

The Laplace Equation: 12 – RNRR (Robin-Neumann-Robin-Robin)

The diagram shows a rectangular domain with a light gray interior and a dark gray border. The domain is labeled with M at the top-left corner, L at the bottom-right corner, 0 at the bottom-left corner, and 1 at the top-right corner. The boundary conditions are as follows:

- Top boundary ($y=M$):** Neumann condition: $\frac{\partial u}{\partial y}\Big|_{y=M} = f_2(x)$
- Bottom boundary ($y=0$):** Robin condition: $\left[-\frac{\partial u}{\partial y} + H_1 u\right]_{y=0} = f_1(y)$
- Left boundary ($x=0$):** Robin condition: $\left[-\frac{\partial u}{\partial x} + H_3 u\right]_{x=0} = f_3(y)$
- Right boundary ($x=L$):** Robin condition: $\left[\frac{\partial u}{\partial x} + H_4 u\right]_{x=L} = f_4(y)$

The interior of the domain is labeled with the Laplace equation: $\nabla^2 u = 0$.

Supplemental problems:

Four square domains representing different boundary conditions for the Poisson equation $\nabla^2 u_i = 0$:

- Domain 1: Top boundary is 0, Bottom boundary is $f_1(x)$, Left boundary is 0, Right boundary is 0.
- Domain 2: Top boundary is $f_2(x)$, Bottom boundary is 0, Left boundary is 0, Right boundary is 0.
- Domain 3: Top boundary is 0, Bottom boundary is 0, Left boundary is $f_3(y)$, Right boundary is 0.
- Domain 4: Top boundary is 0, Bottom boundary is 0, Left boundary is 0, Right boundary is $f_4(y)$.

Solution of supplemental problems:

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \cosh[\lambda_n(y - M)] [\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)]$$

$$a_n = \frac{\int_0^L f_1(x) [\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)] dx}{[\lambda_n \sinh(\lambda_n M) + H_1 \cosh(\lambda_n M)] \left[\frac{(\lambda_n^2 + H_3^2)}{2} \left(L + \frac{H_4}{\lambda_n^2 + H_4^2} \right) + \frac{H_3}{2} \right]}$$

$$u_2(x, y) = \sum_{n=1}^{\infty} b_n \left[\cosh(\lambda_n y) + \frac{H_1}{\lambda_n} \sinh(\lambda_n y) \right] [\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)]$$

$$b_n = \frac{\int_0^L f_2(x) [\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)] dx}{[\lambda_n \sinh(\lambda_n M) + H_1 \cosh(\lambda_n M)] \left[\frac{(\lambda_n^2 + H_3^2)}{2} \left(L + \frac{H_4}{\lambda_n^2 + H_4^2} \right) + \frac{H_3}{2} \right]}$$

Where λ_n are positive roots of

$$(H_3 H_4 - \lambda^2) \sin(\lambda L) + (H_3 + H_4) \lambda \cos(\lambda L) = 0$$

$$u_3(x, y) = \sum_{n=1}^{\infty} c_n \cos[\lambda_n(y-M)] \left[\cosh[\lambda_n(x-L)] - \frac{H_4}{\lambda_n} \sinh[\lambda_n(x-L)] \right]$$

$$c_n = \frac{\int_0^M f_3(y) \cos[\lambda_n(y-M)] dy}{\left[\left(\lambda_n - \frac{H_3 H_4}{\lambda_n} \right) \sinh(\lambda_n L) + (H_3 + H_4) \cosh(\lambda_n L) \right] \left[\frac{M}{2} + \frac{\sin(2\lambda_n M)}{4\lambda_n} \right]}$$

$$u_4(x, y) = \sum_{n=1}^{\infty} d_n \cos[\lambda_n(y-M)] \left[\cosh(\lambda_n x) + \frac{H_3}{\lambda_n} \sinh(\lambda_n x) \right]$$

$$d_n = \frac{\int_0^M f_4(y) \cos[\lambda_n(y-M)] dy}{\left[\left(\lambda_n - \frac{H_3 H_4}{\lambda_n} \right) \sinh(\lambda_n L) + (H_3 + H_4) \cosh(\lambda_n L) \right] \left[\frac{M}{2} + \frac{\sin(2\lambda_n M)}{4\lambda_n} \right]}$$

Where λ_n are positive roots of

$$\lambda \sin(\lambda M) - H_1 \cos(\lambda M) = 0$$

Solution of BVP problem:

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

POISSON'S EQUATION: 12 – RNRR

$$\begin{array}{c}
 \left. \frac{\partial u}{\partial y} \right|_{y=M} = f_2(x) \\
 \begin{array}{ccc}
 M & \boxed{\nabla^2 u = F(x, y)} & \\
 \left[-\frac{\partial u}{\partial x} + H_3 u \right]_{x=0} = f_3(y) & & \left[\frac{\partial u}{\partial x} + H_4 u \right]_{x=L} = f_4(y) \\
 0 & & L \\
 \left[-\frac{\partial u}{\partial y} + H_1 u \right]_{y=0} = f_1(y) & &
 \end{array}
 \end{array}$$

Supplemental problems:

$$\begin{array}{ccc}
 f_2(x) & & 0 \\
 \downarrow & & \\
 \begin{array}{ccc}
 f_3(y) & \boxed{\nabla^2 u_5 = 0} & f_4(y) \\
 f_1(x) & &
 \end{array}
 & &
 \begin{array}{ccc}
 0 & \boxed{\nabla^2 u_6 = F} & 0 \\
 0 & & 0
 \end{array}
 \end{array}$$

Solution of supplemental problems:

Solution of Laplace's homogeneous equation with non-homogeneous b.c.'s:

$$u_5(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

Solution of Poisson's equation with homogeneous boundary conditions

$$u_6(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m} [\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)] \cos[\mu_m (y - M)]$$

$$A_{n,m} = \frac{\int_0^L \int_0^M F(x, y) [\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)] \cos[\mu_m (y - M)] dx dy}{(\lambda_n^2 + \mu_m^2) \left[\frac{(\lambda_n^2 + H_3^2)}{2} \left(L + \frac{H_4}{\lambda_n^2 + H_4^2} \right) + \frac{H_3}{2} \right] \left[\frac{M}{2} + \frac{\sin(2\mu_m M)}{4\mu_m} \right]}$$

Where λ_n and μ_m are positive roots of

$$(H_3 H_4 - \lambda^2) \sin(\lambda L) + (H_3 + H_4) \lambda \cos(\lambda L) = 0$$

$$\mu \sin(\mu M) - H_1 \cos(\mu M) = 0$$

Solution of BVP for Poisson's Equation (superposition principle):

$$u(x, y) = u_5(x, y) + u_6(x, y)$$