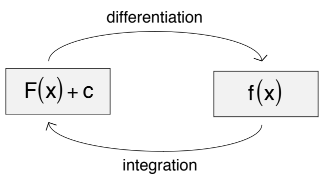
integration

indefinite integral

$F(x)$ is **antiderivative** of $f(x)$ if $F'(x) = f(x)$

if $F(x)$ is **antiderivative** of $f(x)$ then

$F(x) + c$ is also antiderivative of $f(x)$



indefinite integral (operation to find antiderivative)

$$\int f(x)dx = F(x) + c$$

table of antiderivatives

$f(x)$	$F(x)$
x^n	$\frac{x^{n+1}}{n+1}$ $n \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{e^{ax}}{a}$
a^x	$\frac{a^x}{\ln a}$
$\ln x$	$x\ln x - x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\lvert\cos x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

u - substitution

$$\int f[u(x)]u'(x)dx = \int f(u)du$$

$$\int_a^b f[u(x)]u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

integration by parts

$$\int udv = uv - \int vdu$$

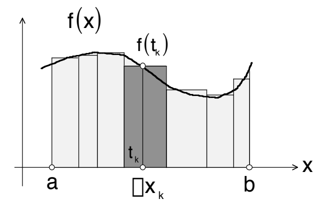
$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

definite integral

partition $\Delta = \{a = x_0, x_1, ..., x_k, ..., x_{n-1}, x_n = b\}$

norm of partition $|\Delta| = \max_k \Delta x_k$

t_k is an arbitrary point in the subinterval $[x_{k-1}, x_k]$ $\Delta x_k = x_k - x_{k-1}$



definite integral

$$\int_a^b f(x)dx = \lim_{|\Delta| \to 0} \sum_{k=1}^{n} f(t_k)\Delta x_k$$

if the limit of Riemann's sum exists

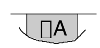
function $f(x)$ is said integrable on $[a,b]$

Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

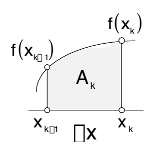
Area

if $f(x) \geq 0$ for all $x \in [a,b]$ then
definite integral is interpreted as the area under $f(x)$ over $[a,b]$



if $f(x) \leq 0$ for all $x \in [a,b]$ then
definite integral is interpreted as the negative area between $f(x)$ and x-axis over $[a,b]$

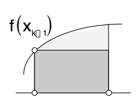
Numerical approximation of definite integral



regular subdivision on n subintervals $\Delta x = \frac{b-a}{n}$ $x_k = a + k\Delta x$, $k = 0,1,...,n$

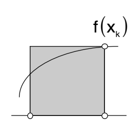
$$A = \int_a^b f(x)dx = A_1 + ... + A_n$$

Left-Hand Sum



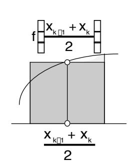
$$A \approx [f(x_0) + ... + f(x_{n-1})]\Delta x$$

Right-Hand Sum



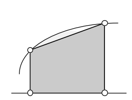
$$A \approx [f(x_1) + ... + f(x_n)]\Delta x$$

Mid-Point Sum



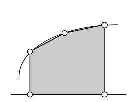
$$A \approx \left[f\left(\frac{x_0 + x_1}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right)\right]\Delta x$$

Trapezoidal Rule



$$A \approx [f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n)]\frac{\Delta x}{2}$$

Simpson's Rule



$$A \approx \left\{f(x_0) + 2[f(x_1) + ... + f(x_{n-1})] + 4\left[f\left(\frac{x_0 + x_1}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right)\right] + f(x_n)\right\}\frac{\Delta x}{6}$$

properties

linearity

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \qquad c \in R$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

limit rules

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

additivity

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad c \in [a,b]$$

differentiation

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f[u(x)]\cdot u'(x) - f[v(x)]\cdot v'(x)$$

comparison

if $f(x) \geq 0$ on $x \in [a,b]$ then

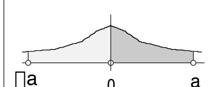
$$\int_a^b f(x)dx \geq 0$$

if $f(x) \geq g(x)$ on $x \in [a,b]$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

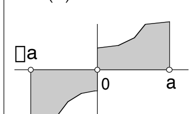
symmetric interval

$f(x)$ is even if $f(-x) = f(x)$



$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

$f(x)$ is odd if $f(-x) = -f(x)$



$$\int_{-a}^a f(x)dx = 0$$

Average Value of Function $f(x)$ over $[a,b]$

$$f_{aver} = \frac{\int_a^b f(x)dx}{b-a}$$

Mean Value Theorem

If function $f(x)$ is continuous on $[a,b]$

then there exists a point $z \in [a,b]$ such that

$$f(z) = f_{average}$$