d^2}{dy^2} g_2(y) = -\phi'(y) + C_1 = -n\left(y+\frac{1}{2}\right)^{n-1} + C_1 = -\psi_n(y) + C_1 .$$

Then, equation (27) can be written for the limiting case as

$$\frac{d^2}{dy^2} g_2(y) = -\psi(y) + C_1$$

Application of it as a starting point for a sequence of further steps of obtaining equations (28) and (29) together with boundary conditions will yield the limiting case of solution directly.

Therefore, the momentum equation should include function $\psi(y)$, which describes spreading only into the domain, instead of Dirac δ, which describes spreading into the domain and outside of the domain.
Sequence of functions $\psi_n(y)$ used, in fact, in the paper, describe correctly approach to the limiting case.

Boundary value problems of heat conduction

Özışık, M. Necati

Scranton: International Textbook Co., 1968

November 24, 2020 Finite Hankel Transform