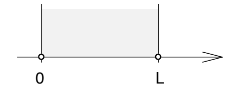
sturm-liouville problem

$H_1 = \dfrac{h_1}{k_1}$ $\quad H_2 = \dfrac{h_2}{k_2}$

$X'' + \lambda X = 0$ $\quad X(x) \quad x \in [0, L]$



boundary conditions	eigenvalues $\lambda_n = \mu_n^2$	eigenfunctions X_n	norm $\|X_n\|^2 = \int_0^L X_n^2(x)\,dx$	kernel $K_n(x) = \dfrac{X_n(x)}{\|X_n\|}$
Dirichlet $X(0) = 0$ Dirichlet $X(L) = 0$	$\mu_n = \dfrac{n\pi}{L}$ $\quad n = 1,2,\ldots$	$\sin\dfrac{n\pi}{L}x$	$\dfrac{L}{2}$	$\sqrt{\dfrac{2}{L}}\sin\dfrac{n\pi}{L}x$
Neumann $X'(0) = 0$ Dirichlet $X(L) = 0$	$\mu_n = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}$ $\quad n = 0,1,2,\ldots$	$\cos\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}x$	$\dfrac{L}{2}$	$\sqrt{\dfrac{2}{L}}\cos\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}x$
Dirichlet $X(0) = 0$ Neumann $X'(L) = 0$	$\mu_n = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}$ $\quad n = 0,1,2,\ldots$	$\sin\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}x$	$\dfrac{L}{2}$	$\sqrt{\dfrac{2}{L}}\sin\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}x$
Neumann $X'(0) = 0$ Neumann $X'(L) = 0$	$\mu_n = \dfrac{n\pi}{L}$ $\quad n = 0,1,2,\ldots$	$\cos\dfrac{n\pi}{L}x$	L $\quad n = 0$ $\dfrac{L}{2}$ $\quad n = 1,2,\ldots$	$\dfrac{1}{\sqrt{L}}$ $\sqrt{\dfrac{2}{L}}\cos\dfrac{n\pi}{L}x$
Dirichlet $X(0) = 0$ Robin $k_2X'(L) + h_2X(L) = 0$	μ_n are positive roots of $\mu\cos\mu L + H_2\sin\mu L = 0$ $\quad n = 1,2,\ldots$	$\sin\mu_n x$	$\dfrac{L}{2} - \dfrac{\sin(2\mu_n L)}{4\mu_n}$	$\dfrac{\sin\mu_n x}{\sqrt{\dfrac{L}{2} - \dfrac{\sin(2\mu_n L)}{4\mu_n}}}$
Neumann $X'(0) = 0$ Robin $k_2X'(L) + h_2X(L) = 0$	μ_n are positive roots of $\mu\sin\mu L - H_2\cos\mu L = 0$ $\quad n = 1,2,\ldots$	$\cos\mu_n x$	$\dfrac{L}{2} + \dfrac{\sin(2\mu_n L)}{4\mu_n}$	$\dfrac{\cos\mu_n x}{\sqrt{\dfrac{L}{2} + \dfrac{\sin(2\mu_n L)}{4\mu_n}}}$
Robin $-k_1X'(0) + h_1X(0) = 0$ Dirichlet $X(L) = 0$	μ_n are positive roots of $\mu\cos\mu L + H_1\sin\mu L = 0$ $\quad n = 1,2,\ldots$	$\sin\mu_n(x - L)$	$\dfrac{L}{2} - \dfrac{\sin(2\mu_n L)}{4\mu_n}$	$\dfrac{\sin\mu_n(x - L)}{\sqrt{\dfrac{L}{2} - \dfrac{\sin(2\mu_n L)}{4\mu_n}}}$
Robin $-k_1X'(0) + h_1X(0) = 0$ Neumann $X'(L) = 0$	μ_n are positive roots of $\mu\sin\mu L - H_1\cos\mu L = 0$ $\quad n = 1,2,\ldots$	$\cos\mu_n(x - L)$	$\dfrac{L}{2} + \dfrac{\sin(2\mu_n L)}{4\mu_n}$	$\dfrac{\cos\mu_n(x - L)}{\sqrt{\dfrac{L}{2} + \dfrac{\sin(2\mu_n L)}{4\mu_n}}}$
Robin $-k_1X'(0) + h_1X(0) = 0$ Robin $k_2X'(L) + h_2X(L) = 0$	μ_n are positive roots of $(H_1H_2 - \mu^2)\sin\mu L + (H_1 + H_2)\mu\cos\mu L = 0$ $n = 1,2,\ldots$	$\mu_n\cos\mu_n x + H_1\sin\mu_n x$	$\dfrac{(\mu_n^2 + H_1^2)}{2}\left[L + \dfrac{H_2}{\mu_n^2 + H_2^2}\right] + \dfrac{H_1}{2}$	$\dfrac{\mu_n\cos\mu_n x + H_1\sin\mu_n x}{\sqrt{\dfrac{(\mu_n^2 + H_1^2)}{2}\left[L + \dfrac{H_2}{\mu_n^2 + H_2^2}\right] + \dfrac{H_1}{2}}}$

$\{X_n(x)\}$ is a complete set of orthogonal functions on [0,L]

$$\int_0^L X_n(x)X_m(x)\,dx = \begin{cases} \|X_n\|^2 & n = m \\ 0 & n \neq m \end{cases}$$

finite Fourier transform

$$F_n = \int_0^L K_n(x)f(x)\,dx$$

generalized Fourier series

$$f(x) = \sum_n a_n X_n(x) \qquad a_n = \frac{\int_0^L X_n(x)f(x)\,dx}{\|X_n\|^2}$$

inverse transform (Fourier series)

$$f(x) = \sum_n F_n K_n(x)$$