1st order ODE

$$y' = f(t,y)$$

Initial Condition

$$y(t_0) = y_0$$

LINEAR

standard form

$$y' + p(t)y = g(t)$$

Integrating Factor $\mu = e^{\int p(t)dt}$

Genearl solution:

$$y(t) = \frac{c}{\mu(t)} + \frac{1}{\mu(t)}\int \mu(t)g(t)dt$$

Solution of IVP $y(t_0) = y_0$:

$$y(t) = y_0\frac{\mu(t_0)}{\mu(t)} + \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)ds$$

Equation with constant coefficients:

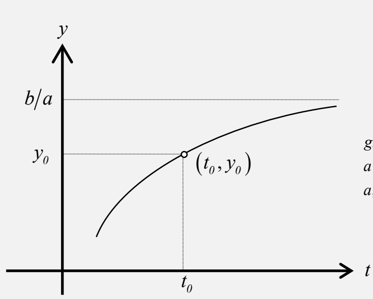
$$y' + ay = b$$

General solution:

$$y = ce^{-at} + \frac{b}{a}$$

Solution of IVP $y(t_0) = y_0$:

$$y = e^{-a(t-t_0)}\left(y_0 - \frac{b}{a}\right) + \frac{b}{a}$$



graph in a case of $a,b>0$

NON-LINEAR

differential form

$$M(x,y)dx + N(x,y)dy = 0$$

differential form is equivalent to a pair of differential equations

$$M(x,y)\frac{dx}{dy} + N(x,y) = 0$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Separable

$$M(x)dx + N(y)dy = 0$$

General solution:

$$\int M(x)dx + \int N(y)dy = c$$

change of variable reduces homogeneous equation to separable

Homogeneous equation

use change of variable

$y = ux$	or	$x = vy$
$dy = udx + xdu$		$dx = vdy + ydv$

back substitution

$u = y/x$	$v = x/y$

Equation is **homogeneous** *if*

$$M(\lambda x, \lambda y) = \lambda^n M(x,y)$$

$$N(\lambda x, \lambda y) = \lambda^n N(x,y)$$

Test for Exact (*ode is exact if*)

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$$

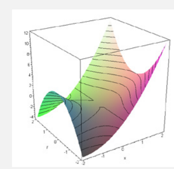
Find $f(x,y)$ *such that*

$$df(x,y) = M(x,y)dx + N(x,y)dy$$

$$\frac{\partial f}{\partial x} = M(x,y), \quad \frac{\partial f}{\partial y} = N(x,y)$$

General solution

$$f(x,y) = c$$



multiplication by integrating factor μ *makes equation exact*

Integrating Factor

if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = h(x)$ *then* $\mu = e^{\int h(x)dx}$

does not depend on y

if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ *then* $\mu = e^{\int g(y)dy}$

does not depend on x

To check that y(x) is a solution, substitute it into $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$

To check that $x(y)$ *is a solution, substitute it into* $M(x,y)\dfrac{dx}{dy} + N(x,y) = 0$

1.3 Classification of Differential Equations

Differential equation

ODE

PDE

Systems of differential equations

Order of ODE

Normal ODE

Linear ODE

Non-Linear ODE

Solutionof ODE

IVP

Existence of a solution

General solution

Particular solution

Uniqueness of a solution

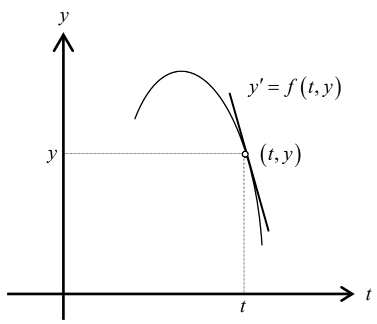
1.1 Direction fields

Consider the differential equation of the first order explicitly written for the derivative of unknown function y'

$$\frac{dy}{dt} = f(t,y)$$

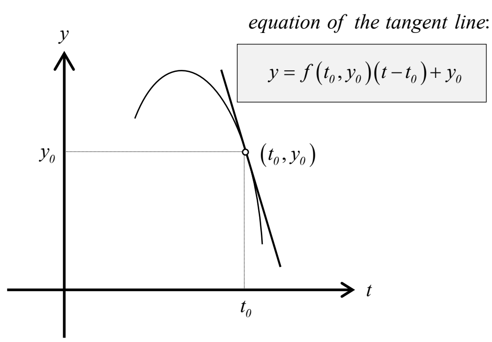
where $f(t,y)$ is some given function of variables t and y. Then we say that equation is written in **normal form**.

Derivative $y'(t)$ defines the slope of the tangent line to the curve $y = y(t)$ at the point *t*.



If derivative $y'(t)$ is given by the differential equation in normal form $y' = f(t,y)$,

then equation the tangent line to the curve $y = y(t)$ at some fixed point (t_0, y_0) can be written as



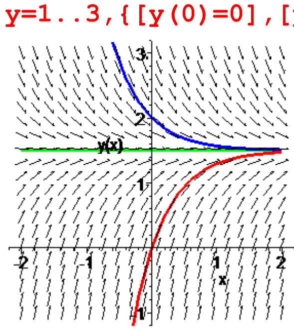
Construction of the direction field with Maple Problem *1.1 #1* $\frac{dy}{dt} = 3 - 2y$

```
> restart;
> with(DEtools):
> DE:=diff(y(x),x)=3-2*y(x);
```

$$DE := \frac{d}{dx}\,y(x) = 3 - 2\,y(x)$$

Plotting the direction field and the solution curves satisfying the initial conditions with the help of DEplot command:

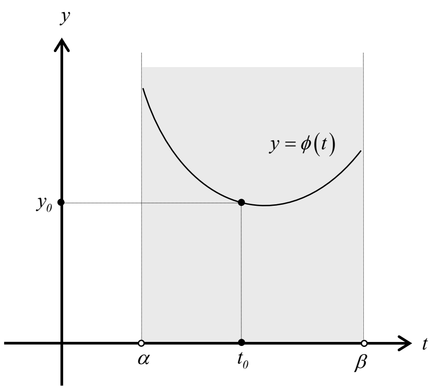
```
>DEplot(DE,y(x),x=-2..2,y=1..3,{[y(0)=0],[y(0)=2],[y(0)=1.5]},color=black);
```



(method of isoclines)

2.4 Differences between linear and non-linear equations

Theorem 2.4.1 (existence and uniqueness of the solution of IVP for **linear** 1st order ODE $y' + p(t)y = g(t)$)



Let $t_0 \in (\alpha, \beta)$ and

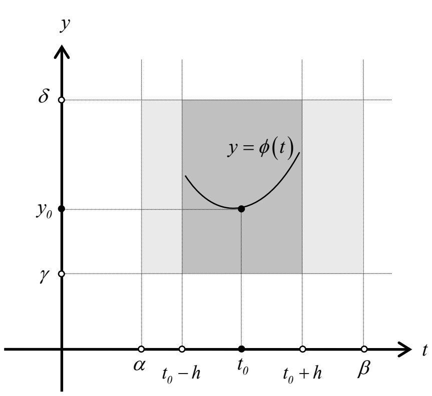
let $p(t),\ g(t) \in C(\alpha, \beta)$ (continuous functions)

then the linear differential equation $y' + p(t)y = g(t)$

has a unique solution $y = \phi(t),\ t \in (\alpha, \beta)$

such that $y(t_0) = y_0$

Theorem 2.4.2 (existence and uniqueness of the solution of IVP for **non-linear** 1st order ODE $y' = f(y,t)$)



Let $f(y,t),\ \dfrac{\partial f(y,t)}{\partial y} \in C((\alpha,\beta)\times(\gamma,\delta))$ (continuous)

and let $t_0 \in (\alpha, \beta)$ and $y_0 \in (\gamma, \delta)$

then the non-linear differential equation $y' = f(y,t)$

has a unique solution $y = \phi(t),\ t \in (t_0 - h, t_0 + h) \subseteq (\alpha, \beta),\ h > 0$

such that $y(t_0) = y_0$. Here, $h>0$ is some positive number.

Remark: if only $f(y,t) \in C((\alpha,\beta)\times(\gamma,\delta))$ is continuous, then solution of IVP exists but is not necessarily unique.

Note: the theorems guarantee only that under given conditions there exists a unique solution of the IVP, but they do not claim that the solution does not exist if the conditions of the theorems are violated.

Constant solutions

$y' = f(x,y)$ if $f(x,b) = 0$, then $y = b$ is a solution

$x' = g(x,y)$ if $g(a,y) = 0$, then $x = a$ is a solution

General Solution (implicit)

$F(x,y,c) = 0$ Solution which includes an arbitrary constant.

General solution of linear ODE includes all possible solutions (**complete solution**).

For non-linear ODE, there can be some additional solutions.

Suppressed solutions Solutions not described by the general solution

2.1 Linear ODE – Integrating Factor

Standard form $y' + p(t)y = g(t)$ **Initial Condition:** $y(t_0) = y_0$

Integrating factor $\mu(t) = e^{\int p(t)dt}$

General solution $y = \dfrac{c}{\mu(t)} + \dfrac{1}{\mu(t)}\int \mu(t)g(t)\,dt$

$$y = \frac{c}{\mu(t)} + \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds$$ (integral form of solution)

Solution of IVP $y = y_0\dfrac{\mu(t_0)}{\mu(t)} + \dfrac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds$

Case of a constant coefficient $y' + ay = g(t)$

Integrating factor $\mu(t) = e^{at}$

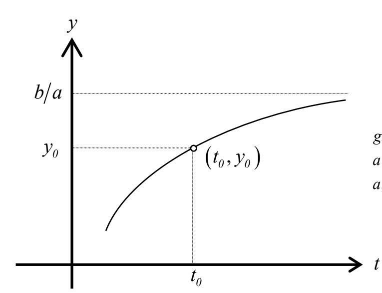
General solution $y = ce^{-at} + e^{-at}\int e^{at}g(t)dt$

Solution of IVP $y = y_0e^{-a(t-t_0)} + e^{-at}\int_{t_0}^{t} e^{as}g(s)\,ds$

Case $g(t) = b = const$ $y' + ay = b$

General solution $y = ce^{-at} + \dfrac{b}{a}$

Solution of IVP $y = e^{-a(t-t_0)}\left(y_0 - \dfrac{b}{a}\right) + \dfrac{b}{a}$



graph in a case of $a,b>0$

Exercise: Solve $ty' + 2y = 4t^2$ $y(1) = 2$ and sketch the solution curve

Example: Find a general solution of equation

$$y' + (\cot x)\, y = \sin 2x$$

and sketch the solution curves.

Solution: The integrating factor for this equation is

$$\mu(x) = e^{\int \cot x dx} = e^{\ln|\sin x|} = \sin x$$

Then a general solution is

$$y = \frac{c}{\sin x} + \frac{1}{\sin x}\int \sin(x)\sin(2x)dx$$

$$= \frac{c}{\sin x} + \frac{2}{\sin x}\int \sin(x)\sin(x)\cos(x)dx \qquad \text{(double angle formula)}$$

$$= \frac{c}{\sin x} + \frac{2}{\sin x}\int \sin^2(x)d\sin(x) \qquad \text{(u-substitution)}$$

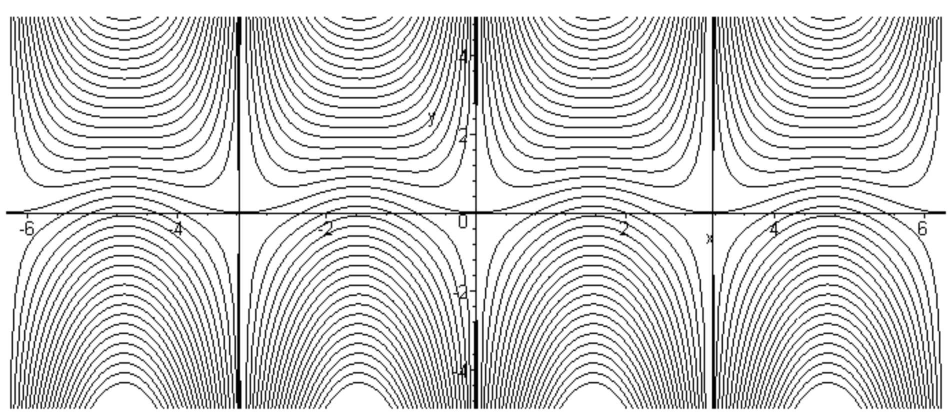
$$= \frac{c}{\sin x} + \frac{2\sin^2 x}{3}$$

Maple: create a sequence of particular solutions by varying the constant c, and then plot the graph of solution curves:

```
> y(x):=2*sin(x)^2/3+c/sin(x);
```

$$y(x) := \frac{2}{3}\sin(x)^2 + \frac{c}{\sin(x)}$$

```
> p:={seq(subs(c=i/4,y(x)),i=-20..20)}:
> plot(p,x=-2*Pi..2*Pi,y=-5..5);
```



$x = -\pi$ $\qquad x = 0$ $\qquad x = \pi$

2.2 Separable equation

Differential form of ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Note that equation in differential form has no distinction between independent and dependent variable

differential form is equivalent to a pair of differential equations

$$M(x,y)\frac{dx}{dy} + N(x,y) = 0$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Separable ODE	$M(x)\,dx + N(y)\,dy = 0$
General Solution	$\int M(x)\,dx + \int N(y)\,dy = c$
Solution of IVP $y(x_0) = y_0$	$\int_{x_0}^{x} M(x)\,dx + \int_{y_0}^{y} N(y)\,dy = 0$

Equation is ***homogeneous*** of order m if

$$M(\lambda x, \lambda y) = \lambda^m M(x,y)$$

$$N(\lambda x, \lambda y) = \lambda^m N(x,y)$$

are homogeneous functions of order m

Homogeneous equation can be reduced to ***separable*** — Back substitution:

by a change of variable y to $y = ux$ $\quad dy = u\,dx + x\,du$ $\quad u = \dfrac{y}{x}$

or by a change of variable x to $x = vu$ $\quad dx = v\,dy + y\,dv$ $\quad v = \dfrac{x}{y}$

By change to polar coordinates $x = r\cos\theta \quad y = r\sin\theta$

Example Problem 2.2 #1, p.47: Solve $y' = x^2/y$ subject to initial condition $y(3) = 2$

Example Problem 2.2 #31

Example Solve the differential equation $y^2 + 2x^2 + xyy' = 0$

$(y^2 + 2x^2)dx + xydy = 0$ ***differential form***

Solution: M and N are homogeneous functions of degree 2.

Change of variable: $y = ux$ $\quad dy = xdu + udx$

$$(u^2x^2 + 2x^2)dx + xux(xdu + udx) = 0$$

$$(u^2x^2 + 2x^2 + u^2x^2)dx + ux^3du = 0$$

$$2x^2(u^2 + 1)dx + ux^3du = 0 \qquad \textit{separable}$$

$$2\frac{dx}{x} + \frac{udu}{u^2+1} = 0 \qquad \textit{separate variables, } x \neq 0$$

$$2\frac{dx}{x} + \frac{1}{2}\frac{d(u^2+1)}{u^2+1} = 0 \qquad \textit{integrate}$$

$$\ln x^4 + \ln(u^2 + 1) = \ln c \qquad \textit{solution}$$

$$x^4(u^2 + 1) = c \qquad \textit{back substitution } u = \frac{y}{x}$$

$$\boxed{(y^2 + x^2)x^2 = c} \qquad \textit{general solution(implicit)}$$

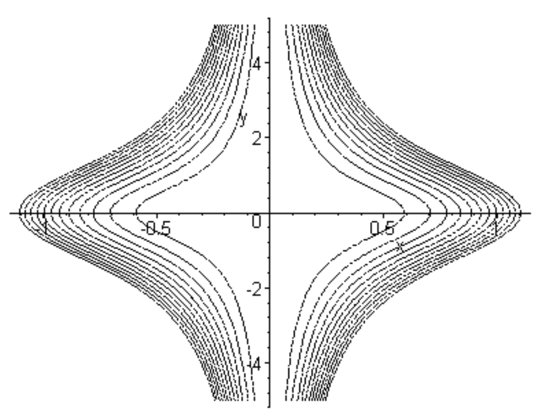
Check for suppressed solutions: earlier we assumed that $x \neq 0$, then check that

$x = 0$ *is also a solution*

but this solution is a particular case of general solution when $c = 0$.

Use Maple to plot the solution curves:

```
> f:={seq(x^2*(y^2+x^2)=i/8,i=0..12)}:
> implicitplot(f,x=-2..2,y=-5..5)
```



2.6 Exact Equations and Integrating Factors

Differential of $f(x,y)$ is

$$d\,f(x,y) = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

Exact equation

$$M(x,y)dx + N(x,y)dy = 0$$

is ***exact*** if there exists some $f(x,y)$ such that

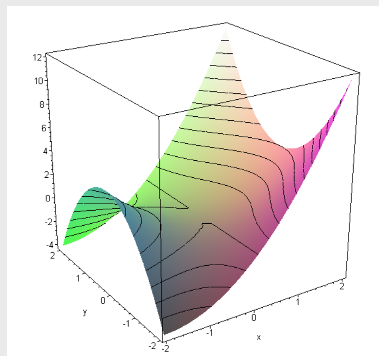
$$d\,f(x,y) = M(x,y)dx + N(x,y)dy$$

Test on exact equation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

General solution

$f(x,y) = c$ level curves of the surface defined by $f(x,y)$



Finding $f(x,y)$

1) $\frac{\partial f}{\partial x} = M(x,y) \quad \Rightarrow \quad f(x,y) = \int M(x,y)dx + k(y)$

2) $\frac{\partial f}{\partial y} = N(x,y) \quad \Rightarrow \quad \frac{\partial}{\partial y}\int M(x,y)dx + \frac{d}{dy}k(y) = N(x,y)$

$\Rightarrow \quad find\ \ k(y)$

3) General Solution: $f(x,y) = \int M(x,y)dx + k(y) = c$

Integrating factor μ

Equation multiplied by an integration factor μ becomes **exact**.

i) if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = h(x)$ then $\mu(x) = e^{\int h(x)dx}$

ii) if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ then $\mu(y) = e^{\int g(y)dy}$

Example (p.95) Solve $2x + y^2 + 2xyy' = 0$

Rewrite in differential form $\overbrace{\left(2x + y^2\right)}^{M} dx + \overbrace{2xy}^{N}\ dy = 0$

Test for exact: $\dfrac{\partial M}{\partial y} = \dfrac{\partial\left(2x + y^2\right)}{\partial y} = 2y$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial\left(2xy\right)}{\partial x} = 2y \qquad \Rightarrow \qquad$ Exact

Find $f\left(x, y\right)$ 1) $\dfrac{\partial f}{\partial x} = M$

$$\frac{\partial f}{\partial x} = 2x + y^2$$

$$f = x^2 + y^2 x + k\left(y\right)$$

2) $\dfrac{\partial f}{\partial y} = N$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2 + y^2 x + k\left(y\right)\right] = 2yx + \frac{dk}{dy}$$

$$2yx + \frac{dk}{dy} = 2xy$$

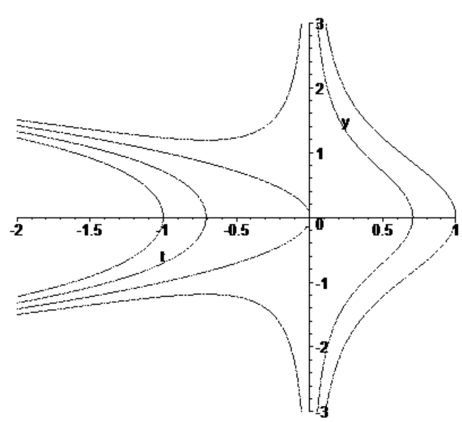
$$\frac{dk}{dy} = 0$$

$$k = c$$

Therefore, $f = x^2 + y^2 x + c$

$x^2 + y^2 x + c = c_1$ *combine constants, then*

General solution: $\boxed{\left(x + y^2\right)x = c}$



2.5 Autonomous Equations

$$y' = f(y)$$

independent variable t is not in equation explicitly

Critical points $\{y_k : \; y'(y_k) = 0\}$

critical points $y = y_k$ are constant solutions

They are called the ***equilibrium solutions***.

Second derivative $y'' = \dfrac{\partial}{\partial t} y' = \dfrac{\partial}{\partial t} f\left[y(t)\right] \overset{\text{chain rule}}{=} \dfrac{df}{dy}\dfrac{dy}{dt} = \dfrac{df}{dy} y' = \dfrac{df}{dy} \cdot f$

Hypercritical points $\{y_k : \; y''(y_k) = 0\}$

inflection can occur only at hypercritical points

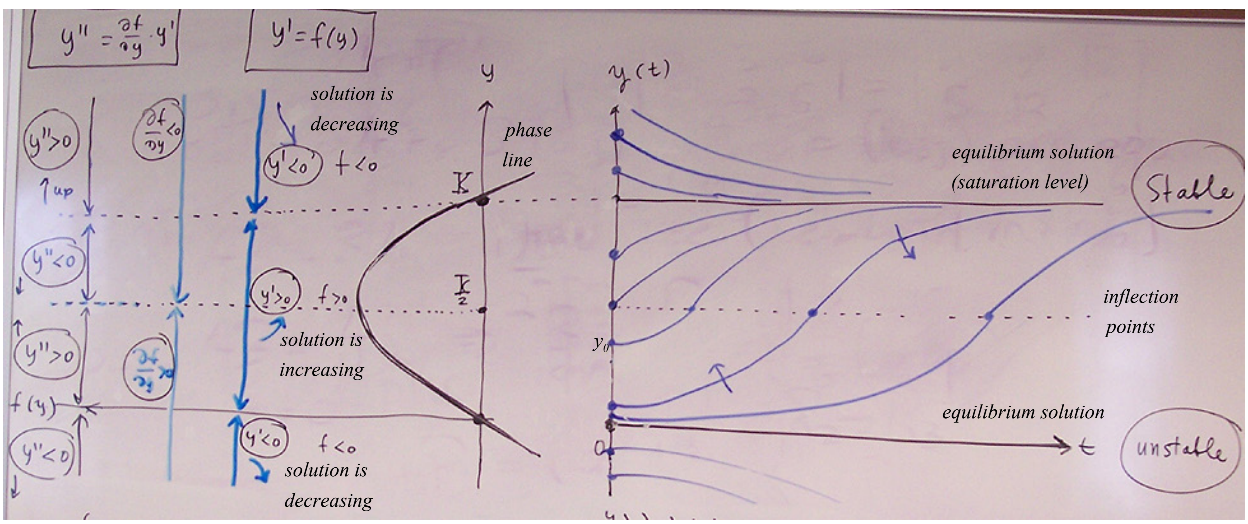
second derivative test *first derivative test*

concave up

concave down

concave up

concave down



Logistic Equation $y' = r\left(1 - \dfrac{1}{K}y\right)y \qquad y(0) = y_0 \qquad (7)$

critical points:

First derivative $y' = f(y) \qquad y' = r\left(1 - \dfrac{1}{K}y\right)y$

$y = 0$ and $y = K$

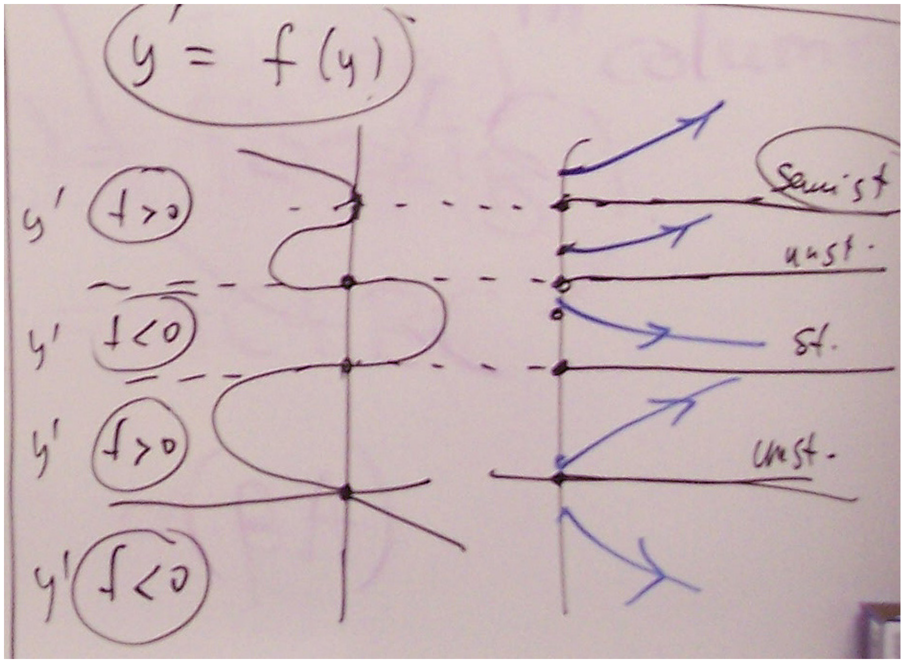
Second derivative $y'' = f \cdot \dfrac{df}{dy} \qquad \dfrac{df}{dy} = r\left(1 - \dfrac{2}{K}y\right)$

$y'' = r^2 y\left(1 - \dfrac{1}{K}y\right)\left(1 - \dfrac{2}{K}y\right)$

hypercritical points:

$y = 0$, $y = K$ and $y = \dfrac{K}{2}$

Stability of Autonomous Equation stability of equilibrium solutions $y = y_k$, $\{y_k : f(y_k) = 0\}$



semistable

unstable

asymptotically stable

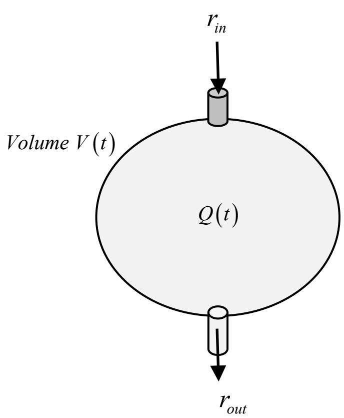
unstable

Example:2.5 #22 $\dfrac{dy}{dt} = \alpha y(1-y)$ $y(0) = y_0$ $\alpha > 0$

2.3 Mathematical modeling

■ **Tank problem** (1.1- #21)

$$m_2 - m_1 = m_{in} - m_{out}$$ *mass balance*



$$Q(t+\Delta t) - Q(t) = \overbrace{\dot{Q}_{in}\Delta t}^{Q_{in}} - \overbrace{\dot{Q}_{out}\Delta t}^{Q_{out}}$$ *mass balance for process during time interval* Δt

$$\frac{dQ}{dt} = \dot{Q}_{in} - \dot{Q}_{out}$$

$$\frac{dQ}{dt} = c_{in} r_{in} - c_{out} r_{out}$$

$Q(t)\ [g]$		*mass*
$\dot{Q}\left[\frac{g}{\min}\right]$	$= r \cdot c$	*mass flow rate*
$c\left[\frac{g}{gal}\right]$		*concentration*
$c_{out} = \frac{Q(t)}{V(t)}$		
$r\left[\frac{gal}{\min}\right]$		*volume flow rate*
$V\ [gal]$		*volume*

for $r_{in} = r_{out} = r = const$
and $V = const$

$$\frac{dQ}{dt} = -\frac{r}{V}Q + c_{in} r$$

■ **Exponential decay** (1.2-12,13) $\quad \frac{dQ}{dt} = -rQ \quad r > 0$	
■ **Exponential growth** $\quad \frac{dQ}{dt} = rQ \quad r > 0$	
■ **Population model** (1.1, p.5) $\quad \frac{dp}{dt} = rp - k \quad r, k > 0$	
■ **Newton's Law of Cooling** (1.1-23, 1.2-15) $\quad \frac{du}{dt} = -k \cdot (u - T) \quad k > 0$	

- **Exponential decay** $$\frac{dp}{dt}=-rp$$

  $r\left[\frac{1}{\sec}\right]$ = *decay rate, r > 0*

- **Exponential growth** $$\frac{dp}{dt}=rp$$

  $r\left[\frac{1}{\sec}\right]$ = *growth rate, r > 0*

- **Tank problem**

  $Q(t)[lbm]$ *amount of salt* $$\frac{dQ}{dt}=-\frac{r}{V}Q+c_{in}r$$

- **Population model** (mice-owl)

  $p(t)[mice]$ *population* $$\frac{dp}{dt}=rp-k$$

  $r\left[\frac{1}{year}\right]$ = *growth rate / reproduction rate*

  $k\left[\frac{mice}{year}\right]$ = *continuous rate of killing mice*

  $p(0)=p_0\ [mice]$ *initial population*

- **Bank model**

  $S(t)\ [\$]$ = *investment or debt*

  $r\left[\frac{1}{year}\right]$ = *annual interest rate / rate of return*

  $k\left[\frac{\$}{year}\right]$ = *continuous annual rate of deposits* $\quad k = 12\cdot m$ $$\frac{dS}{dt}=rS+k \qquad S(t)=S_0e^{rt}+\frac{k}{r}\left(e^{rt}-1\right) \qquad S(t)=S_0e^{rt}+\frac{12m}{r}\left(e^{rt}-1\right)$$

  $w\left[\frac{\$}{year}\right]$ = *continuous annual rate of withdrawls* $\quad w = 12\cdot m$ $$\frac{dS}{dt}=rS-w \qquad S(t)=S_0e^{rt}-\frac{w}{r}\left(e^{rt}-1\right) \qquad S(t)=S_0e^{rt}-\frac{12m}{r}\left(e^{rt}-1\right)$$

  $p\left[\frac{\$}{year}\right]$ = *continuous annual rate of payments* $\quad p = 12\cdot m$ $$\frac{dS}{dt}=rS-p \qquad S(t)=S_0e^{rt}-\frac{p}{r}\left(e^{rt}-1\right) \qquad S(t)=S_0e^{rt}-\frac{12m}{r}\left(e^{rt}-1\right)$$

  $m\left[\frac{\$}{month}\right]$ = *monthly deposits, withdrawls or payments* $$S(t)=\left(S_0-\frac{12m}{r}\right)e^{rt}+\frac{12m}{r}$$

  $S(0)=S_0\ [\$]$ *initial deposit or a loan*

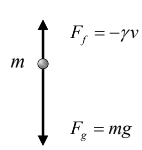
- **Newton's Law**

  $u(t)\left[{}^oF\right]$ *temperature* $$\frac{du}{dt}=-k\cdot(u-T)$$

  $k\left[\frac{1}{\sec}\right]=\frac{1}{\tau}$

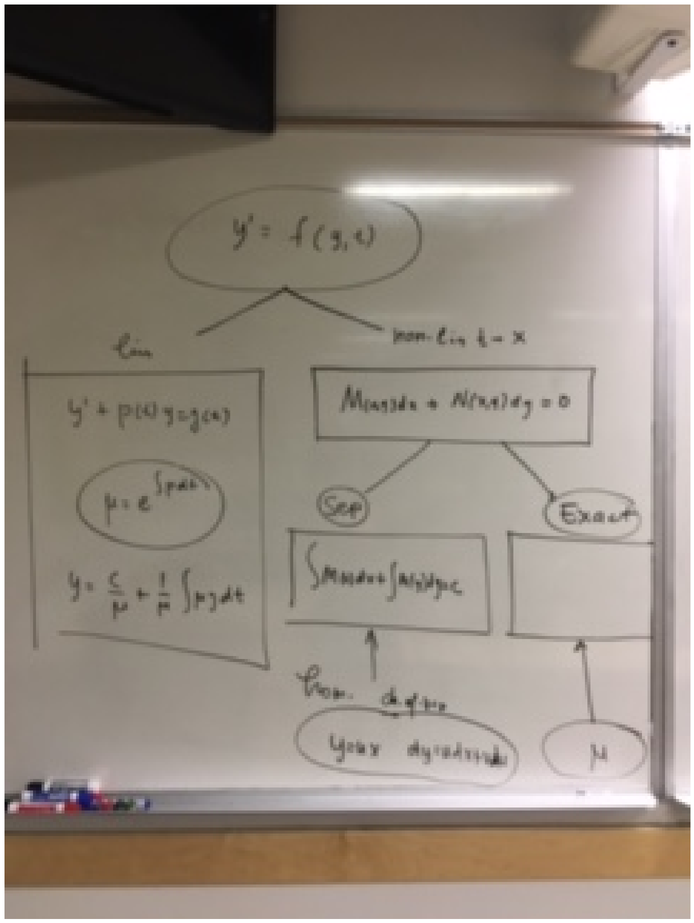
  $\tau\ [\sec]=\frac{\rho c_p V}{hA}>0$ *time constant*

- **Falling ball**



  $v(t)\left[\frac{ft}{s}\right]$ *velocity* $$\frac{dv}{dt}=-\frac{\gamma}{m}v+g$$

  $v(0)=v_0\left[\frac{ft}{s}\right]$ *initial velocity*


y' = f(y,t)
y' + p(t)y = g(t)
Sep
Exact

Integrals Expected Known

1. $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

2. $\int \csc x \, dx = \ln|\csc x - \cot x| + c$

3. $\int \tan x \, dx = -\ln|\cos x| + c$

4. $\int \cot x \, dx = \ln|\sin x| + c$

5. $\int \ln x \, dx = x \ln x - x + c$

6. $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan \frac{x}{a} + c$

7. $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a} + c$

8. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}\left(a \sin bx - b \cos bx\right)}{a^2 + b^2} + c$

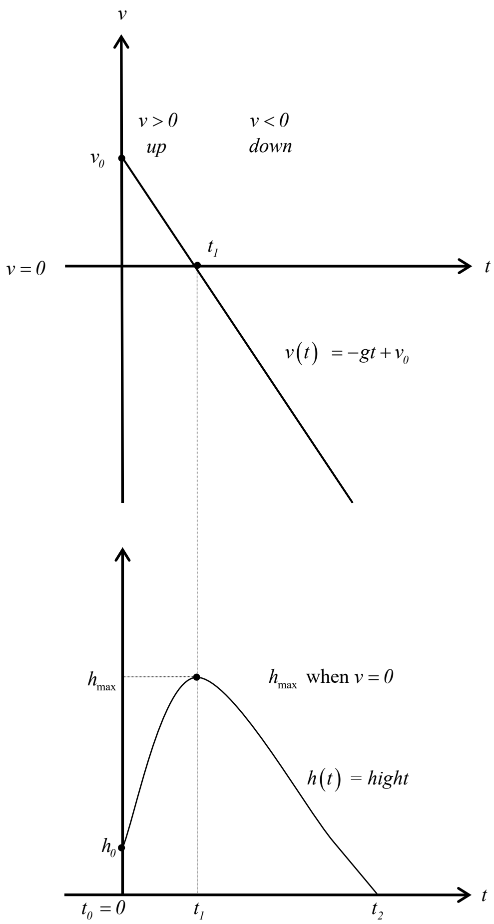
9. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}\left(a \sin bx + b \cos bx\right)}{a^2 + b^2} + c$

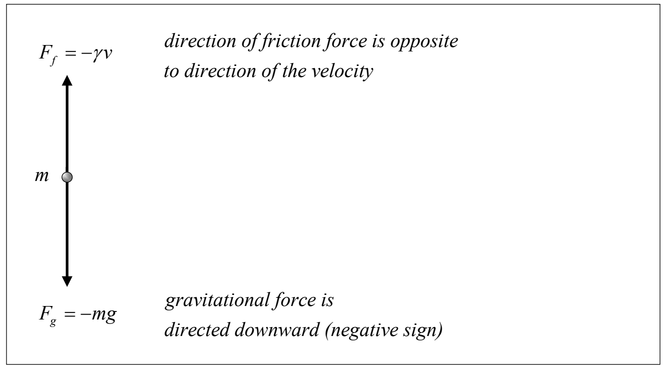
10. $\int \sin^2 mx \, dx = \frac{mx - \sin mx \cos mx}{2m} + c$

11. $\int \cos^2 mx \, dx = \frac{mx + \sin mx \cos mx}{2m} + c$

2.3 Falling ball

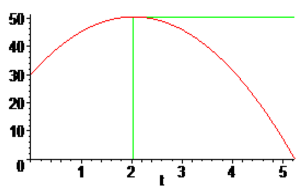
For convenience, assume that the **positive** direction is **UPWARD** (in contrast to textbook, see p.2)





$F = F_g + F_f$ *resulting force* $\qquad F = m\dfrac{dv}{dt}$ *Newton's Law*

$m\dfrac{dv}{dt} = -mg - \gamma v$	*governing equation for velocity* $v(t)$	
$\dfrac{dh}{dt} = v(t)$	*integrate from 0 to t* $\Rightarrow$	$h(t) = h_0 + \int_0^t v(t)dt$



conservation of energy $\quad mg\left(h_{max} - h_0\right) = \dfrac{mv_0^2}{2} \quad \Rightarrow \quad$ *elevation* $\left(h_{max} - h_0\right) = \dfrac{v_0^2}{2g}$

2.3 #16 $\quad m\dfrac{dv}{dt} = \cancel{\gamma v} - mg \qquad v(0) = v_0, \quad \gamma = 0 \qquad$ *no friction force*

$$\frac{dv}{dt} = -g$$

$$v = -gt + c_1 \qquad v_0 = -g\cancel{t} + c_1 \Rightarrow c_1 = v_0$$

$$\boxed{v(t) = -gt + v_0}$$

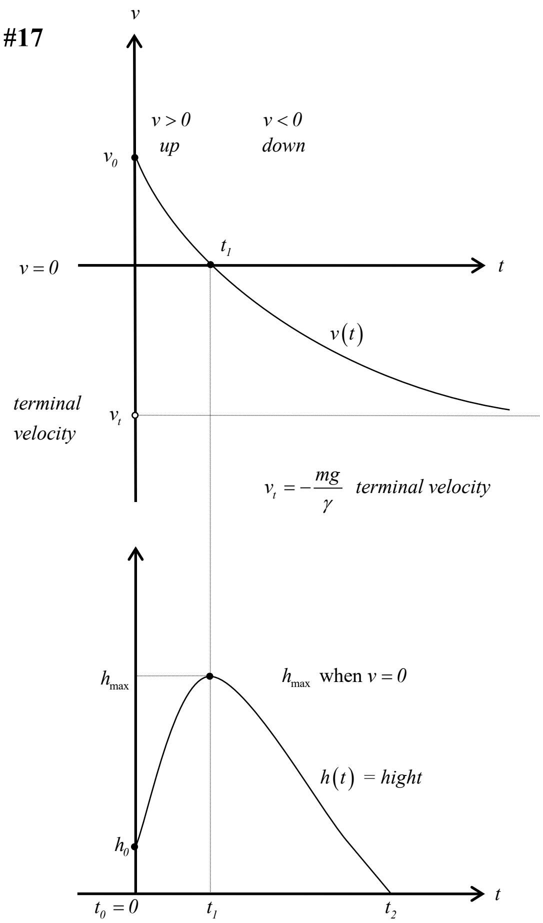
At $h = h_{max}, \quad v(t_1) = 0 \qquad \Rightarrow 0 = -gt_1 + v_0 \Rightarrow \boxed{t_1 = \dfrac{v_0}{g}}$

$$\frac{dh}{dt} = v(t) = -gt + v_0 \qquad \Rightarrow h(t) = -\frac{gt^2}{2} + v_0 t + c_2$$

$$h(0) = h_0 = -\cancel{\frac{gt^2}{2}} + \cancel{v_0 t} + c_2 \qquad \Rightarrow c_2 = h_0$$

$$\boxed{h(t) = -\frac{gt^2}{2} + v_0 t + h_0} \qquad h(t_2) = 0 = -\frac{gt_1^2}{2} + v_0 t_1 + h_0 \qquad \Rightarrow \boxed{t_2 = \frac{v_0}{g} + \sqrt{\left(\frac{v_0}{g}\right)^2 + \frac{2h_0}{g}}}$$

2.3 #17



friction force is opposite to velocity,
automatically changes direction

$$m\frac{dv}{dt} = -\gamma v - mg \qquad v(0) = v_0$$

$$\frac{dv}{dt} + \frac{\gamma}{m}v = -g$$

$$v = c_1 e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma} \qquad v_0 = c_1 - \frac{mg}{\gamma} \Rightarrow c_1 = v_0 + \frac{mg}{\gamma}$$

$$v(t) = \left(v_0 + \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}$$

At $h = h_{max}$, $v(t_1) = 0$ $\Rightarrow$ $0 = v(t_1) = \left(v_0 + \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t_1} - \frac{mg}{\gamma}$ $\Rightarrow$ $t_1 = -\frac{m}{\gamma}\ln\left(\frac{1}{1+\frac{\gamma v_0}{mg}}\right) = 5.24s$

$$\frac{dh}{dt} = v(t) = \left(v_0 + \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}$$

$$h(t) = -\frac{m}{\gamma}\left(v_0 + \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}t + c_2$$

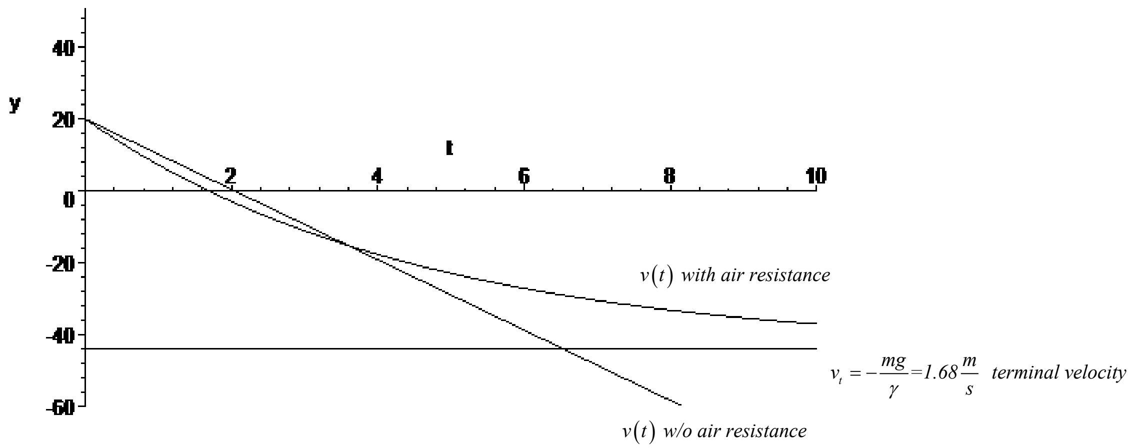
$$h(0) = h_0 = -\frac{m}{\gamma}\left(v_0 + \frac{mg}{\gamma}\right) + c_2 \qquad c_2 = h_0 + \frac{m}{\gamma}\left(v_0 + \frac{mg}{\gamma}\right)$$

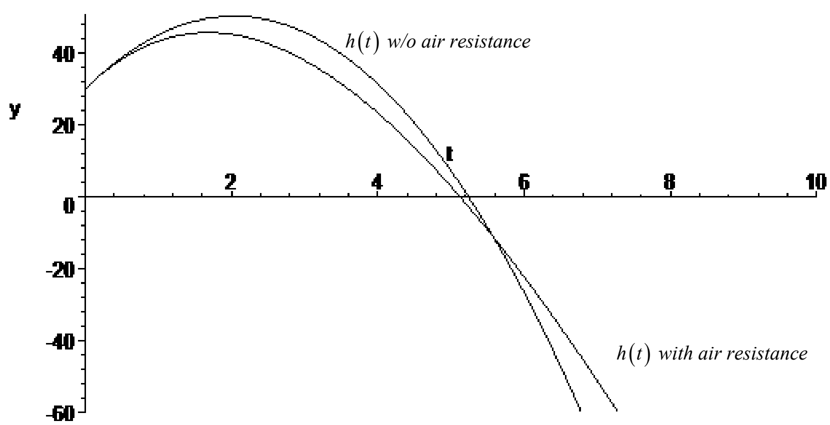
$$h(t) = \frac{m}{\gamma}\left(v_0 + \frac{mg}{\gamma}\right)\left(1 - e^{-\frac{\gamma}{m}t}\right) - \frac{mg}{\gamma}t + h_0$$

$$h_{max} = h(t_1) = \frac{m}{\gamma}\left(v_0 + \frac{mg}{\gamma}\right)\left(1 - e^{-\frac{\gamma}{m}t_1}\right) - \frac{mg}{\gamma}t_1 + h_0$$

use calculator solver

$h(t_2) = 0 = \left[1 - \frac{m}{\gamma}\left(v_0 + \frac{mg}{\gamma}\right)\right]e^{-\frac{\gamma}{m}t_2} - \frac{mg}{\gamma}t_2 + h_0$ solve for $t_2 = 5.12s$


y
t
40
20
0
-20
-40
-60
2
4
6
8
10
v(t) with air resistance
$v_t = -\frac{mg}{\gamma} = 1.68\frac{m}{s}$ terminal velocity
v(t) w/o air resistance


y
t
40
20
0
-20
-40
-60
2
4
6
8
10
h(t) w/o air resistance
h(t) with air resistance

2.3 #18 **Part I** $(v>0)$ $m\frac{dv}{dt}=-\gamma v^2-mg$ $0\le t\le t_1$ $v(0)=v_0>0$ *friction force is downward*

$$h(t)=h_0+\int_0^{t_1}v(t)dt$$

Part II $(v<0)$ $m\frac{dv}{dt}=+\gamma v^2-mg$ $t\ge t_1$ $v(t_1)=0$ *friction force is upward*

$$h(t)=h_{\max}+\int_{t_1}^{t_2}v(t)dt$$