# convergence of infinite series

infinite series $\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots$

partial sums $s_n = a_0 + a_1 + a_2 + \cdots + a_n$

**Definition** infinite series is convergent if and only if the sequence of partial sums is convergent

$\sum_{k=0}^{\infty} a_k = L$ if and only if $s_k \to L$

**basic test**

$a_k \underset{n\to\infty}{\not\to} 0 \Rightarrow \sum a_k$ diverges

**geometric**

$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \Rightarrow$

- $|x| < 1 \Rightarrow \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$
- $|x| \ge 1 \Rightarrow$ diverges

**p-series**

$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \Rightarrow$

- $p > 1 \Rightarrow$ converges
- $p \le 1 \Rightarrow$ diverges

harmonic series $(p = 1)$

$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges

**telescoping**

$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} [f(k) - f(k+1)] = f(0) - f(n+1) \Rightarrow$

- $f(n+1) \to c \Rightarrow \sum_{k=0}^{\infty} a_k = f(0) - c$ converges
- $f(n+1) \to \pm\infty \Rightarrow \sum_{k=0}^{\infty} a_k$ diverges

**absolute convergence test**

$\sum |a_k|$ converges $\Rightarrow \sum a_k$ converges

**alternating**

$\sum_{k=0}^{\infty} (-1)^k a_k \quad a_k \ge 0 \Rightarrow$

$a_k \to 0$, $a_k > a_{k+1} \Rightarrow \sum_{k=0}^{\infty} (-1)^k a_k$ converges

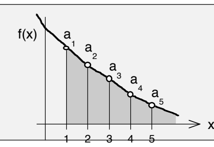
remainder

$\sum_{k=0}^{\infty} (-1)^k a_k = L \Rightarrow |L - s_n| < a_{n+1}$

## series with non-negative terms $a_k \ge 0$

**integral test**

$\int_1^{\infty} f(x)\,dx$ converges $\Leftrightarrow \sum_{k=1}^{\infty} f(k)$ converges




**ratio test**

if $\frac{a_{k+1}}{a_k} \to \rho$ then

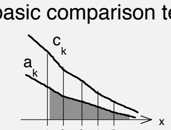
- $\rho < 1 \Rightarrow \sum a_k$ converges
- $\rho = 1 \Rightarrow$ no conclusion
- $\rho > 1 \Rightarrow \sum a_k$ diverges

**root test**

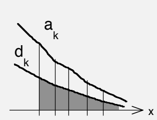
if $(a_k)^{\frac{1}{k}} \to \rho$ then

- $\rho < 1 \Rightarrow \sum a_k$ converges
- $\rho = 1 \Rightarrow$ no conclusion
- $\rho > 1 \Rightarrow \sum a_k$ diverges

**basic comparison test**




$c_k \ge 0$, $c_k \ge a_k$, $\sum c_k$ converges $\Rightarrow \sum a_k$ converges




$d_k \ge 0$, $d_k \le a_k$, $\sum d_k$ diverges $\Rightarrow \sum a_k$ diverges

**limit comparison test**

$a_k \ge 0$, $b_k > 0$, $L > 0$ if $\frac{a_k}{b_k} \to L \Rightarrow$ both $\sum a_k, \sum b_k$ converge or both $\sum a_k, \sum b_k$ diverge

**limits of sequences**

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|---|---|---|---|---|---|---|
| $x^n \to 0 \quad \lvert x\rvert < 1$ | $x^{\frac{1}{n}} \to 1 \quad x > 0$ | $\frac{1}{n^{\alpha}} \to 0 \quad \alpha > 0$ | $\frac{x^n}{n!} \to 0 \quad \forall x$ | $\frac{\ln n}{n} \to 0$ | $n^{\frac{1}{n}} \to 1$ | $\left(1 + \frac{x}{n}\right)^n \to e^x$ |