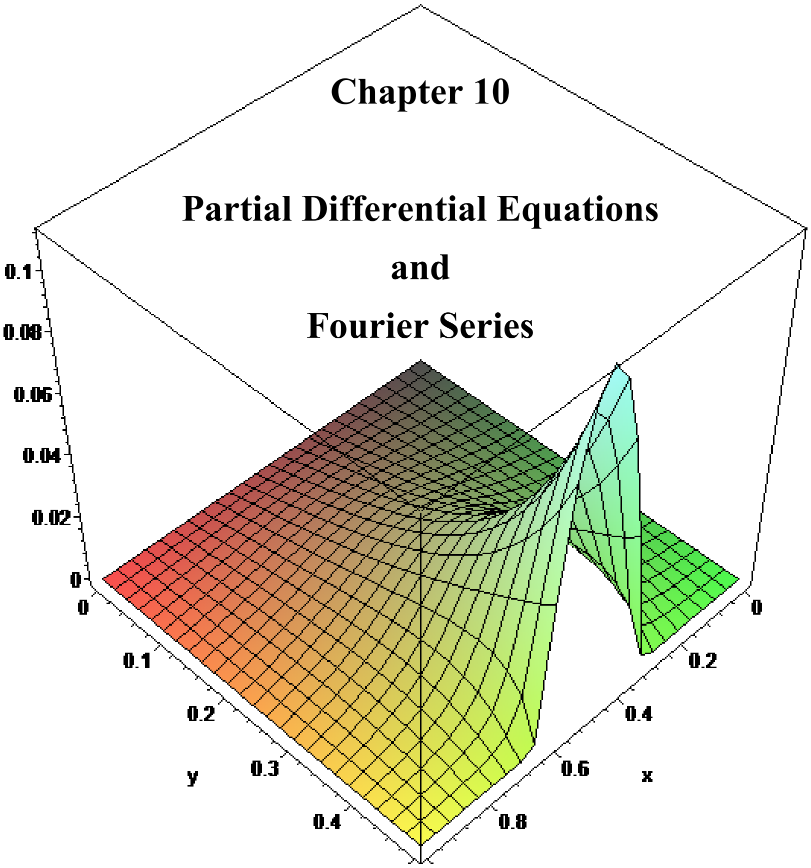
Chapter 10

Partial Differential Equations
and
Fourier Series

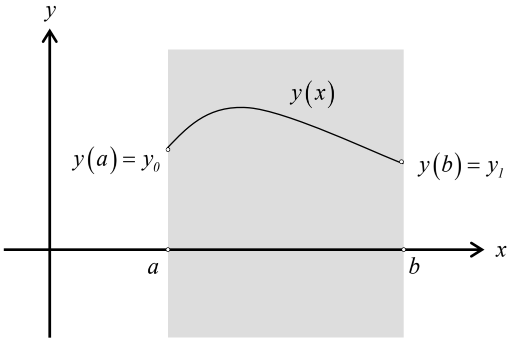


10.1 Boundary Value Problems for 2nd order ODE – One-Dimensional Boundary Value Problems

$$y'' + p(x)y' + q(x)y = g(x), \qquad x \in (a,b)$$

2nd order linear ODE

Domain: open interval $a < x < b$ **Boundary** of the interval consists of two points: $x = a$ and $x = b$



Coefficients $k, h_1, h_2 \geq 0$

The possible types of conditions at $x = a$ and at $x = b$:

Type of boundary condition	***Non-homogeneous*** *boundary conditions:*	
I *1st kind (Dirichlet)*	$y(a) = y_0$	$y(b) = y_1$
II *2nd kind (Neumann)*	$y'(a) = y_0$	$y'(b) = y_1$
III *3rd kind (Robin, mixed)*	$-ky'(a) + h_1 y(a) = y_0$	$ky'(b) + h_2 y(b) = y_1$
	Homogeneous *boundary conditions:*	
I	$y(a) = 0$	$y(b) = 0$
II	$y'(a) = 0$	$y'(b) = 0$
III	$-ky'(a) + h_1 y(a) = 0$	$ky'(b) + h_2 y(b) = 0$

Note the different direction of the gradient y' at the left and the right boundary points of the domain in the 3rd kind of the boundary conditions (***important!***).

Special Equation (homogeneous 2nd order linear ODE with constant coefficients)

$$y'' - \lambda y = 0, \quad x \in (0,L)$$

Characteristic equation $m^2 = \lambda$ **General Solution** can be written in the following forms**:**

1) $\lambda = \mu^2 > 0$ $m_{1,2} = \pm\mu$

$$y(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}$$

$$y(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

$$y(x) = c_1 \cosh[\mu(x-L)] + c_2 \sinh[\mu(x-L)]$$

2) $\lambda = -\mu^2 < 0$ $m_{1,2} = \pm\mu i$

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

$$y(x) = c_1 \cos[\mu(x-L)] + c_2 \sin[\mu(x-L)]$$

3) $\lambda = 0$ $m_{1,2} = 0$

$$y(x) = c_1 + c_2 x$$

$$y(x) = c_1 + c_2 \cdot (x-L)$$

Eigenvalue Problem: find the values of λ for which the differential equation

$$y'' = \lambda y, \qquad x \in (0,L)$$

subject to one of the nine possible combinations of the **homogeneous** boundary conditions

I	$y(0) = 0$
II	$y'(0) = 0$
III	$-y'(0) + H_1 y(0) = 0$

I	$y(L) = 0$
II	$y'(L) = 0$
III	$y'(L) + H_2 y(L) = 0$

has a **non-zero solution** $y(x)$**.**

This eigenvalue problem has a non-zero solution **only if** * $\lambda = -\mu^2 < 0$

There exist infinitely many real values $0 < \mu_1 < \mu_2 < \mu_3 < \ldots$ (***eigenvalues***)

and corresponding non-zero solutions $y_1,\ y_2,\ y_3,\ \ldots$ (***eigenfunctions***)

which satisfy the boundary conditions

(*in a case when both b.c.'s are of the 2nd kind: $y'(0) = 0$, $y'(L) = 0$, there is also $\mu_0 = 0$ with $y_0 = 1$)

Eigenfunctions $y_1,\ y_2,\ y_3,\ \ldots$ are mutually ***orthogonal*:** $(y_m, y_n) = \int_0^L y_m(x) y_n(x)\, dx = 0$ for $m \neq n$

There ***nine*** possible eigenvalue problems $y'' + \mu^2 y = 0$ subject to different combinations of boundary conditions:

Case 1: $y'' + \mu^2 y = 0$ $x \in (0, L)$

I $y(0) = 0$

I $y(L) = 0$

$\mu_n = \dfrac{n\pi}{L}$ $n = 1, 2, \ldots$

$y_n = \sin \dfrac{n\pi}{L} x$

$y = c_1 \cos \mu x + c_2 \sin \mu x$ (p.581)

$x = 0 \quad y(0) = 0 \;\Rightarrow\; c_1 \cos 0 + c_2 \sin 0 = 0 \quad \Rightarrow\; c_1 = 0$

$y = c_2 \sin \mu x$

$x = L \quad y(L) = 0 \;\Rightarrow\; \sin \mu L = 0 \quad \Rightarrow\; \mu L = n\pi$

$(y_n, y_n) = \dfrac{L}{2}$

Case 2: $y'' + \mu^2 y = 0$ $x \in (0, L)$

10.1 #18

II $y'(0) = 0$

II $y'(L) = 0$

$\mu_0 = 0$ $\mu_n = \dfrac{n\pi}{L}$

$y_0 = 1$ $y_n = \cos \dfrac{n\pi}{L} x$

$(y_0, y_0) = L$

$(y_n, y_n) = \dfrac{L}{2}$

Case 3: $y'' + \mu^2 y = 0$ $x \in (0, L)$

10.1 #14

I $y(0) = 0$

II $y'(L) = 0$

$\mu_n = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}$ $n = 0, 1, 2, \ldots$

$y_n = \sin\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L} x$

$(y_n, y_n) = \dfrac{L}{2}$

Case 4: $y'' + \mu^2 y = 0$ $x \in (0, L)$

10.1 #15,17

II $y'(0) = 0$

I $y(L) = 0$

$\mu_n = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L}$ $n = 0, 1, 2, \ldots$

$y_n = \cos\left(n + \dfrac{1}{2}\right)\dfrac{\pi}{L} x$

$(y_n, y_n) = \dfrac{L}{2}$

The following possible examples are important but they are not considered in our class (here, H>0 is a constant):

Case 5: $y'' + \mu^2 y = 0 \qquad x \in (0, L)$

I $y(0) = 0$

III $y'(L) + Hy(L) = 0$

Case 6: $y'' + \mu^2 y = 0 \qquad x \in (0, L)$

III $-y'(0) + Hy(0) = 0$

I $y(L) = 0$

Case 7: $y'' + \mu^2 y = 0 \qquad x \in (0, L)$

II $y'(0) = 0$

III $y'(L) + Hy(L) = 0$

Case 8: $y'' + \mu^2 y = 0 \qquad x \in (0, L)$

III $-y'(0) + Hy(0) = 0$

II $y'(L) = 0$

Case 9: $y'' + \mu^2 y = 0 \qquad x \in (0, L)$

III $-y'(0) + H_1 y(0) = 0$

III $y'(L) + H_2 y(L) = 0$

Exercise: Write expansion into the generalized Fourier series of $f(x) = 1 - x$ on the interval $[0, 1]$ with the help of Eigenfunctions of the *Case 4* with $L = 1$.

10.2 Fourier Series

Periodic function $f(x+T)=f(x)$ for all x. T is a period.

The smallest period is called the *fundamental period*.

Even function $f(-x)=f(x)$

Odd function $u(-x)=-u(x)$

Properties: Let f,g be **even** functions, and let u,v be **odd** functions, then

$f(x)+g(x)$	even
$f(x)\cdot g(x)$	even
$u(x)+v(x)$	odd
$u(x)\cdot v(x)$	even
$f(x)\cdot u(x)$	odd

$f(x)$ is ***even***, then $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$

$u(x)$ is ***odd***, then $\int_{-L}^{L} u(x)dx = 0$

Inner product of two functions on $[a,b]$: $(u,v)=\int_a^b uvdx$

Functions u and v are said to be ***orthogonal*** if: $(u,v)=\int_a^b u(x)v(x)dx=0$

Generalized Fourier Series:

If $\{y_k(x)\}$ is a complete set of functions mutually orthogonal on $[a,b]$, then $f(x)$ can be represented by

$$f(x) = c_1y_1(x)+c_2y_2(x)+\dots = \sum_{k=1}^{\infty} c_ky_k(x),$$ where coefficients $$c_k=\frac{(f,y_k)}{(y_k,y_k)}$$

Set of functions $\left\{ 1,\ cos\frac{n\pi}{L}x,\ sin\frac{n\pi}{L} \right\}$ is orthogonal on $[-L,L]$. Verify orthogonality

$\int_{-L}^{L} 1\cdot 1\, dx = \quad 2L$	$\int_{-L}^{L} cos\frac{m\pi}{L}x\, cos\frac{n\pi}{L}xdx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$
$\int_{-L}^{L} 1\cdot cos\frac{n\pi}{L}xdx = \quad \frac{L}{\pi}\left[sin\frac{n\pi}{L} \right]_{-L}^{L} = 0$	$\int_{-L}^{L} sin\frac{m\pi}{L}x\, sin\frac{n\pi}{L}xdx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$
$\int_{-L}^{L} 1\cdot sin\frac{n\pi}{L}xdx = \quad 0$	$\int_{-L}^{L} cos\frac{m\pi}{L}x\, sin\frac{n\pi}{L}xdx = \quad 0$

The Euler-Fourier Formulas (p.599):

Fourier Series

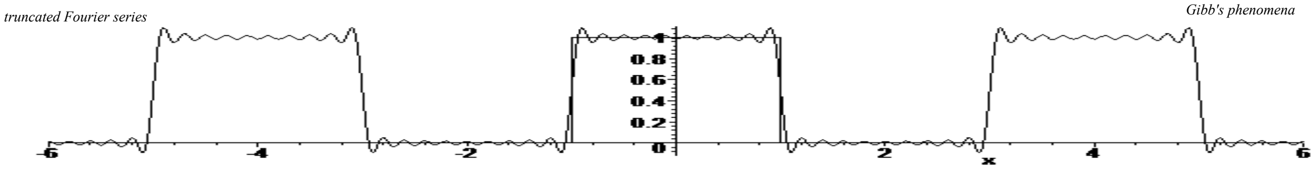
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\, cos\frac{m\pi}{L}x + b_m\, sin\frac{m\pi}{L}x \right) \tag{9}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{12}$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi}{L}xdx \tag{13}$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi}{L}xdx \tag{14}$$

10.3 Convergence of the Fourier Series



Let $f(x)$ be a piece-wise continuous with a finite number of finite jumps on $[-L,L)$, then

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\, cos\frac{m\pi}{L}x + b_n\, sin\frac{m\pi}{L}x \right) \text{ converges to } \begin{cases} f(x), & \text{if } f(x) \text{ is continuous at } x \\ \dfrac{f(x^-)+f(x^+)}{2}, & \text{if } f(x) \text{ is discontinuous at } x \end{cases}$$

Maple Fourier series expansion of $f(x)=\begin{cases} 0 & -2<x<-1 \\ 1 & -1<x<\ \ 1 \\ 0 & \ \ 1<x<\ \ 2 \end{cases}$

```
> L:=2;
```

$$L := 2$$

```
> f:=Heaviside(x+1)-Heaviside(x-1);
```

$$f := \text{Heaviside}(x+1) - \text{Heaviside}(x-1)$$

```
> a[0]:=1/L*int(f,x=-L..L);
```

$$a_0 := 1$$

```
> a[k]:=1/L*int(f*cos(k*Pi/L*x),x=-L..L);
```

$$a_k := \frac{2\sin\left(\frac{k\pi}{2}\right)}{k\pi}$$

```
> b[k]:=1/L*int(f*sin(k*Pi/L*x),x=-L..L);
```

$$b_k := 0$$

```
> u:=a[0]/2+sum(a[k]*cos(k*Pi/L*x)+b[k]*sin(k*Pi/L*x),k=1..20):
```

The Fourier Series

```
> plot({f,u},x=-L..L);
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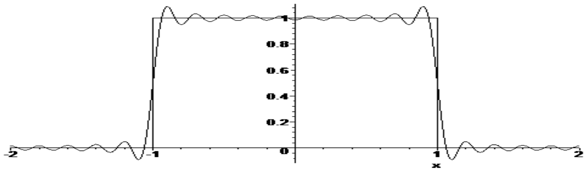

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> plot({abs(f-u)},x=-L..L);
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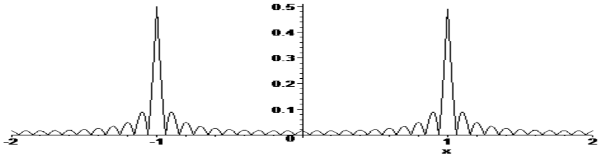

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> plot({u},x=-5*L..5*L);    periodic extension:
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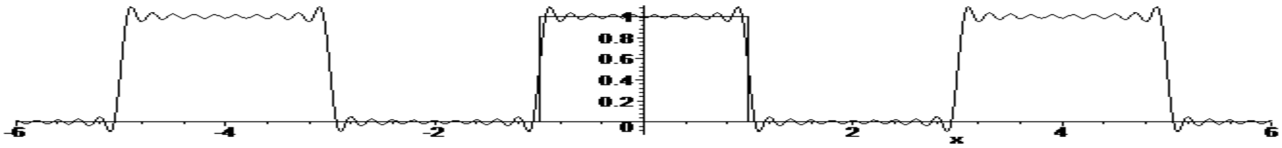
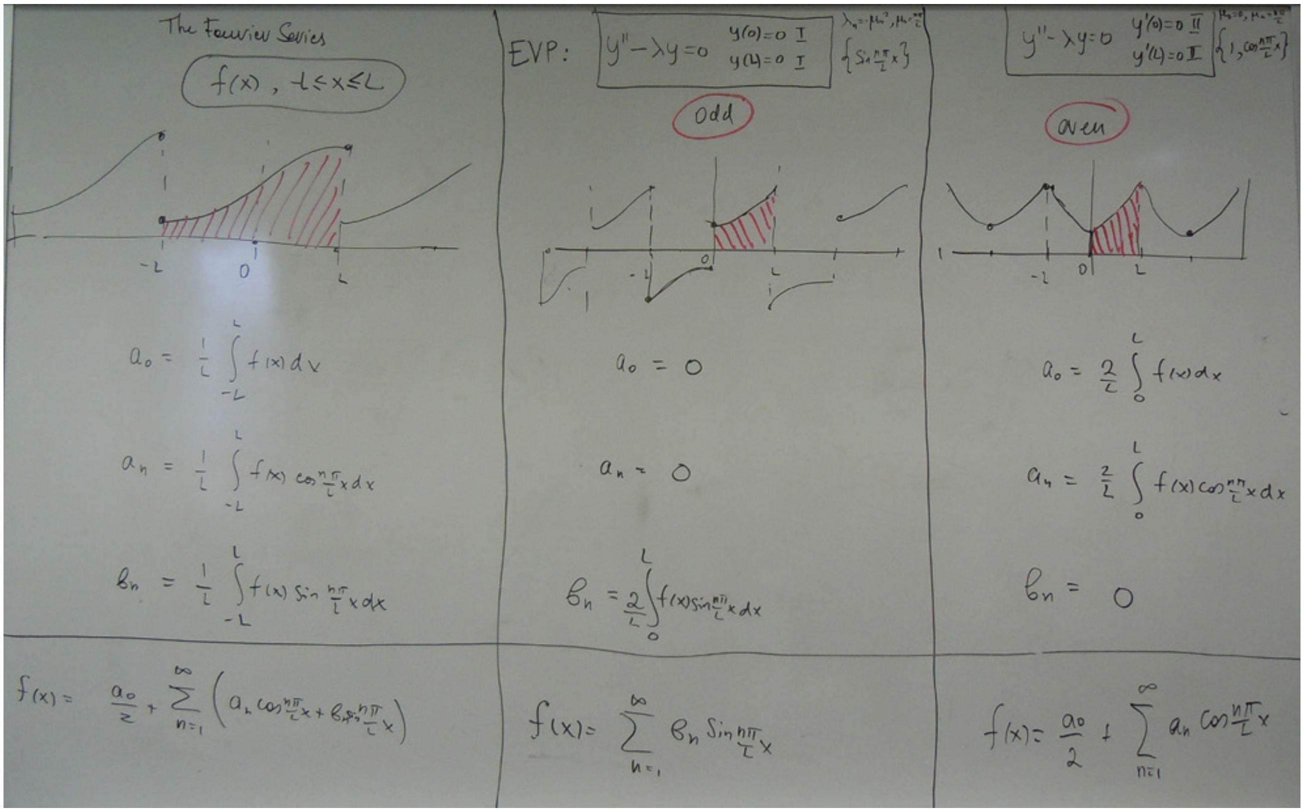


Useful facts :

$\sin n\pi = 0$	$\cos n\pi = (-1)^n$	$\sin\frac{n\pi}{2} =$ $\cos\frac{n\pi}{2} =$

10.4 The Fourier Series of Odd and Even Functions (Sine Fourier Series and Cosine Fourier Series)



The Fourier Series

$\boxed{f(x)\ ,\ -L \le x \le L}$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$$

EVP: $\boxed{y'' - \lambda y = 0 \quad \begin{matrix} y(0)=0 & \text{I} \\ y(L)=0 & \text{I} \end{matrix}}$ $\lambda_n = -\mu_n^2,\ \mu_n = \frac{n\pi}{L}$ $\left\{\sin\frac{n\pi}{L}x\right\}$

Odd

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

$\boxed{y'' - \lambda y = 0 \quad \begin{matrix} y'(0)=0 & \text{II} \\ y'(L)=0 & \text{II} \end{matrix}}$ $\mu_0 = 0,\ \mu_n = \frac{n\pi}{L}$ $\left\{1, \cos\frac{n\pi}{L}x\right\}$

even

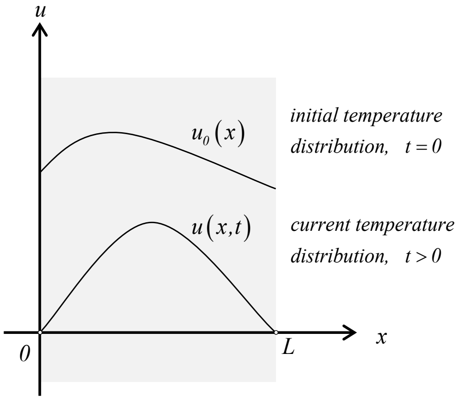
$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

10.5 The Heat Equation – Homogeneous Boundary Conditions (I – I) Initial-Boundary Value Problem:



Heat Equation	$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{a^2}\dfrac{\partial u}{\partial t}$	$0 < x < L,\ t > 0$	
Initial condition:	$u(x,0) = u_0(x)$	$0 \le x \le L$	
Boundary conditions:	$u(0,t) = 0$	$t > 0$	**(I)**
	$u(L,t) = 0$	$t > 0$	**(I)**

1) Assume

$$u(x,t) = X(x)T(t)$$

$$\frac{\partial^2}{\partial x^2}u(x,t) = X''(x)T(t)$$

$$\frac{\partial}{\partial t}u(x,t) = X(x)T'(t)$$

2) Separate variables

$$X''T = \frac{1}{a^2}XT' \quad \Rightarrow \quad \frac{X''}{X} = \frac{1}{a^2}\frac{T'}{T} = \lambda$$

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

$$u(L,t) = X(L)T(t) = 0 \quad \Rightarrow \quad X(L) = 0$$

3) Solve eigenvalue problem

$$\frac{X''}{X} = \lambda \quad \Rightarrow \quad X'' = \lambda X$$

$$X(0) = 0 \qquad \text{(I)}$$

$$X(L) = 0 \qquad \text{(I)}$$

$$\mu_n = \frac{n\pi}{L} \qquad n = 1, 2, \ldots$$

$$X_n(x) = \sin(\mu_n x) = \sin\left(\frac{n\pi}{L}x\right)$$

4) Solve

$$\frac{1}{a^2}\frac{T'}{T} = \lambda = -\mu_n^2 \quad \Rightarrow \quad T' + \mu_n^2 a^2 T = 0$$

$$T_n(t) = e^{-\mu_n^2 a^2 t}$$

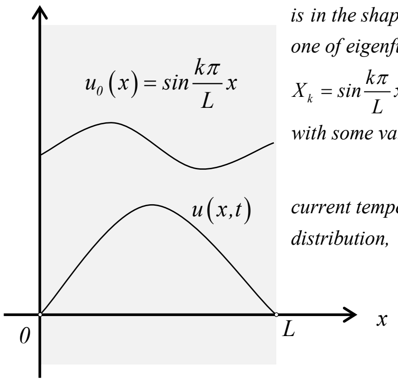
5) Solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

where

$$c_n = \frac{\int_0^L u_0(x) X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx} = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

10.5 The Heat Equation – (I – I) Single term solution



initial temperature distribution at $t = 0$ is in the shape of one of eigenfunctions $X_k = sin\frac{k\pi}{L}x$ *with some value* k

current temperature distribution, $t > 0$

Heat Equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}$ $0 < x < L,\ t > 0$

Initial condition: $u(x,0) = sin\frac{k\pi}{L}x$ $0 \le x \le L$

Boundary conditions: $u(0,t) = 0$ $t > 0$ **(I)**

$u(L,t) = 0$ $t > 0$ **(I)**

1) Assume $u(x,t) = X(x)T(t)$

$$\frac{\partial^2}{\partial x^2}u(x,t) = X''(x)T(t)$$

$$\frac{\partial}{\partial t}u(x,t) = X(x)T'(t)$$

2) Separate variables $X''T = \frac{1}{a^2}XT'$ $\Rightarrow$ $\frac{X''}{X} = \frac{1}{a^2}\frac{T'}{T} = \lambda$

$u(0,t) = X(0)T(t) = 0$ $\Rightarrow$ $X(0) = 0$

$u(L,t) = X(L)T(t) = 0$ $\Rightarrow$ $X(L) = 0$

3) Solve eigenvalue problem $\frac{X''}{X} = \lambda$ $\Rightarrow$ $X'' = \lambda X$

$X(0) = 0$ (I)

$X(L) = 0$ (I)

$\mu_n = \frac{n\pi}{L}$ $n = 1, 2, \ldots$

$X_n(x) = sin(\mu_n x) = sin\left(\frac{n\pi}{L}x\right)$

4) Solve $\frac{1}{a^2}\frac{T'}{T} = \lambda = -\mu_n^2$ $\Rightarrow$ $T' + \mu_n^2 a^2 T = 0$

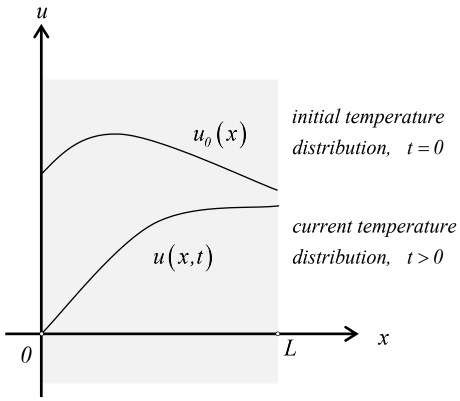
$T_n(t) = e^{-\mu_n^2 a^2 t}$

5) **Solution:** $u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$ where $c_k = \frac{2}{L}\int_0^L sin\left(\frac{k\pi}{L}x\right) sin\left(\frac{k\pi}{L}x\right)dx = 1$

$c_n = 0$ for all $n \ne k$

Therefore, $u(x,t) = X_k(x)T_k(t)$

10.5 The Heat Equation – Homogeneous Boundary Conditions (I – II) Initial-Boundary value Problem:



Heat Equation	$\frac{\partial^2 u}{\partial x^2}=\frac{1}{a^2}\frac{\partial u}{\partial t}$	$0<x<L,\ t>0$	
Initial condition:	$u(x,0)=u_0(x)$	$0\le x\le L$	
Boundary conditions:	$u(0,t)=0$	$t>0$	(I)
	$u'(L,t)=0$	$t>0$	(II)

1) Assume

$$u(x,t)=X(x)T(t)$$

$$\frac{\partial^2}{\partial x^2}u(x,t)=X''(x)T(t)$$

$$\frac{\partial}{\partial t}u(x,t)=X(x)T'(t)$$

substitute into equation and boundary conditions

2) Separate variables

$$X''T=\frac{1}{a^2}XT' \quad\Rightarrow\quad \boxed{\frac{X''}{X}=\frac{1}{a^2}\frac{T'}{T}=\lambda}$$

$$u(0,t)=X(0)T(t)=0 \quad\Rightarrow\quad X(0)=0$$

$$u'(L,t)=X'(L)T(t)=0 \quad\Rightarrow\quad X(L)=0$$

3) Solve eigenvalue problem

$$\frac{X''}{X}=\lambda \quad\Rightarrow\quad X''=\lambda X$$

$$X(0)=0 \qquad \text{(I)}$$

$$X'(L)=0 \qquad \text{(II)}$$

$$\mu_n=\frac{\pi}{L}\left(n+\frac{1}{2}\right),\ n=0,1,...$$

$$X_n(x)=sin(\mu_n x)=sin\left[\frac{\pi}{L}\left(n+\frac{1}{2}\right)x\right]$$

4) Solve

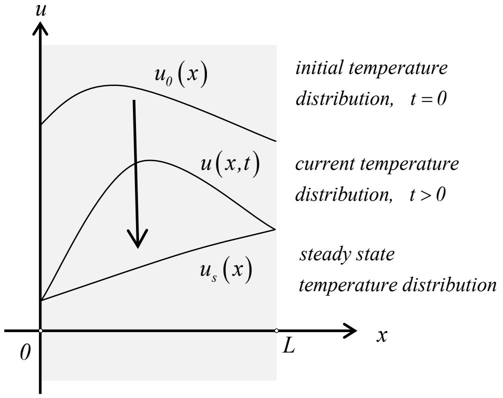
$$\frac{1}{a^2}\frac{T'}{T}=\lambda \quad\Rightarrow\quad T'+\mu_n^2a^2T=0$$

$$\boxed{T_n(t)=e^{-\mu_n^2a^2t}}$$

5) Solution:

$$u(x,t)=\sum_{n=0}^{\infty}c_nX_n(x)T_n(t), \qquad c_n=\frac{\int_0^L u_0(x)X_n(x)dx}{\int_0^L X_n^2(x)dx}=\frac{2}{L}\int_0^L u_0(x)sin\left(\frac{\pi}{L}\left(n+\frac{1}{2}\right)x\right)dx$$

10.6 The Heat Equation – Non-Homogeneous Boundary Conditions Initial-Boundary Value Problem:



Heat Equation	$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{a^2}\dfrac{\partial u}{\partial t}$	$0<x<L,\ t>0$	
Initial condition	$u(x,0)=u_0(x)$	$0\le x\le L$	
Boundary conditions	$\left[u(x,t)\right]_{x=0}=u_1$	$t>0$	I
	$\left[u(x,t)\right]_{x=L}=u_2$	$t>0$	I

1) ***Steady state problem*** $\dfrac{\partial^2 u_s(x)}{\partial x^2}=0$ [definition of steady state: $u_s(x)=\lim\limits_{t\to\infty}u(x,t)$]

Boundary conditions: $u_s(0)=u_1$ (I)

$u_s(L)=u_2$ (I)

Steady State Solution:

$$\boxed{u_s(x)=\frac{u_2-u_1}{L}x+u_1}$$

2) Define ***transient solution:*** $U(x,t)=u(x,t)-u_s(x)$

Initial-Boundary problem	$\dfrac{\partial^2 U}{\partial x^2}=\dfrac{1}{a^2}\dfrac{\partial U}{\partial t}$	$0<x<L,\ t>0$
Initial condition	$U(x,0)=u_0(x)-u_s(x)$	$0\le x\le L$
Boundary conditions	$\left[U(x,t)\right]_{x=0}=0$	$t>0$
	$\left[U(x,t)\right]_{x=L}=0$	$t>0$

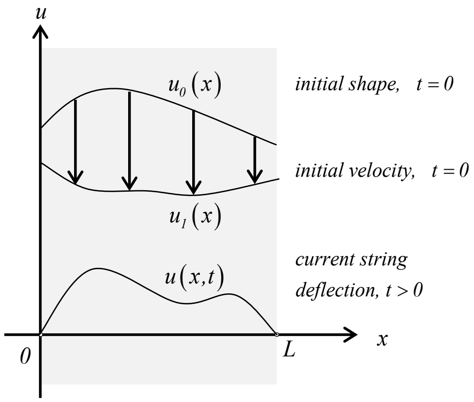
(Homogenous B.C.'s)

3) Solution of IBVP: $u(x,t)=U(x,t)+u_s(x)$ (consists of transient and steady state solutions)

Solution:

$$u(x,t)\ =\ \frac{u_2-u_1}{L}x+u_1\ +\ \sum_{n=1}^{\infty}c_nX_n(x)T_n(t),\qquad c_n=\frac{\int_0^L\left[u_0(x)-u_s(x)\right]X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}\ ,\qquad T_n(t)=e^{-\mu_n^2a^2t}$$

10.7 The Wave Equation – Basic Case – Homogeneous Boundary Condition



Wave Equation	$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$	$0<x<L,\ t>0$	
Initial condition:	$u(x,0) = u_0(x)$	$0 \le x \le L$	
	$\frac{\partial}{\partial t}u(x,0) = u_1(x)$	$0 \le x \le L$	
Boundary conditions:	$\left[u(x,t)\right]_{x=0} = 0$	$t>0$	I
	$\left[u(x,t)\right]_{x=L} = 0$	$t>0$	I

1) Assume $u(x,t) = X(x)T(t)$

$$\frac{\partial^2}{\partial x^2}u(x,t) = X''(x)T(t)$$

$$\frac{\partial}{\partial t}u(x,t) = X(x)T''(t)$$ substitute

2) Separate variables $X''T = \frac{1}{a^2}XT''$ $\Rightarrow$ $\boxed{\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = \lambda}$

$u(0,t) = X(0)T(t) = 0$ $\Rightarrow$ $X(0) = 0$

$u(L,t) = X(L)T(t) = 0$ $\Rightarrow$ $X(L) = 0$

3) Solve eigenvalue problem $\frac{X''}{X} = \lambda$ $\Rightarrow$

$X'' = \lambda X$

$X(0) = 0$ (I)

$X(L) = 0$ (I)

$$\mu_n = \frac{n\pi}{L}, \quad n = 1,2,...$$

$$X_n(x) = sin(\mu_n x) = sin\left(\frac{n\pi}{L}x\right)$$

4) Solve $\frac{1}{a^2}\frac{T''}{T} = \lambda = -\mu_n^2$ $\Rightarrow$ $T'' + \mu_n^2 a^2 T = 0$

$$\boxed{T_n(t) = c_{1,n}\cos(a\mu_n t) + c_{2,n}\sin(a\mu_n t)}$$

5) Solution:

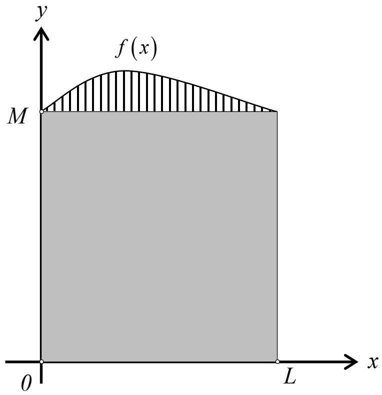
$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(a\mu_n t) + b_n \sin(a\mu_n t)\right]\cdot \sin\left(\frac{n\pi}{L}x\right),$$

$$a_n = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi}{L}x\right)dx\ , \quad b_n = \frac{2}{an\pi}\int_0^L u_1(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

The Wave Equation – Homogeneous Boundary Conditions

Standing Waves

10.8 The Laplace Equation – Basic Boundary Value Problem – 3 homogeneous, 1 non-homogenous b.c.'s :



The Laplace Equation $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<L,\ 0<y<M$$

Boundary conditions: $$[I,II,III]_{x=0}=0$$
$$[I,II,III]_{x=L}=0$$
$$[I,II,III]_{y=0}=0$$
$$[I,II,III]_{y=M}=f(x)$$

1) Assume $u(x,y)=X(x)Y(y)$

substitute into equation and b.c.'s

2) Separate variables $X''Y+XY''=0 \quad \Rightarrow \quad \boxed{\frac{X''}{X}=-\frac{Y''}{Y}=\lambda}$

$$\left[u(0,y)\right]=\left[X(0)\right]Y(y)=0 \quad \Rightarrow \quad \left[X(0)\right]=0$$
$$\left[u(L,y)\right]=\left[X(L)\right]Y(y)=0 \quad \Rightarrow \quad \left[X(L)\right]=0$$
$$\left[u(x,0)\right]=X(x)\left[Y(0)\right]=0 \quad \Rightarrow \quad \left[Y(0)\right]=0$$

3) Solve eigenvalue problem $\frac{X''}{X}=\lambda \quad \Rightarrow$

$X''=\lambda X$

$\left[X(0)\right]=0$ (I, II or III)

$\left[X(L)\right]=0$ (I, II or III)

Non-zero solutions exist only if $\lambda=-\mu_n^2$

$\mu_n =$

$X_n(x) =$

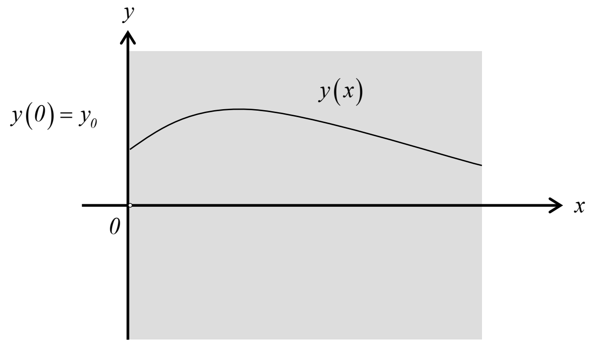
4) Solve $-\frac{Y''}{Y}=\lambda \quad \Rightarrow \quad Y''-\mu_n^2 Y=0$

$$\boxed{Y_n(y)=c_{1,n}\cosh(\mu_n y)+c_{2,n}\sinh(\mu_n y)}$$

$$\left[Y(0)\right]=0 \quad \Rightarrow \quad Y_n(y)=c_{2,n}\sinh(\mu_n y)$$

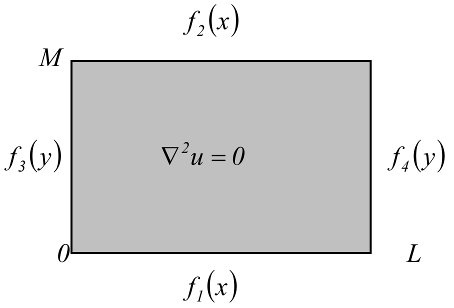
5) Solution: $$u(x,y)=\sum_{n=1}^{\infty}a_n \sinh(\mu_n y)X_n(x), \qquad a_n=\frac{1}{\sinh(\mu_n M)}\frac{\int_0^L f(x)X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

Semi-infinite layer domain: $x > 0$, boundary is defined by a single point: $x = 0$

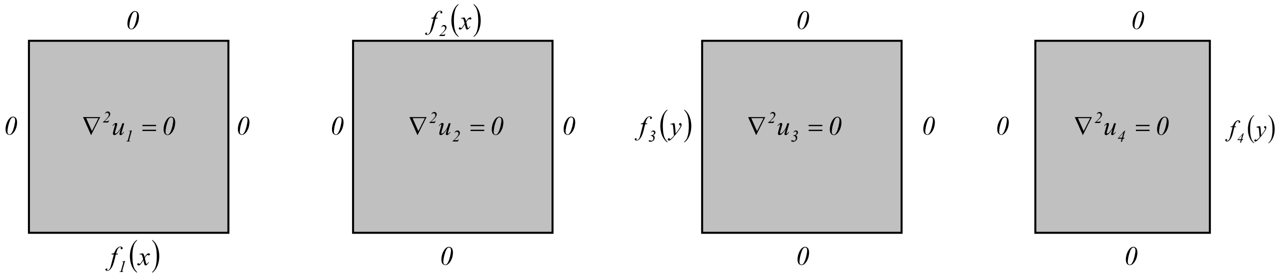


The Laplace Equation – 4 non-homogeneous boundary conditions– all of the 1st kind

(I) Dirichlet Problem $\left[u\right]_S = f$



Split into supplemental basic problems **(superposition principle)**:



Solution of supplemental basic problems:

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left[\frac{n\pi}{L}(y-M)\right] \qquad a_n = \frac{-\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi}{L}x\right)dx}{\sinh\left(\frac{n\pi}{L}M\right)}$$

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}y\right) \qquad b_n = \frac{\frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi}{L}x\right)dx}{\sinh\left(\frac{n\pi}{L}M\right)}$$

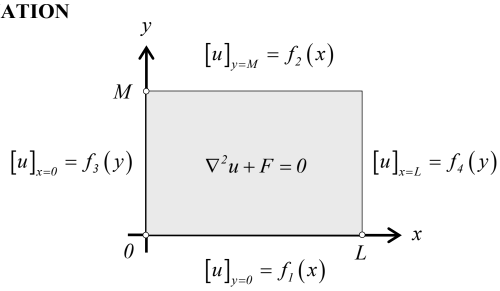
$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left[\frac{n\pi}{M}(x-L)\right] \sin\left(\frac{n\pi}{M}y\right) \qquad c_n = \frac{-\frac{2}{M}\int_0^M f_3(y)\sin\left(\frac{n\pi}{M}y\right)dy}{\sinh\left(\frac{n\pi}{M}L\right)}$$

$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sinh\left(\frac{n\pi}{M}x\right) \sin\left(\frac{n\pi}{M}y\right) \qquad d_n = \frac{\frac{2}{M}\int_0^M f_4(y)\sin\left(\frac{n\pi}{M}y\right)dy}{\sinh\left(\frac{n\pi}{M}L\right)}$$

Solution of Dirichlet problem – superposition of supplemental solutions:

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

POISSON'S EQUATION



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+F(x,y)=0 \qquad (x,y)\in(0,L)\times(0,M)\subset\mathbb{R}^2$$

$$[u]_S=f$$

brackets mean that condition at the boundary can be either of I^{st}, II^{nd} or III^{rd} kind

Supplemental Sturm-Liouville Problems

(in a case if 3-D problem is considered)

$$\boxed{\begin{array}{lcl} X''-\mu X=0 & & X_n''=-\lambda_n^2X_n \\ [X]_{x=0}=0 & \overset{SLP}{\Rightarrow} & \mu_n=-\lambda_n^2 \\ [X]_{x=L}=0 & & X_n(x) \end{array}} \quad \boxed{\begin{array}{lcl} Y''-\eta Y=0 & & Y_m''=-\nu_m^2Y_m \\ [Y]_{y=0}=0 & \overset{SLP}{\Rightarrow} & \eta_m=-\nu_m^2 \\ [Y]_{y=M}=0 & & Y_m(y) \end{array}} \quad \boxed{\begin{array}{lcl} Z''-\gamma Z=0 & & Z_k''=-\omega_k^2Z_k \\ [Z]_{z=0}=0 & \overset{SLP}{\Rightarrow} & \gamma_k=-\omega_k^2 \\ [Z]_{z=K}=0 & & Z_k(z) \end{array}}$$

LAPLACE'S EQUATION (homogeneous eqn, non-homogeneous boundary conditions)

$$\begin{array}{ccc} & f_2 & \\ f_3 & \boxed{\nabla^2 u_5=0} & f_4 \\ & f_1 & \end{array}$$

Solution of Basic Cases of Laplace's Equation with one non-homogeneous b.c. by Separation of Variables (SV)

$$\begin{array}{ccc} & 0 & \\ 0 & \boxed{\nabla^2 u_1=0} & 0 \\ & f_1 & \end{array} \quad \begin{array}{c} X_n \\ Y_n^{(1)} \end{array} \quad u_1(x,y)=\sum_n a_nX_nY_n^{(1)} \quad a_n=\frac{1}{Y_n^{(1)}(0)}\frac{\int_0^L f_1X_n dx}{\|X_n\|^2}$$

$$\begin{array}{ccc} & f_2 & \\ 0 & \boxed{\nabla^2 u_2=0} & 0 \\ & 0 & \end{array} \quad \begin{array}{c} X_n \\ Y_n^{(2)} \end{array} \quad u_2(x,y)=\sum_n b_nX_nY_n^{(2)} \quad b_n=\frac{1}{Y_n^{(2)}(M)}\frac{\int_0^L f_2X_n dx}{\|X_n\|^2}$$

$$\begin{array}{ccc} & 0 & \\ f_3 & \boxed{\nabla^2 u_3=0} & 0 \\ & 0 & \end{array} \quad \begin{array}{c} X_m^{(1)} \\ Y_m \end{array} \quad u_3(x,y)=\sum_m c_mX_m^{(1)}Y_m \quad c_m=\frac{1}{X_m^{(1)}(0)}\frac{\int_0^M f_3Y_m dy}{\|Y_m\|^2}$$

$$\begin{array}{ccc} & 0 & \\ 0 & \boxed{\nabla^2 u_4=0} & f_4 \\ & 0 & \end{array} \quad \begin{array}{c} X_m^{(2)} \\ Y_m \end{array} \quad u_4(x,y)=\sum_m d_mX_m^{(2)}Y_m \quad d_m=\frac{1}{X_m^{(2)}(L)}\frac{\int_0^M f_4Y_m dy}{\|Y_m\|^2}$$

$$\boxed{u_5=u_1+u_2+u_3+u_4}$$

Superposition Principle (SP)

Solution of Basic Case of Poisson's Eqn (homogeneous b.c.'s)

$$\begin{array}{ccc} & 0 & \\ 0 & \boxed{\nabla^2 u_6=-F} & 0 \\ & 0 & \end{array} \quad \begin{array}{c} X_n \\ Y_m \end{array} \quad u_6(x,y)=\sum_n\sum_m A_{nm}X_nY_m \quad A_{mn}=\frac{\int_0^M\int_0^L FX_nY_m dxdy}{(\lambda_n^2+\nu_m^2)\|X_n\|^2\|Y_m\|^2}$$

Eigenvalue Expansion (EE)

Solution of Poisson's Equation with non-homogeneous b.c.'s

$$\boxed{u(x,y)=u_5+u_6}$$

Superposition Principle (SP)

Eigenvalue Problem $y'' + \mu^2 y = 0\,,\quad 0 < x < L$ subject to homogeneous boundary conditions at $x = 0$ and $x = L$.

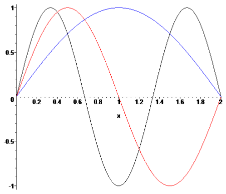
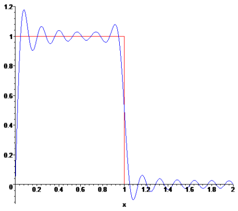
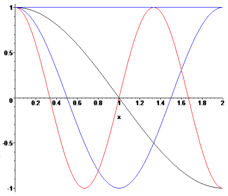
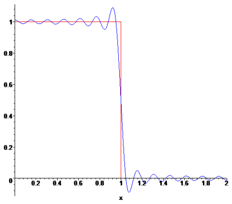
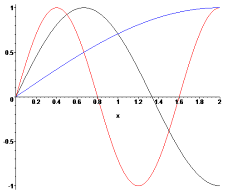
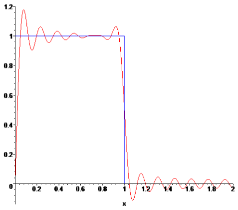
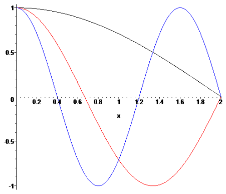
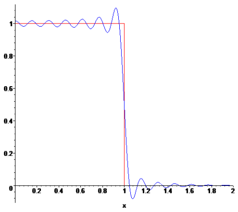
$y'' = \lambda y$

has solutions only if $\lambda = \mu^2 \ge 0$

There exist infinitely many real values $0 \le \mu_1 < \mu_2 < \mu_3 < \ldots$ (***eigenvalues***)

and corresponding non-zero solutions $y_1,\ y_2,\ y_3,\ \ldots$ (***eigenfunctions***)

which satisfy the boundary conditions.

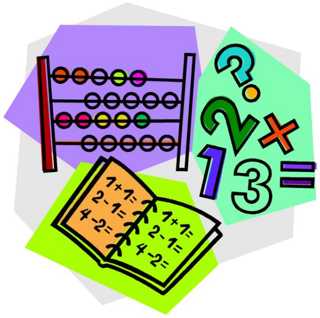
Boundary cs.	*Eigenvalues, eigenfunctions*	*Fourier series*
I $y(0)=0$ **I** $y(L)=0$	$\mu_n = \frac{n\pi}{L}$ $y_n = \sin\frac{n\pi}{L}x$ $n = 1,2,\ldots$ 	$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$ $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$ 
II $y'(0)=0$ **II** $y'(L)=0$	$\mu_0 = 0\,,\ \mu_n = \frac{n\pi}{L}$ $y_0 = 1\,, y_n = \cos\frac{n\pi}{L}x$ $n = 1,2,\ldots$ 	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$ $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ 
I $y(0)=0$ **II** $y'(L)=0$	$\mu_n = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$ $y_n = \sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x$ $n = 0,1,2,\ldots$ 	$f(x) = \sum_{n=0}^{\infty} c_n \sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x$ $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}xdx$ 
II $y'(0)=0$ **I** $y(L)=0$	$\mu_n = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$ $y_n = \cos\left(n + \frac{1}{2}\right)\frac{\pi}{L}x$ $n = 0,1,2,\ldots$ 	$f(x) = \sum_{n=0}^{\infty} d_n \cos\left(n + \frac{1}{2}\right)\frac{\pi}{L}x$ $d_n = \frac{2}{L}\int_0^L f(x)\cos\left(n + \frac{1}{2}\right)\frac{\pi}{L}xdx$ 

The Generalized Fourier series $f(x) = \sum_{n} c_n y_n(x)$ $\quad c_n = \frac{(f, y_n)}{(y_n, y_n)} = \frac{2}{L}\int_0^L f(x)\ y_n(x)dx$

Orthogonality of eigenfunctions $(y_m, y_n) = \int_0^L y_m(x)\ y_n(x)dx = 0 \quad \text{if} \quad m \ne n$

Chapter 10

EXERCISES



Eigenvalue Problem $y'' + \mu^2 y = 0\,, \quad x \in (0, L)$ subject to homogeneous boundary conditions.

There exist infinitely many real values $0 \le \mu_1 < \mu_2 < \mu_3 < \ldots$ (***eigenvalues***)

and corresponding non-zero solutions $y_1,\ y_2,\ y_3,\ \ldots$ (***eigenfunctions***)

which satisfy the boundary conditions.

Boundary cs.	Eigenvalues, eigenfunctions	Fourier series
I $y(0) = 0$ **I** $y(L) = 0$		
II $y'(0) = 0$ **II** $y'(L) = 0$		
I $y(0) = 0$ **II** $y'(L) = 0$		
II $y'(0) = 0$ **I** $y(L) = 0$		

The Generalized Fourier series $f(x) = \sum_{n}^{\infty} c_n y_n(x)$ $\quad c_n = \frac{(f, y_n)}{(y_n, y_n)} = \frac{2}{L} \int_0^L f(x)\ y_n(x)\, dx$

Orthogonality $(y_m, y_n) = \int_0^L y_m(x)\ y_n(x)\, dx = 0 \quad if \quad m \ne n$

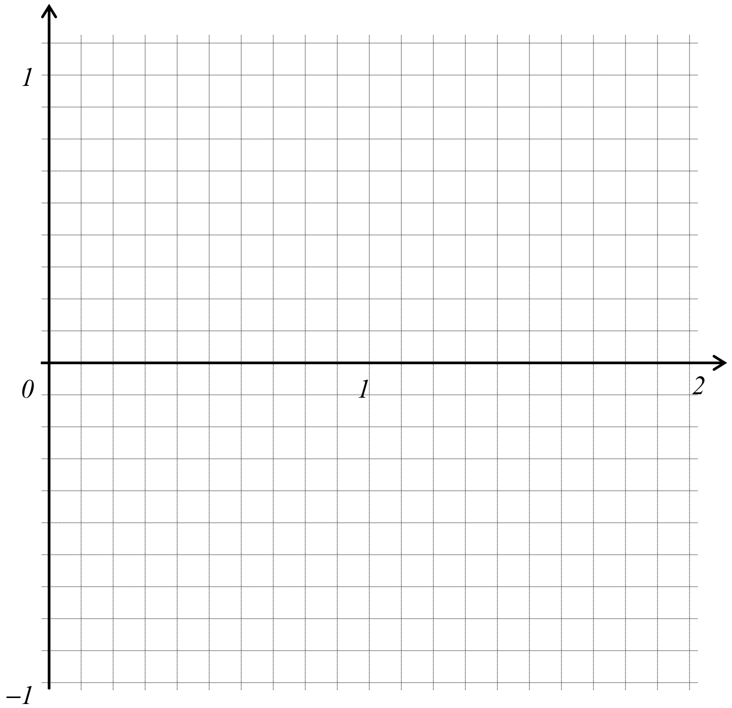
1) The Heat Equation – homogeneous boundary conditions:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t},\ 0 < x < 2,\ t > 0$$

$$u(0,t) = 0,\ u(2,t) = 0,\ t > 0$$

$$u(x,0) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

and sketch the graph of the initial temperature distribution and qualitatively sketch the graph of the temperature profiles for different moments of time.



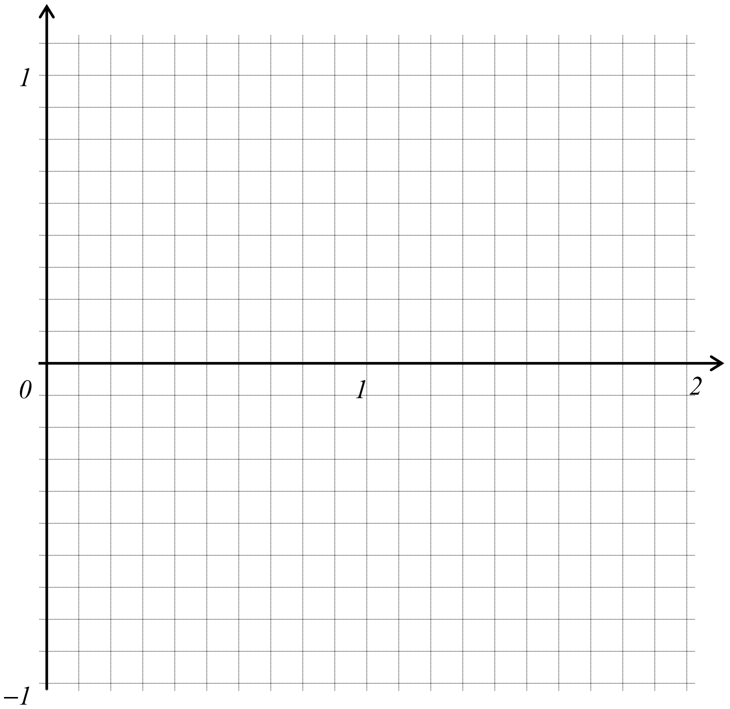
2) The Heat Equation – non-homogeneous boundary conditions:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}, \ 0 < x < 2, \ t > 0$$

$$u(0,t) = 0, \ u(2,t) = 1, \ t > 0$$

$$u(x,0) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

and sketch the graph of the initial temperature distribution and qualitatively sketch the graph of the temperature profiles for different moments of time.



3) The Wave Equation – homogeneous boundary conditions (string with the fixed ends):

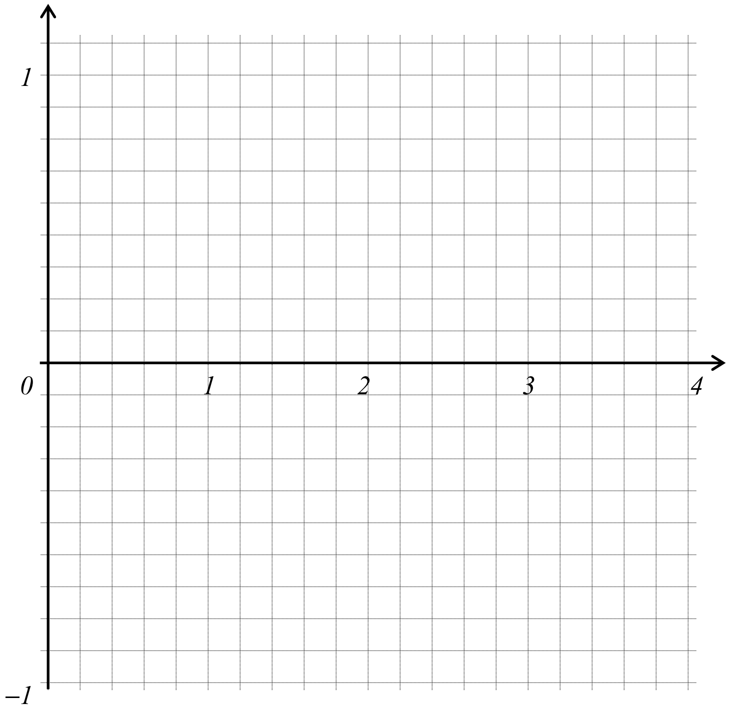
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial^2 u}{\partial t^2},\ 0 < x < 4,\ t > 0$$

$$u(0,t) = 0,\ u(4,t) = 0,\ t > 0$$

$$u(x,0) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 < x < 2 \\ 0 & 2 < x < 4 \end{cases}$$

$$\frac{\partial}{\partial t}u(x,0) = 0$$

and sketch the graph of the initial string shape and qualitatively sketch the graph of the string deflection for different moments of time.



4) The Wave Equation – homogeneous boundary conditions (string with the fixed ends) – standing waves

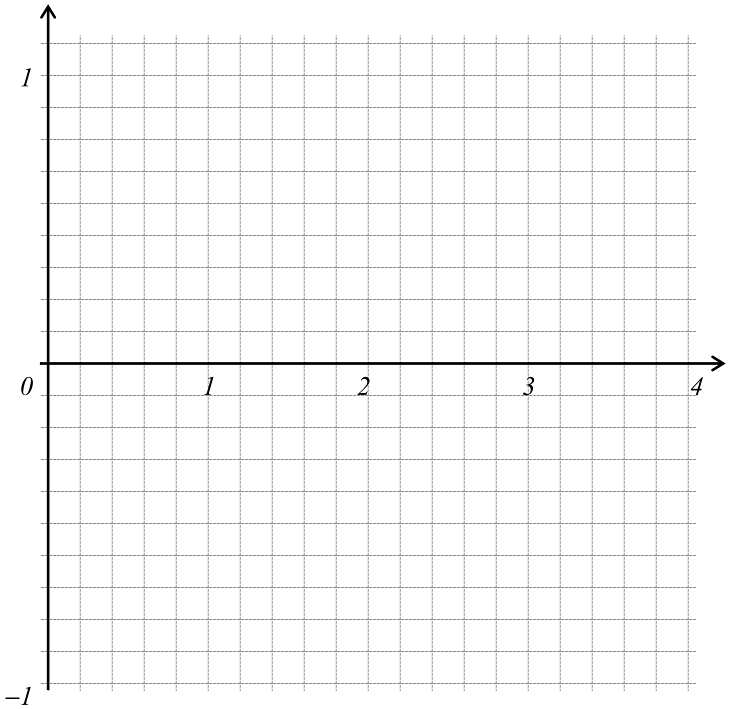
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial^2 u}{\partial t^2},\ 0 < x < 4,\ t > 0$$

$$u(0,t) = 0,\ u(4,t) = 0,\ t > 0$$

$$u(x,0) = sin\frac{\pi}{4}x$$

$$\frac{\partial}{\partial t}u(x,0) = 0$$

and sketch the graph of the initial string shape and qualitatively sketch the graph of the string deflection for different moments of time.



5) The Laplace Equation – 3 homogeneous boundary conditions, 1 non-homogeneous

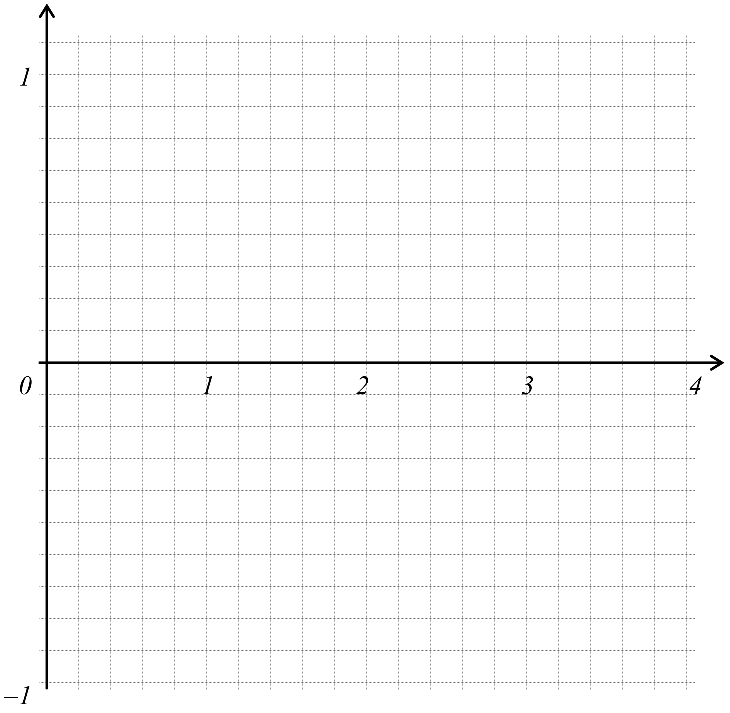
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\ 0<x<4,\ 0<y<1$$

$$u(0,y)=0, \qquad 0<y<1$$
$$u(4,y)=0, \qquad 0<y<1$$

$$u(x,0)=0, \qquad 0<x<4$$
$$u(x,1)=f(x), \quad 0<x<4$$

$$f(x)=\begin{cases}1 & 0<x<1\\ 0 & 1<x<4\end{cases}$$

and qualitatively sketch the graph of the surface defined by $u(x,y)$



6) The Laplace Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\ 0<x<4,\ 0<y<1$$

$$u(0,y)=0,\ 0<y<1$$
$$u(4,y)=sin(\pi x),\ 0<y<1$$

$$u(x,0)=0,\ 0<x<4$$
$$u(x,1)=0,\ 0<x<4$$

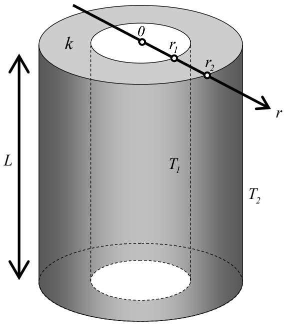
and qualitatively sketch the graph of the surface defined by $u(x,y)$

Write the formal solution of the given BVP as a superposition of the basic BVP's.

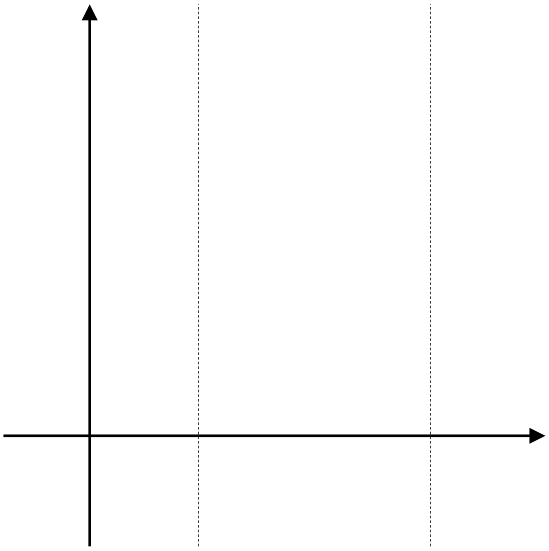
10.1. Boundary value problems

Consider steady state conduction in the cylindrical region between two isothermal surfaces $r = r_1$ at temperature T_1 and $r = r_2$ at temperature T_2. Radial temperature distribution under assumption of angular symmetry in the absence of heat generation is described by differential equation

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = 0 \qquad \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$



Find the temperature distribution $T(r)$, $r_1 < r < r_2$.



10.1. Boundary value problems

Consider steady state conduction in the spherical region between two isothermal surfaces $r = r_1$ at temperature T_1 and $r = r_2$ at temperature T_2. Radial temperature distribution under assumption of angular symmetry in the absence of heat generation is described by differential equation

$$r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0 \qquad\qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

Find the temperature distribution $T(r)$, $r_1 < r < r_2$.