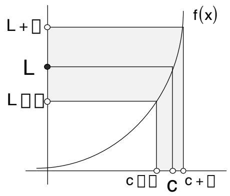
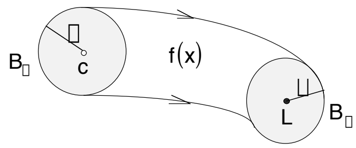
limits

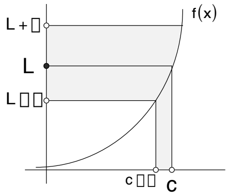
definition of limit

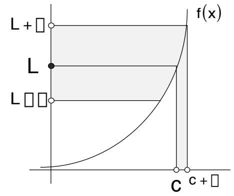
$\lim_{x\to c} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$





one-sided limits





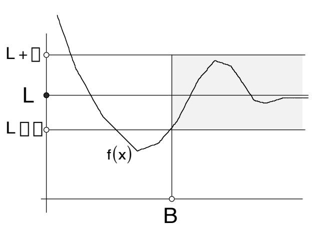
$\lim_{x\to c^-} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $c - \delta < x < c$ then $|f(x) - L| < \varepsilon$

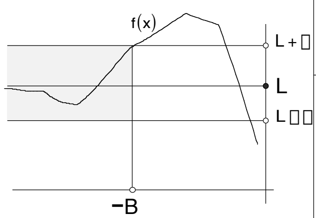
$\lim_{x\to c^+} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $c < x < c + \delta$ then $|f(x) - L| < \varepsilon$

theorem

$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$

limits at infinity (horizontal assymtotes)

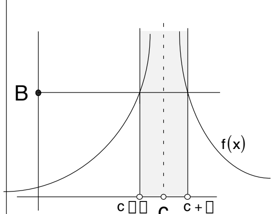


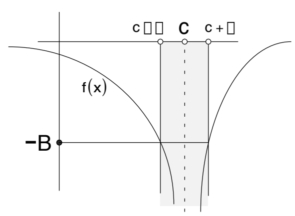


$\lim_{x\to\infty} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $B > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > B$

$\lim_{x\to -\infty} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $B > 0$ such that $|f(x) - L| < \varepsilon$ for all $x < -B$

infinite limits (vertical assymtotes)





$\lim_{x\to c} f(x) = \infty \iff$ for any $B > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) > B$

$\lim_{x\to c} f(x) = -\infty \iff$ for any $B > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) < -B$

uniqueness

if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$ then $L = M$

algebra of limits

in the following statements we assume that the given limits $\lim_{x\to c} f(x)$ exist where c can be $c, c^+, c^-, \pm\infty$, or $|\infty|$

$\lim k = k$

$\lim kf(x) = k\lim f(x)$

$\lim[f(x) + g(x)] = \lim f(x) + \lim g(x)$

$\lim[f(x) - g(x)] = \lim f(x) - \lim g(x)$

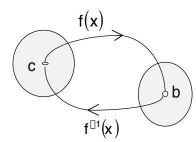
$\lim f(x)g(x) = \lim f(x)\lim g(x)$

$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$ if $\lim g(x) \neq 0$

if $\lim f(x) \neq 0$ and $\lim g(x) = 0$ then $\frac{\lim f(x)}{\lim g(x)}$ DNE

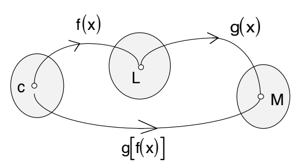
$\lim f(x) = L \iff \lim[f(x) - L] = 0$

inverse function



$\lim_{x\to c} f(x) = b \iff \lim_{x\to b} f^{-1}(x) = c$

composite function

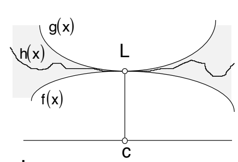


$\lim_{x\to c} f(x) = L$, $\lim_{x\to L} g(x) = M$ then $\lim_{x\to c} g[f(x)] = M$

comparison

$\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, $f(x) \le g(x)$ then $L \le M$

squizze play



$\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = L$, $f(x) \le h(x) \le g(x)$ then $\lim_{x\to c} h(x) = L$

L'Hospital's Rule (indeterminate $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

B_c = open neighborhood of point c

$f(x), g(x)$ are differentiable on B_c

Let $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \pm\infty$, $\lim_{x\to c} g(x) = \pm\infty$

then

if $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = L$

if $\lim_{x\to c} \frac{f'(x)}{g'(x)} = \infty \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = \infty$

indeterminates and reduction to L'Hospital's rule

1) $f \cdot g \sim 0 \cdot \infty$ $\Rightarrow f \cdot g = \frac{f}{\frac{1}{g}} \sim \frac{0}{0}$

$\Rightarrow f \cdot g = \frac{g}{\frac{1}{f}} \sim \frac{\infty}{\infty}$

2) $f^g \sim 0^0$, $f^g \sim \infty^0$, $f^g \sim 1^\infty$ $\Rightarrow f^g = e^{g\ln f}$

3) $f - g \sim \infty - \infty$

a) multiply and devide by conjugate

$\frac{(f-g)(f+g)}{f+g}$

b) apply L'Hospital's rule to

$f - g = \frac{\frac{1}{g} - \frac{1}{f}}{\frac{1}{f \cdot g}}$

remarkable limits

$\lim_{x\to c} x = c$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} x^x = 1$

$\lim_{x\to 0} x\ln x = 0$

$\lim_{x\to 0} (1 + ax)^{1/x} = e^a$

$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$

$\lim_{x\to\infty} x^{\frac{1}{x}} = 1$

$\lim_{x\to\infty} \frac{\ln x}{x} = 0$

rational function

$$\lim_{x\to\infty} \frac{p_n x^n + \ldots + p_1 x + p_0}{q_m x^m + \ldots + q_1 x + q_0} = \begin{cases} 0 & \text{if } n < m \\ \frac{p_n}{q_n} & \text{if } n = m \\ \pm\infty & \text{if } n > m \end{cases}$$