linear o.d.e.

LINEAR O.D.E.

$a_i(x), f(x) \in C(D)$ $\quad D \subset R$

nth ORDER

$$L_n y \equiv a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

linear o.d.e. is normal in D if $a_0(x) \neq 0$ for all $x \in D$

initial value problem

$$L_n y = f(x)$$
$$y(x_0) = k_1 \quad y'(x_0) = k_2 \quad \cdots \quad y^{(n-1)}(x_0) = k_n$$

Theorem

if equation is normal, then IVP has the unique solution

FUNDAMENTAL SET

any set of n linearly independent solutions $y_1(x), \dots, y_n(x)$

of homogeneous linear o.d.e. $L_n y = 0$

is said to be a fundamental set (basis functions)

COMPLIMENTARY SOLUTION

complimentary solution (complete solution) of homogeneous o.d.e. $L_n y = 0$

is given by a linear combinatnation of basis functions

$$y_c(x) = c_1 y_1(x) + \dots + c_n y_n(x) \qquad c_i \in R$$

complimentary solution is a vector space spanned by basis functions

COMPLETE SOLUTION

complete solution of non-homogeneous linear o.d.e. $L_n y = f(x)$

consists of complimentary solution (complete solution of homogeneous equation) and any particular solution of non-homogeneous equation

$$y(x) = c_1 y_1(x) + \dots + c_n y_n(x) + y_p(x)$$

HOMOGENEOUS LINEAR O.D.E. WITH CONSTANT COEFFICIENTS

nth ORDER

$$L_n y = 0$$

looking for solution in the form $y = e^{mx}$

substitution into equation yields an auxilary equation which has n roots (real or complex)

auxilary equation

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_m = 0$$

fundamental set includes

case ① real root of multiplicity k $\quad m \quad$ $e^{mx},\ xe^{mx},\ \dots,\ x^{k-1}e^{mx}$

case ② conjugate pair complex roots $\quad m_1 = a + ib,\ m_2 = a - ib \quad$ $e^{ax}\cos bx$ and $e^{ax}\sin bx$

2nd ORDER

$$a_0 y'' + a_1 y' + a_2 = 0$$

auxialary equation $a_0 m^2 + a_1 m + a_2 = 0$ $\quad$ roots $m_{1,2} = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$

complete solution

case ① real roots $m_1 \neq m_2$ $\qquad y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

case ② repeated real root $m_1 = m_2 = m = \dfrac{-a_1}{2a_0}$ $\qquad y = c_1 e^{mx} + c_2 x e^{mx}$

case ③ conjugate pair complex roots $m_1 = a + ib$, $m_2 = a - ib$ $\qquad y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

REDUCTION OF ORDER

2nd ORDER

if y_1 is a particular solution of homogeneous equation $L_n y = 0$

then second linear independent solution can be found by formula

$$y_2 = y_1 \int \frac{e^{-\int \frac{a_1}{a_0} dx}}{y_1^2} dx$$

WRONSKIAN

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem

$\{y_1, \dots, y_n\}$ are linearly independent in D

if $W(y_1, \dots, y_n) \neq 0$ for all $x \in D$

PARTICULAR SOLUTION OF NON-HOMOGENEOUS O.D.E.

Ⅰ variation of parameter (Lagrange's method)

2nd ORDER

$L_2 y = f(x)$ $\quad$ looking for solution in the form $y_p = u_1 y_1 + u_2 y_2$

where $\{y_1, y_2\}$ is a fundamental set

unknown functions are determined by equations

$$u_1 = -\int \frac{y_2}{W(y_1, y_2)} \frac{f(x)}{a_0(x)} dx \qquad u_2 = \int \frac{y_1}{W(y_1, y_2)} \frac{f(x)}{a_0(x)} dx$$

nth ORDER

$L_n y = f(x)$ $\quad$ looking for solution in the form $y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$

where $\{y_1, \dots, y_n\}$ is a fundamental set

unknown functions are determined by equations

$$u_m = (-1)^m \int \frac{W(y_1, \dots, y_{m-1}, y_{m+1}, \dots, y_n)}{W(y_1, \dots, y_n)} \frac{f(x)}{a_0(x)} dx \qquad m = 1, \dots, n$$

Ⅱ method of undetermined coefficients

nth ORDER

$L_n y = f(x)$ $\quad$ where $f(x) = e^{ax}\left[p_i(x)\cos bx + q_j(x)\sin bx\right]$

1) if $a \pm ib$ is not a root of auxilary equation $a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_m = 0$

then looking for solution in the form $y_p = e^{ax}\left[P_k(x)\cos bx + Q_k(x)\sin bx\right]$ $\quad k = \max\{i, j\}$

2) if $a \pm ib$ is a root of auxilary equation of multiplicity s

then looking for solution in the form $y_p = x^s e^{ax}\left[P_k(x)\cos bx + Q_k(x)\sin bx\right]$ $\quad k = \max\{i, j\}$

where

$$P_k(x) = A_0 x^k + A_1 x^{k-1} + \cdots + A_{k-1} x + A_k$$
$$Q_k(x) = B_0 x^k + B_1 x^{k-1} + \cdots + B_{k-1} x + B_k$$

are polynomials with unknown coefficients

which are found by substitution of trial y_p into equation $L_n y = f(x)$

EULER-CAUCHY EQUATION

Ⅰ

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = f(x)$$

change of variable $\quad x = e^z \quad z = \ln x$

yields a linear o.d.e. with constant coefficients

1st ORDER $\quad a_0 y' + a_1 y = f(z)$

2nd ORDER $\quad a_0 y'' + (a_1 - a_0) y' + a_2 y = f(z)$

3rd ORDER $\quad a_0 y''' + (a_1 - 3a_0) y'' + (2a_0 - a_1 + a_2) y' + a_3 y = f(z)$

Ⅱ homogeneous equation

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = 0$$

looking for solution in the form $\quad y = x^m \quad y' = m x^{m-1} \quad y'' = m(m-1)x^{m-2}$

auxialary equation $\quad a_0 m(m-1)\cdots(m-n+1) + a_1 m(m-1)\cdots(m-n+2) + \cdots + a_n = 0$

2nd ORDER

$$a_0 m^2 + (a_1 - a_0) m + a_2 = 0$$

		independent solutions	complete solution
case ①	$m_1 \neq m_2$	$y_1 = \lvert x\rvert^{m_1}$, $y_2 = \lvert x\rvert^{m_2}$	$y = c_1 \lvert x\rvert^{m_1} + c_2 \lvert x\rvert^{m_2}$
case ②	$m_1 = m_2 = m$	$y_1 = \lvert x\rvert^{m}$, $y_2 = \lvert x\rvert^{m} \ln\lvert x\rvert$	$y = c_1 \lvert x\rvert^{m} + c_2 \lvert x\rvert^{m} \ln\lvert x\rvert$
case ③	$m_1 = a + ib$, $m_2 = a - ib$	$y_1 = \lvert x\rvert^{a} \cos(b \ln\lvert x\rvert)$, $y_2 = \lvert x\rvert^{a} \sin(b \ln\lvert x\rvert)$	$y = \left[c_1 \cos(b\ln\lvert x\rvert) + c_2 \sin(b\ln\lvert x\rvert)\right]\lvert x\rvert^{a}$