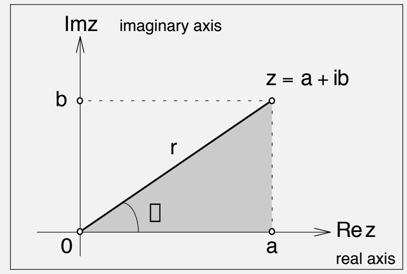
complex numbers

complex number	$z \in \mathbb{C}$	we need complex numbers to be able to solve algebraic equations such as $x^2+1=0$ which has no solution in real numbers
standard form	$z = a + ib$	where i is imaginary unit with property $i^2=-1$ and a and b are real numbers $a, b \in \mathbb{R}$ $\operatorname{Re} z = a$ real part of z $\operatorname{Im} z = b$ imaginary part of z
exponential (polar) form	$z = a + ib = re^{i\varphi}$	
trigonometric form	$z = a + ib = r(\cos\varphi + i\sin\varphi)$	
vector form	$z = (a, b)$	
Euler's formula	$e^{i\varphi} = \cos\varphi + i\sin\varphi$	

complex plane

any point in a plane with coordinates (a,b) is associated with a complex number a+ib

polar coordinates (r,φ) of the point (a,b) can be used for representation of complex numbers



absolute value or modulus of z: $r = |z|$

amplitude or argument of z: $\varphi = \arg z$

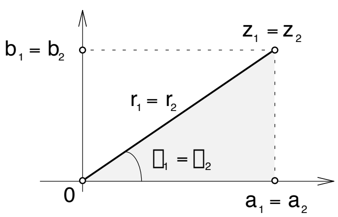
conversion formulas: $r^2 = a^2 + b^2$ $\quad a = r\cos\varphi$ $\quad \tan\varphi = \dfrac{b}{a}$ $\quad b = r\sin\varphi$

trigonometric functions in complex form: $\cos\varphi = \dfrac{e^{i\varphi} + e^{-i\varphi}}{2}$ $\quad \sin\varphi = \dfrac{e^{i\varphi} - e^{-i\varphi}}{2i}$

algebra of complex numbers

$z_1 = a_1 + ib_1 = r_1 e^{i\varphi_1}$ $\qquad z_2 = a_2 + ib_2 = r_2 e^{i\varphi_2}$

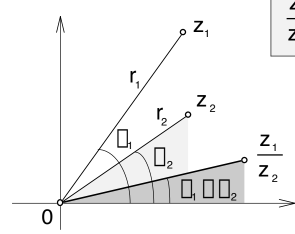
equality



$z_1 = z_2 \iff b_1 = b_2,\ a_1 = a_2$

$\iff r_1 = r_2,\ \varphi_1 = \varphi_2$

quotent

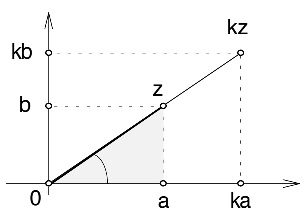


$$\frac{z_1}{z_2} = \frac{(a_1a_2 + b_1b_2) + i(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

$$= \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right]$$

multiplication by a scalar



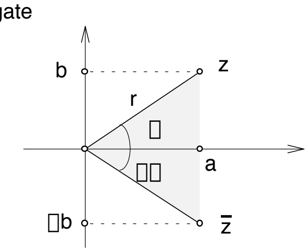
$$kz = ka + i(kb)$$

$$= kre^{i\varphi}$$

$$= kr(\cos\varphi + i\sin\varphi)$$

$$= (ka, kb)$$

conjugate



$$\bar{z} = a - ib$$

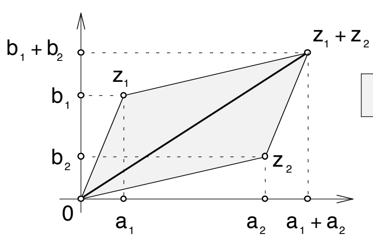
$$= re^{-i\varphi}$$

$$= r(\cos\varphi - i\sin\varphi)$$

$$= (a, -b)$$

$z\bar{z} = a^2 + b^2$ $\qquad r = |z| = \sqrt{z\bar{z}}$

sum

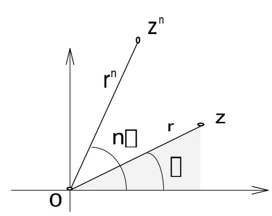


$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$= r_1 e^{i\varphi_1} + r_2 e^{i\varphi_2}$$

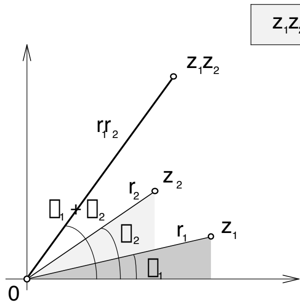
$$= (a_1 + a_2, b_1 + b_2)$$

powers (De Moivre's Formula)



$$z^n = r^n e^{in\varphi} = r^n(\cos n\varphi + i\sin n\varphi)$$

product



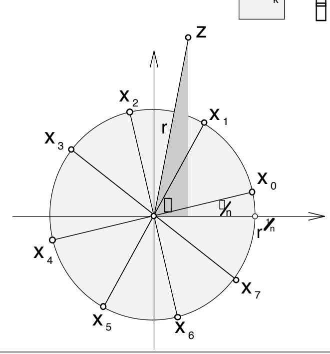
$$z_1 z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$$

$$= r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$$

$$= r_1 r_2\left[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right]$$

$$= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)$$

roots



$$x_k = \left(z^{\frac{1}{n}}\right)_k = r^{\frac{1}{n}} e^{i\frac{\varphi + k2\pi}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\left(\frac{\varphi}{n} + \frac{k2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{k2\pi}{n}\right)\right]$$

$k = 0, 1, 2, \ldots, n-1$

- $z^{1/n}$ can be treated as the solutions of algebraic equation $x^n = z$ which has exactly n roots
- all roots are evenly distributed on the circle with radius $r^{1/n}$
- if z is a real number, then complex roots appear in conjugate pairs