linear systems of first order o.d.e.

STANDARD FORM

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + f_1 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n + f_2 \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n + f_n \end{aligned}$$

$a_{ij}(t)$, $f_i(t)$ i,j=1,..,n $t \in I \subset R$ are continuous functions

MATRIX FORM

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

SOLUTION VECTOR

is a column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ which satisfies the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$

HOMOGENEOUS SYSTEMS

$$\mathbf{x}' = A\mathbf{x}$$

COMPLETE SOLUTION set of all solutions to homogeneous system is a vector space of dimension n

FUNDAMENTAL SET any set of *n* linearly independent solution vectors

$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ (a basis for solution space)

FUNDAMENTAL MATRIX columns are vectors from fundamental set

$$X = [\mathbf{x}_1 \ \ \mathbf{x}_2 \ \ \cdots \ \ \mathbf{x}_n]$$

wronskian $W = \det X(t) \neq 0 \quad \forall \ t \in I$

COMPLETE SOLUTION (complementary function)

$$\mathbf{x}_c = X\mathbf{c}$$

where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a vector of coefficients

INITIAL-VALUE PROBLEM

$$\mathbf{x}' = A\mathbf{x} \qquad \mathbf{x}(t_0) = \mathbf{k}$$

SOLUTION OF IVP

$$\mathbf{x} = X(t)X^{-1}(t_0)\mathbf{k}$$

NON-HOMOGENEOUS SYSTEMS

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

COMPLETE SOLUTION

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$$

where $\mathbf{x}_p$ is any particular solution of non-homogeneous system

VARIATION OF PARAMETER

$$\mathbf{x}_p = X(t)\int X^{-1}(t)\mathbf{f}(t)dt$$

COMPLETE SOLUTION

$$\mathbf{x} = X(t)\mathbf{c} + X(t)\int X^{-1}(t)\mathbf{f}(t)dt$$

INITIAL-VALUE PROBLEM

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \qquad \mathbf{x}(t_0) = \mathbf{k}$$

SOLUTION OF IVP

$$\mathbf{x} = X(t)X^{-1}(t_0)\mathbf{k} + X(t)\int_{t_0}^{t} X^{-1}(s)\mathbf{f}(s)ds$$

FUNDAMENTAL SET OF SYSTEM WITH CONSTANT COEFFICIENTS

$$\mathbf{x}' = A\mathbf{x}$$

A is an $n \times n$ matrix with real constant coefficients

looking for solution in the form

$$\mathbf{x} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t} = \mathbf{K}e^{\lambda t}$$

EIGENVALUE PROBLEM

egenvalues λ are the roots of algebraic characteristic equation $\det(A - \lambda I) = 0$

eigenvector $\mathbf{K}$ corresponding to λ is non-trivial solution of equation $(A - \lambda I)\mathbf{K} = \mathbf{0}$

1 DISTINCT REAL EIGENVALUES

$\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $\mathbf{K}_1, \ldots, \mathbf{K}_m$

FUNDAMENTAL SET

$$\mathbf{K}_1 e^{\lambda_1 t}, \quad \mathbf{K}_2 e^{\lambda_2 t} \quad , \ldots, \quad \mathbf{K}_n e^{\lambda_n t}$$

2 REPEATED EIGENVALUES

λ is eigenvector of multiplicity m

case 1 there are linearly independent eigenvectors $\mathbf{K}_1, \ldots, \mathbf{K}_m$ corresponding to λ

FUNDAMENTAL SET includes

$$\mathbf{K}_1 e^{\lambda t}, \quad \mathbf{K}_2 e^{\lambda t} \quad , \ldots, \quad \mathbf{K}_m e^{\lambda t}$$

case 2 there is only one eigenvector $\mathbf{K}$ corresponding to λ

find vectors $\mathbf{K}, \mathbf{P}, \mathbf{Q}, \ldots$ solutions of the matrix equations

$$(A - \lambda I)\mathbf{K} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{P} = \mathbf{K}$$

$$(A - \lambda I)\mathbf{Q} = \mathbf{P}$$

FUNDAMENTAL SET includes

$$\mathbf{K}e^{\lambda t}, \quad \mathbf{K}te^{\lambda t} + \mathbf{P}e^{\lambda t}, \quad \mathbf{K}\frac{t^2}{2}e^{\lambda t} + \mathbf{P}te^{\lambda t} + \mathbf{Q}e^{\lambda t} \quad \ldots$$

3 COMPLEX EIGENVALUES

$\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ corresponding eigenvectors $\mathbf{K}_1$, $\overline{\mathbf{K}}_1$

solution vectors $\mathbf{x}_1 = \mathbf{K}_1 e^{\lambda_1 t}$ $\quad \mathbf{x}_2 = \overline{\mathbf{K}}_1 e^{\overline{\lambda}_1 t}$

FUNDAMENTAL SET includes real form of solution vectors

$$\mathbf{x}_1 = (\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t)e^{\alpha t}$$

$$\mathbf{x}_2 = (\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t)e^{\alpha t}$$

where $\mathbf{K}_1 = \mathrm{Re}(\mathbf{K}_1) + i\,\mathrm{Im}(\mathbf{K}_1) = \mathbf{B}_1 + i\mathbf{B}_2$

MATRIX EXPONENTIAL

is a matrix represented by a power series

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{(t\mathbf{A})^k}{k!} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots$$

general solution $\mathbf{x} = e^{t\mathbf{A}}\mathbf{c} + e^{t\mathbf{A}}\int e^{-t\mathbf{A}}\mathbf{f}(t)dt$

solution of IVP $\mathbf{x} = e^{(t-t_0)\mathbf{A}}\mathbf{k} + e^{t\mathbf{A}}\int_{t_0}^{t} e^{-s\mathbf{A}}\mathbf{f}(s)ds$