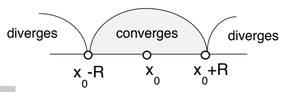
power-series solution

POWER SERIES

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

radius of convergence R



$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

TAYLOR SERIES

$$y(x) = y(x_0) + \frac{y'(x_0)}{1}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n$$

MACLAUREN SERIES

$$y(x) = y(0) + \frac{y'(0)}{1}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!}x^n$$

ANALYTIC FUNCTION

function f(x) is called analytic at x_0 if it can be represented by Taylor series about x_0

BINOMIAL EXPANSION

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

THEOREM 2.6 (identity theorem)

a) $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x if and only if $a_n = 0$

b) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all x if and only if $a_n = b_n$

ALGEBRAIC OPERATIONS

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

sum of two power series

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

product of two power series (Cauchy product)

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n = \sum_{n=0}^{\infty} c_n x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad -1 < x < 1$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \qquad 0 < x < 2$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad |x| < \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad |x| < \infty$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad |x| < \infty$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad |x| < \infty$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad |x| < 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \qquad |x| < \infty$$

$$a^x = \sum_{n=0}^{\infty} \frac{(\ln a)^n x^n}{n!} \qquad |x| < \infty$$

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n} \qquad 0 < x \le 2$$

$$\ln x = \sum_{n=0}^{\infty} \frac{2}{(2n+1)}\left(\frac{x-1}{x+1}\right)^{2n+1} \qquad x > 0$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \qquad |x| < 1$$

HOMOGENEOUS LINEAR SECOND ORDER O.D.E.

$$(*) \qquad a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

initial conditions:

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

TAYLOR SERIES SOLUTION

determine values of function y(x) and its derivatives at the point of expansion x_0:

from initial conditions

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

from equation

$$y''(x_0) = -\frac{a_1(x_0)}{a_0(x_0)} y'(x_0) - \frac{a_2(x_0)}{a_0(x_0)} y(x_0)$$

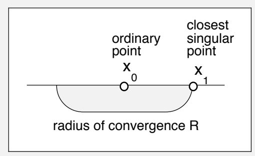
differentiate consequently original equation and substitute x=x_0 . . .

definition

x_0 is ordinary point if $a_0(x_0) \neq 0$

x_0 is singular point if $a_0(x_0) = 0$

THEOREM 2.10 (power-series solution about ordinary point)



Let $a_k(x)$ be analytic, then

two linearly independent solutions of Eq.(∗) can be found as a power series about ordinary point x_0:

1) assume:

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

2) calculate derivatives:

$$y'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x - x_0)^{n-2}$$

3) substitute into Eq. (∗)

4) comparison of coefficients (identity theorem) yeilds the recursive equation for coefficients c_n

c_0, c_1 can be taken arbitrary (or as parameters for complete solution)

standard form of Eq. (∗)

$$y'' + p(x)y' + q(x)y = 0 \qquad (**)$$

definition 2.2 singular point x_0=0 is regular if

$$xp(x) = p_0 + p_1 x + p_2 x^2 + \cdots$$

$$x^2 q(x) = q_0 + q_1 x + q_2 x^2 + \cdots$$

are analytic at x=0

indicial equation

$$r^2 + (p_0 - 1)r + q_0 = 0$$

roots

$$r_{1,2} = \frac{1 - p_0 \pm \sqrt{(1 - p_0)^2 - 4q_0}}{2}$$

THEOREM 2.11 (Frobenius method for regular singular point)

Let x_0=0 be a regular singular point of Eq. (∗∗)

and r_1 and r_2 be the roots of indicial equation, then

two linearly independent solutions y_1 and y_2 of Eq. (∗∗)

can be found in the form: (for 0<|x|<R)

case	condition	y_1	y_2
1	$r_1 - r_2$ is not integer	$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$, $c_0 \neq 0$	$y_2 = \sum_{n=0}^{\infty} d_n x^{n+r_2}$, $d_0 \neq 0$
2	$r_1 - r_2$ is positive integer	$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$, $c_0 \neq 0$	$y_2 = \sum_{n=0}^{\infty} d_n x^{n+r_2} + c y_1 \ln\lvert x\rvert$, $d_0 \neq 0$
3	$r_1 = r_2 = r$	$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$	$y_2 = \sum_{n=1}^{\infty} d_n x^{n+r} + y_1 \ln\lvert x\rvert$, all d_n can be zero