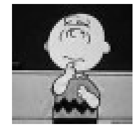
2.9 Reduction of Order (p.134)



1. Reduction of the 2nd order ODE $y'' = f(y', y, t)$ to the 1st order ODE:

a) No y

Given equation:	$y'' = f(y', t)$	
Change of variable:	$y' = v(t)$ $y'' = v'(t)$	$y' = v$ $y'' = v'$
Reduce to 1st order:	$v' = f(v, t)$	
Solve for	$v(t)$	(*solution includes* c_1)
Solution:	$y = \int v(t)dt + c_2$	

b) No t

Given equation:	$y'' = f(y', y)$	
Change of variable:	$\frac{dy}{dt} = v(y)$ $y'' = \frac{d}{dt}v(y) = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy} = vv'$	$y' = v$ $y'' = vv'$
Reduce to 1st order:	$vv' = f(v, y)$	(y *is now treated as an independet variable*)
Solve for	$v(y)$	(*solution includes* c_1)
Then	$\frac{dy}{dt} = v(y)$	
Separate variables:	$\frac{dy}{v(y)} = dt$	
Integrate:	$\int \frac{dy}{v(y)} = t + c_2$	

<u>*Examples:*</u> $y'' + t(y')^2 = 0$ (2.9: 38) $y'' + y(y')^3 = 0$ (2.9: 44)

2. Reduction formula for the 2nd order ***linear*** ODE (p.171), when one solution is known:

Let $y_1(t)$ be a solution of $y'' + p(t)y' + q(t)y = 0$,

then the second solution can be found by

$$y_2 = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2} dt$$

Ch3 Higher Order Linear Differential Equations

Linear ODE's: $Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = g(t)$ *standard form* (4.2)

Initial Conditions:

$$y(0) = y_0$$
$$y'(0) = y_1$$
$$\vdots$$
$$y^{(n-1)}(0) = y_{n-1} \qquad (4.3)$$

$Ly = 0$ Homogeneous equation

$Ly = g$ Non-homogeneous equation

Equations with constant coefficients

Homogeneous linear ODE $a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$

Trial form of solution $y = e^{rt}$

Characteristic equation $a_0 r^n + a_1 r^{n-1} + \ldots + a_n r = 0 \quad \Rightarrow \quad (r-r_1)(r-r_2)\cdots(r-r_n) = 0\,, \quad r_k \in \mathbb{C}$

Wronskian Let $Ly_1 = 0,\ \ldots\ ,\ Ly_n = 0$, then the Wronskian of $y_1(t),\ \ldots\ ,\ y_n(t)$ is defines as

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \qquad 2^{nd} \text{ order}$$

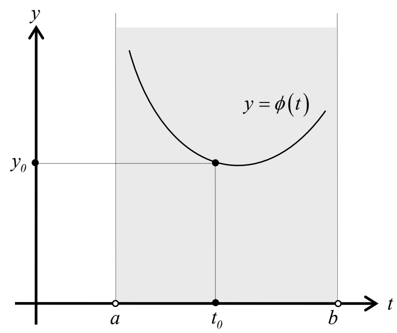
$$W(y_1,\ldots,y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \qquad n^{th} \text{ order}$$

$$W(y_1, y_2, \ldots, y_n)(t) = ce^{-\int p_1(t)dt}$$ **Abel's Formula**

3.2 2[nd] Order Linear Equations

$$Ly \equiv y'' + p(t)y' + q(t)y = g(t) \qquad (3)$$
$$y(0) = y_0\,,\ y'(0) = y_1$$

Theorem 3.2.1 and 4.1.1 (Existence and Uniqueness of the solution of the IVP for linear ODE)



Let $t_0 \in (a,b)$ and

Let $p_k(t), g(t) \in C(a,b)$ (continuous functions)

Then the linear differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = g(t)$$

has a unique solution $y = \phi(t)$, $t \in (a,b)$

such that $y(t_0) = y_0\,, y'(t_0) = y_1, \ldots,\ y^{(n-1)}(t_0) = y_{n-1}$

Theorem 3.2.2 (Principle of Superposition for homogeneous equation)

Let $Ly_1 = 0$ and $Ly_2 = 0$, then $L(c_1y_1 + c_2y_2) = 0$ for any $c_1, c_2 \in \mathbb{R}$.

Theorem 3.2.3 (Solution of initial value problem for homogeneous equation)

Let $Ly_1 = 0$ and $Ly_2 = 0$, then there exist constants $c_1, c_2 \in \mathbb{R}$ such that $y = c_1y_1 + c_2y_2$ satisfies the given initial conditions $y(t_0) = y_0$, $y'(t_0) = y_1$

if and only if $W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$.

Theorem 3.2.4 (General solution of homogeneous equation – a fundamental set of solutions)

Let $Ly_1 = 0$ and $Ly_2 = 0$, then $y = c_1y_1 + c_2y_2\,,\ c_1, c_2 \in \mathbb{R}$
is a ***general solution*** *(all solutions)* of $Ly = 0$,

if and only if $W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$ at some point t_0.

The set $\{y_1, y_2\}$ is called a ***fundamental set***, if $W(y_1,y_2) \neq 0$.

Theorem 3.2.5 (Existence of a fundamental set of solutions)

Let $t_0 \in (a,b)$ and let $p(t), q(t) \in C(a,b)$ (continuous functions).

Let $y_1(t)$ be a solution of the IVP: $Ly = 0$, $y(t_0) = 1$, $y'(t_0) = 0$.

Let $y_2(t)$ be a solution of the IVP: $Ly = 0$, $y(t_0) = 0$, $y'(t_0) = 1$.

Then $\{y_1(t), y_2(t)\}$ is a ***fundamental set*** of $Ly = 0$.

Theorem 3.2.6

$$W(y_1,y_2)(t) = ce^{-\int p(t)dt}$$

Abel's Formula

FUNDAMENTAL SETS of the 2nd Order Linear ODEs with Constant Coefficients

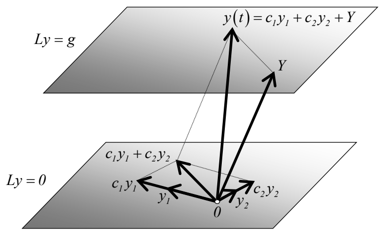
Homogeneous equation $Ly \equiv a_0 y'' + a_1 y' + a_2 y = 0\,,$ $a_0, a_1, a_2 = const \in \mathbb{R}$ (3)

Characteristic equation $a_0 r^2 + a_1 r + a_2 = 0$ $r_{1,2} = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$

Roots of characteristic equation	**Fundamental Solutions** $W\left[y_1(t), y_2(t)\right] \neq 0$			
	$y_1(t)$	$y_2(t)$	$y_1(t-t_0)$	$y_2(t-t_0)$
I $m_1 \neq m_2 \in \mathbb{R}$ distinct $(r-m_1)(r-m_2) = 0$	$e^{m_1 t}$	$e^{m_2 t}$	$e^{m_1(t-t_0)}$	$e^{m_2(t-t_0)}$
case of symmetric roots $(a_1 = 0)$ $r^2 - m^2 = 0$	e^{-mt}	e^{mt}	$e^{-m(t-t_0)}$	$e^{m(t-t_0)}$
$r = \pm m$	$cosh(mt)$	$sinh(mt)$	$cosh\left[m(t-t_0)\right]$	$sinh\left[m(t-t_0)\right]$
II $m_1 = m_2 = m \in \mathbb{R}$ repeated $(r-m)^2 = 0$	e^{mt}	te^{mt}	$e^{m_1(t-t_0)}$	$(t-t_0)e^{m_2(t-t_0)}$
III $a,b \in \mathbb{R}$, $m_{1,2} = a \pm ib$ complex $(r-m_1)(r-m_2) = 0$	$e^{at}\cos bt$	$e^{at}\sin bt$	$e^{a(t-t_0)}\cos b(t-t_0)$	$e^{a(t-t_0)}\sin b(t-t_0)$

Non-homogeneous equation: $Ly \equiv \; y'' + p(t)y' + q(t)y \; = \; g(t)$

Theorem 3.5.2 (General solution of non-homogeneous equation)



Let $\{y_1(t), y_2(t)\}$ be a ***fundamental set*** of homogeneous equation $Ly = 0$, and let $Y(t)$ be a particular solution of non-homogeneous equation $Ly = g(t)$, then

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

is the ***general solution*** of $Ly = g(t)$.

3.5, 4.3 Method of Undetermined Coefficients for $LY = g(t)$. Let $\{y_1, y_2\}$ be a fundamental set of $LY = 0$.

$g(t)$	Trial form for Y	If Y repeats y_1 *or* y_2, then try $t \cdot Y$
5	A	tA
$3t$	$At + B$	$t(At + B)$
$2t^2 - t$	$At^2 + Bt + C$	$t(At^2 + Bt + C)$
$2\sin 5t$	$A\sin 5t + B\cos 5t$	$t(A\sin 5t + B\cos 5t)$
$t\sin 2t$	$(A_1t + A_0)\sin 2t + (B_1t + B_0)\cos 2t$	$t(A_1t + A_0)\sin 2t + t(B_1t + B_0)\cos 2t$
$3e^{-2t}$	Ae^{-2t}	tAe^{-2t}
$e^{3t}\sin 2t$	$e^{3t}(A\sin 2t + B\cos 2t)$	$te^{3t}(A\sin 2t + B\cos 2t)$
te^{2t}	$(At + B)e^{2t}$	$t(At + B)e^{2t}$

If the function $g(t)$ is a combination of the exponential, trigonometric, and polynomial functions in the form:

$$g(t) \; = \; e^{at}\left[\left(p_nt^n + p_{n-1}t^{n-1} + \ldots p_1t + p_0\right)\cos bt + \left(q_mt^m + q_{m-1}t^{m-1} + \ldots q_1t + q_0\right)\sin bt\right] \quad \text{and}$$

1) $a \pm ib$ **is not** a root of characteristic equation $a_0r^2 + a_1r + a_2 \; = \; 0$, then look for particular solution in the form:

$$Y(t) \; = \; e^{at}\left[\left(A_kt^k + A_{k-1}t^{k-1} + \ldots + A_1t + A_0\right)\cos bt + \left(B_kt^k + B_{k-1}t^{k-1} + \ldots + B_1t + B_0\right)\sin bt\right], \qquad \text{where } k = \max\{n, m\}$$

the coefficients in which have to be found by substitution into $Ly = g$.

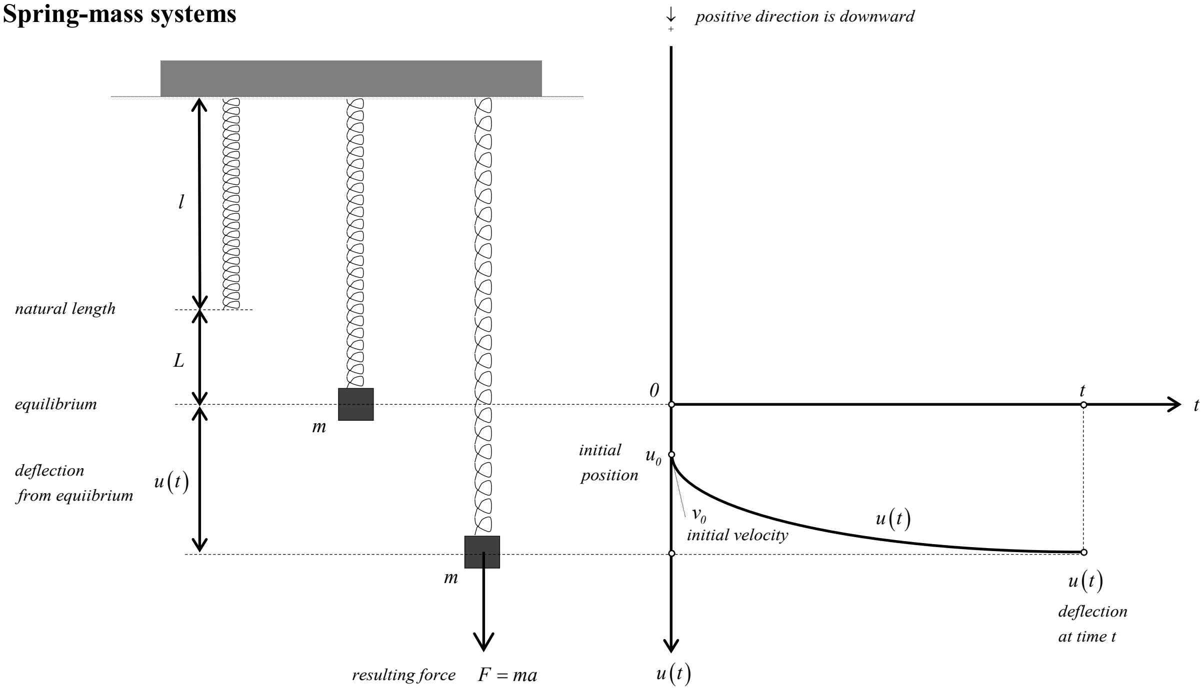
2) $a \pm ib$ **is** a root of characteristic equation of multiplicity *s*, then look for particular solution in the form:

$$t^s \cdot Y(t)$$

3.6 Variation of Parameter Particular solution of $y'' + p(t)y' + q(t)y \; = \; g(t)$

$Y = u_1y_1 + u_2y_2$	$u_1 \; = \; -\int \frac{y_2}{W(y_1, y_2)} g(t)\,dt$	$u_2 \; = \; \int \frac{y_1}{W(y_1, y_2)} g(t)\,dt$

3.7-8 Spring-mass systems



Force balance	resulting force F = spring force F_s + damping force F_d + gravitational force (weight) w + external force $F(t)$ $mu'' = -k(u+L) - \gamma u' + mg + F(t)$	Equilibrium relationship weight is balanced by the restoring force, $w=-F_s$ $mg = kL$
Equation of motion	$mu'' + \gamma u' + ku = F(t)$	$u(0)=u_0$ initial deflection $u'(0)=v_0$ initial velocity

	BRITISH	**METRIC**	**CONVERSION**
weight (Newton's Law)	$w[lb_f]=m[lb_m]\cdot 32\left[\frac{ft}{s^2}\right]$	$w[N]=m[kg]\cdot 9.8\left[\frac{m}{s^2}\right]$	$1[m]=100[cm]$
damping force	$F_d[lb_f]=-\gamma\cdot u'$	$F_d[N]=-\gamma\cdot u'$	$1[in]=\frac{1}{12}[ft]$
spring force (Hook's Law)	$F_s[lb_f]=-k\cdot u$	$F_s[N]=-k\cdot u$	$1[in]=2.54[cm]$
spring constant	$k\left[\frac{lb_f}{ft}\right]$	$k\left[\frac{N}{m}\right]=\left[\frac{kg}{s^2}\right]$	$1[ft]=0.3048[m]$
damping constant	$\gamma\left[\frac{lb_f\cdot s}{ft}\right]$	$\gamma\left[\frac{N\cdot s}{m}\right]=\left[\frac{kg}{s}\right]$	$1[slug]=32[lb_m]$
velocity	$u'\left[\frac{ft}{s}\right]$	$u'\left[\frac{m}{s}\right]$	$1[lb_m]=0.45[kg]$

Comment: *in the textbook (p.195), $[lb]$ is used as a unit of force; $[lb\cdot s^2/ft]$ is used as a unit of mass*
in engineering textbooks: unit of force is $[lbf]$; unit of mass is $[lbm]$

Nomenclature

$u(t)$	deflection of the weight from equilibrium position of the mass-spring system
$\omega_0 = \sqrt{\frac{k}{m}}$	natural frequency
$T = \frac{2\pi}{\omega_0}$	period of undamped unforced vibrations
$\mu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$	quasi-frequency of damped unforced vibrations (for $\gamma < 2\sqrt{km}$)
$T_d = \frac{2\pi}{\mu}$	quasi-period of damped unforced vibrations
$u(t) = R\cos(\omega_0 t - \delta)$	combined modes of vibrations, δ = *phase angle*, δ/ω_0 = *time lag*
$\gamma = 2\sqrt{km}$	critical damping coefficient
$F(t) = F_0 \cos \omega t$	periodic external force
$F(t) = F_0 \sin \omega t$	
ω	frequency of the periodic external force
F_0	amplitude of the periodic external force

Example 1 (p.194) How to interpret the conditions of the problem:

A mass weighing 4 lb	$\Rightarrow$	$4\left[lb_f\right] = m\left[lb_m\right] \cdot 32\left[\frac{ft}{s^2}\right]$	$\Rightarrow$	$m = \frac{1}{8}\ \left[lb_m\right]$
stretches a spring 2 in	$\Rightarrow$	$4\left[lb_f\right] = k \cdot 2[in] \cdot \frac{1}{12}\left[\frac{ft}{in}\right]$	$\Rightarrow$	$k = 24\ \left[\frac{lb}{ft}\right]$
The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s.	$\Rightarrow$	$-6\left[lb_f\right] = -\gamma \cdot 3\left[\frac{ft}{s}\right]$	$\Rightarrow$	$\gamma = 2\ \left[\frac{lb_f \cdot s}{ft}\right]$
Suppose that the mass is displaced an additional 6in in the positive direction	$\Rightarrow$	$u(0) = 6[in] \cdot \frac{1}{12}\left[\frac{ft}{in}\right]$	$\Rightarrow$	$u(0) = \frac{1}{2}\ [ft]$
and then released			$\Rightarrow$	$u'(0) = 0\ \left[\frac{ft}{s}\right]$
Then equation of motion is		$mu'' + \gamma u' + ku = 0$		
		$\frac{1}{8}u'' + 2u' + 24u = 0$		
Initial Value Problem		$u'' + 16u' + 192u = 0$,	$u(0) = \frac{1}{2}$,	$u'(0) = 0$

I Unforced Vibrations $F(t)=0$

1) Undamped ($\gamma=0$)

$$mu''+ku=0$$

$$u''+\frac{k}{m}u=0$$

$$u''+\omega_0^2u=0$$

$$r^2+\omega_0^2=0$$

$$r_{1,2}=\pm\omega_0 i$$

Natural frequency $\omega_0=\sqrt{\dfrac{k}{m}}$

General solution: $u(t)=A\cos(\omega_0 t)+B\sin(\omega_0 t)$

Period $T=\dfrac{2\pi}{\omega_0}$

Solution of IVP: $u(0)=u_0\,,\quad u'(0)=v_0$

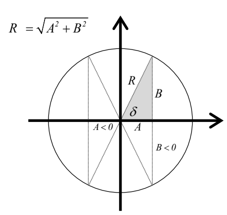
$$u_0=A\cdot 1+B\cdot 0 \qquad \boxed{A=u_0}$$

$$u'(t)=-\omega_0A\sin(\omega_0 t)+\omega_0B\cos(\omega_0 t)$$

$$v_0=-\omega_0A\cdot 0+\omega_0B\cdot 1 \qquad \boxed{B=\frac{v_0}{\omega_0}}$$

$$u(t)=A\cos(\omega_0 t)+B\sin(\omega_0 t)=u_0\cos(\omega_0 t)+\frac{v_0}{\omega_0}\sin(\omega_0 t)$$

Combining the modes: $u(t)=R\cos(\omega_0 t-\delta)$ $\quad R^2=A^2+B^2\,,\quad \delta$ = *phase angle*



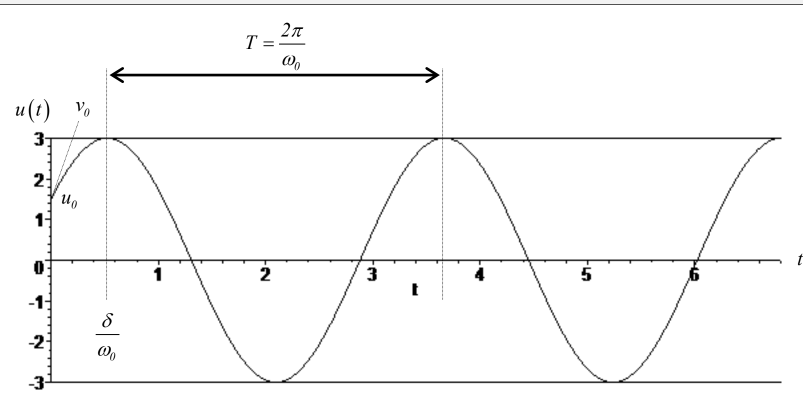
$$\tan\delta=\frac{B}{A}=\frac{v_0}{\omega_0u_0}$$

$$\sin\delta=\frac{B}{R}$$

$$\cos\delta=\frac{A}{R}$$

if $A>0,B>0$	then	$\delta=\tan^{-1}\dfrac{B}{A}$
if $A<0$	then	$\delta=\tan^{-1}\dfrac{B}{A}+\pi$
if $A>0,B<0$	then	$\delta=\tan^{-1}\dfrac{B}{A}+2\pi$

Graphing solution $u(t)=R\cos\left[\omega_0\left(t-\dfrac{\delta}{\omega_0}\right)\right]$, $\quad \dfrac{\delta}{\omega_0}$ = *time lag* , $\quad R$ = *amplitude*



2) Damped ($\gamma > 0$)

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + \gamma r + k = 0$$

$$r_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}$$

a) ***critical damping***

$$\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m} = 0 \qquad \gamma = 2\sqrt{km}$$

$$r_{1,2} = -\frac{\gamma}{2m}$$

$$u(t) = Ae^{-\left(\frac{\gamma}{2m}\right)t} + Bte^{-\left(\frac{\gamma}{2m}\right)t}$$

$$u'(t) = -\left(\frac{\gamma}{2m}\right)Ae^{-\left(\frac{\gamma}{2m}\right)t} + Be^{-\left(\frac{\gamma}{2m}\right)t} - \left(\frac{\gamma}{2m}\right)Bte^{-\left(\frac{\gamma}{2m}\right)t}$$

$$u(0) = u_0 = A \cdot 1 + B \cdot 0$$

$$u'(t) = v_0 = -\left(\frac{\gamma}{2m}\right)A \cdot 1 + B \cdot 1 - 0$$

$$A = u_0$$

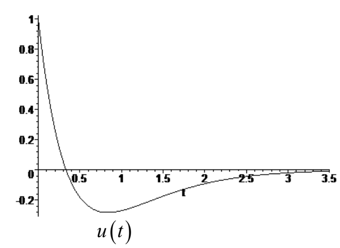
$$B = v_0 + \left(\frac{\gamma}{2m}\right)u_0$$

$$u(t) = \left[u_0 + \left(v_0 + \frac{\gamma}{2m}u_0\right)t\right]e^{-\left(\frac{\gamma}{2m}\right)t}$$

Example: $u'' + 4u' + 4u = 0$

$u(0) = 1,\ u'(0) = -5$

Solution: $u(t) = (1 - 3t)e^{-2t}$



b) ***overdamping***

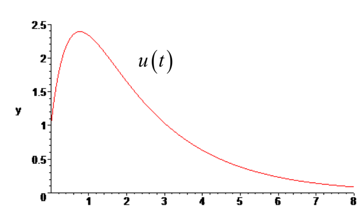
$$\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m} > 0 \qquad \gamma > 2\sqrt{km}$$

Roots: $r_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}} < 0$ $\qquad r_1 < 0,\ r_2 < 0$ *always*

Solution: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$

$$A = \frac{v_0 - r_2 u_0}{r_1 - r_2}$$

$$B = \frac{r_1 u_0 - v_0}{r_1 - r_2}$$



c) **damped vibrations**

$$\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m} < 0 \qquad \gamma < 2\sqrt{km}$$

quasi frequency $\mu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$

quasi period $T_d = \frac{2\pi}{\mu}$

$$r_{1,2} = -\frac{\gamma}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$$

$$u(t) = Ae^{-\left(\frac{\gamma}{2m}\right)t} \cos \mu t + Be^{-\left(\frac{\gamma}{2m}\right)t} \sin \mu t$$

$$u'(t) = \left[Ae^{-\left(\frac{\gamma}{2m}\right)t} \cos \mu t + Be^{-\left(\frac{\gamma}{2m}\right)t} \sin \mu t \right]'$$

$$u(0) = u_0 = A \cdot 1 + B \cdot 0 \qquad A = u_0$$

$$u'(t) = -\left(\frac{\gamma}{2m}\right) Ae^{-\left(\frac{\gamma}{2m}\right)t} \cos \mu t - \mu Ae^{-\left(\frac{\gamma}{2m}\right)t} \sin \mu t - \left(\frac{\gamma}{2m}\right) Be^{-\left(\frac{\gamma}{2m}\right)t} \sin \mu t + \mu Be^{-\left(\frac{\gamma}{2m}\right)t} \cos \mu t$$

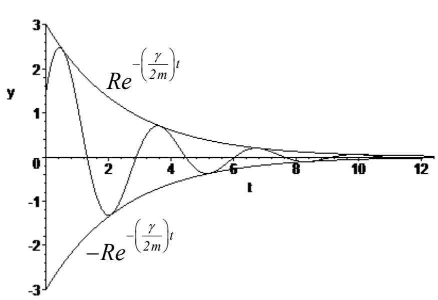
$$v_0 = -\left(\frac{\gamma}{2m}\right) A + \mu B \qquad B = \frac{v_0 + \left(\frac{\gamma}{2m}\right) u_0}{\mu}$$

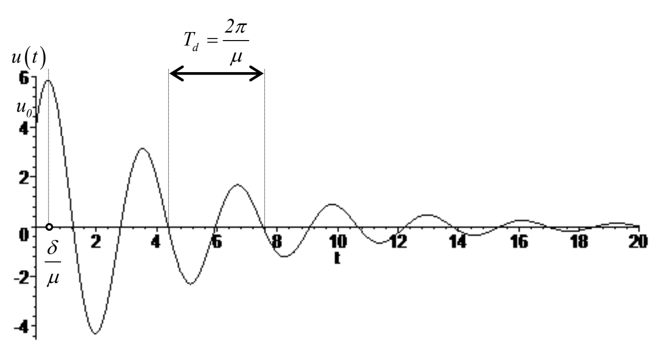
$$u(t) = e^{-\left(\frac{\gamma}{2m}\right)t} \left[A \cos \mu t + B \sin \mu t \right]$$

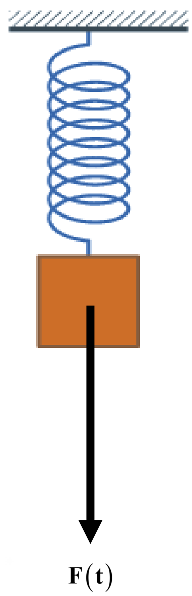
$$u(t) = Re^{-\left(\frac{\gamma}{2m}\right)t} \cos(\mu t - \delta) \qquad R^2 = A^2 + B^2, \quad \tan \delta = \frac{B}{A}$$

Exercise: A mass 1 kg is attached to a spring with spring constant k=8 kg/s^2.
The mass is in a medium that exerts a viscous resistance 4 N when the mass has a velocity 1 m/s. No external force is applied.
Assume that mass is pulled down by 4 m below its equilibrium and released with initial downward velocity 2 m/s.
Find the equation which describes the motion of the spring.

Answer: $u(t) = (4\cos 2t + 5\sin 2t)e^{-2t}$






F(t)

II Forced Vibrations $F(t) \neq 0$

$F(t) = a\cos\omega t + b\sin\omega t$ periodic external force

$$mu'' + \gamma u' + ku = F(t)$$

General solution: $u(t) = u_c(t) + U(t)$

initial conditions: $u(0) = u_0, \quad u'(0) = v_0$

2) Damped ($\gamma > 0$)

$$mu'' + \gamma u' + ku = F_0\cos(\omega t)$$

$u(0) = u_0, \quad u'(0) = v_0$

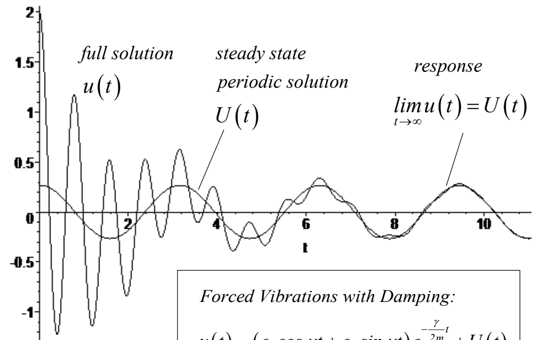
$$r_{1,2} = -\frac{\gamma}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$$

$$\mu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2} = \omega_0\sqrt{1 - \frac{\gamma^2}{4mk}}, \qquad \omega_0^2 = \frac{k}{m}$$

General solution: $u(t) = u_c(t) + U(t)$

$\lim_{t\to\infty} u_c(t) = 0$ (*transient solution*)

$\lim_{t\to\infty} u(t) = U(t)$ (*steady state solution*)



Forced Vibrations with Damping:

$$u(t) = (c_1\cos\mu t + c_2\sin\mu t)e^{-\frac{\gamma}{2m}t} + U(t)$$

Steady state solution $U(t) = A\cos\omega t + B\sin\omega t$ (use undetermined coefficients to find a particular solution)

$$A = \frac{mF_0\left(\omega_0^2 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}$$

$$B = \frac{F_0\omega\gamma}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2} \qquad B > 0 \text{ always}$$

$$U(t) = A\cos\omega t + B\sin\omega t = R\cos\left[\omega\left(t - \frac{\delta}{\omega}\right)\right]$$

$$\tan\delta = \frac{B}{A} = \frac{1}{\left(\omega_0^2 - \omega^2\right)}\frac{\omega\gamma}{m}$$

if $A>0$ *then* $\delta = \tan^{-1}\frac{B}{A}$

if $A<0$ *then* $\delta = \pi + \tan^{-1}\frac{B}{A}$

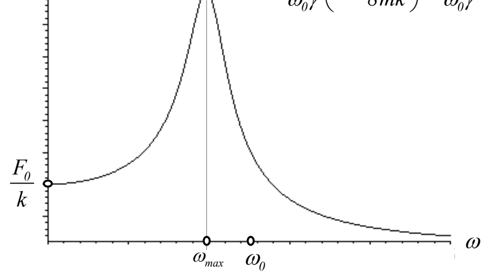
$$R = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}}$$

"Resonance"

Max amplitude of the steady state vibrations when the frequency of external force is

$$\omega_{max} = \underbrace{\omega_0\sqrt{1 - \frac{\gamma^2}{2mk}}}_{\mu}$$

$$R_{max} = \frac{F_0}{\omega_0\gamma}\left(1 + \frac{\gamma^2}{8mk}\right) \approx \frac{F_0}{\omega_0\gamma}$$



2) **Damped** ($\gamma > 0$)

$$mu'' + \gamma u' + ku = F_0 \sin(\omega t)$$

$u(0) = u_0, \quad u'(0) = v_0$

3.8 ## 6,8

Steady state solution $U(t) = A\cos\omega t + B\sin\omega t$ (use undetermined coefficients to find a particular solution)

$$A = \frac{-\omega\gamma F_0}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2} \qquad \omega_0^2 = \frac{k}{m} \qquad A < 0$$

$$B = \frac{mF_0\left(\omega_0^2 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}$$

$$U(t) = A\cos\omega t + B\sin\omega t = R\cos\left[\omega\left(t - \frac{\delta}{\omega}\right)\right]$$

$$\tan\delta = \frac{B}{A} = \frac{-m\left(\omega_0^2 - \omega^2\right)}{\omega\gamma} \qquad \text{since } A<0, \qquad \delta = \pi + \tan^{-1}\frac{B}{A}$$

$$R = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}}$$

General solution:

$$u(t) = u_c(t) + U(t)$$

$$u(t) = c_1 e^{-\left(\frac{\gamma}{2m}\right)t}\cos\mu t + c_2 e^{-\left(\frac{\gamma}{2m}\right)t}\sin\mu t + R\cos\left[\omega\left(t - \frac{\delta}{\omega}\right)\right]$$

Solution of IVP:

$$u_0 = c_1\cdot 1 + 0 + R\cos\delta = c_1\cdot 1 + 0 + R\frac{A}{R}u_0 = c_1 + A \qquad \Rightarrow c_1 = u_0 - A$$

$$u'(t) = -c_1\frac{\gamma}{2m}e^{-\left(\frac{\gamma}{2m}\right)t}\cos\omega_0 t - c_1\omega_0 e^{-\left(\frac{\gamma}{2m}\right)t}\sin\omega_0 t - c_2\frac{\gamma}{2m}e^{-\left(\frac{\gamma}{2m}\right)t}\sin\omega_0 t + c_2\omega_0 e^{-\left(\frac{\gamma}{2m}\right)t}\cos\omega_0 t - R\omega\sin\left[\omega\left(t - \frac{\delta}{\omega}\right)\right]$$

$$v_0 = -c_1\frac{\gamma}{2m} - 0 - 0 + c_2\omega_0 - R\omega\sin(-\delta)$$

$$v_0 = -u_0 + A\frac{\gamma}{2m} + c_2\omega_0 + R\omega\frac{B}{R} \qquad \Rightarrow c_2 = \frac{v_0 + u_0 - A\frac{\gamma}{2m} - \omega B}{\omega_0}$$

$$u(t) = (u_0 - A)e^{-\left(\frac{\gamma}{2m}\right)t}\cos\mu t + \left(\frac{v_0 + u_0 - A\frac{\gamma}{2m} - \omega B}{\omega_0}\right)e^{-\left(\frac{\gamma}{2m}\right)t}\sin\mu t + R\cos\left[\omega\left(t - \frac{\delta}{\omega}\right)\right]$$

1) **Undamped** ($\gamma = 0$) $u'' + \omega_0^2 u = \frac{F_0}{m}\cos(\omega t)$ $\omega_0^2 = \frac{k}{m}$ *natural frequency*

General solution: $u(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + U(t)$

a) $\omega \neq \omega_0$ $U(t) = A\cos\omega t + B\sin\omega t$ (use undetermined coefficients to find a particular solution)

$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$, $B = 0$, $\frac{B}{A} = 0 \quad \Rightarrow \quad \delta = 0$

$$U(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t$$

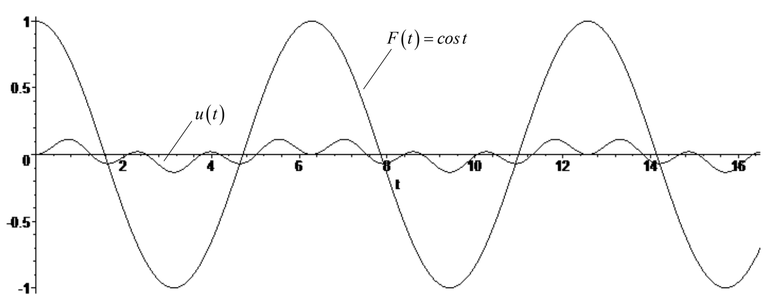
Solution of IVP:

$$u(t) = \left(u_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)}\right)\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t$$

Example: $u(0) = u_0 = 0$, $u'(0) = v_0 = 0$ (vibrations are driven only by external force)

$$\cos x - \cos y = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

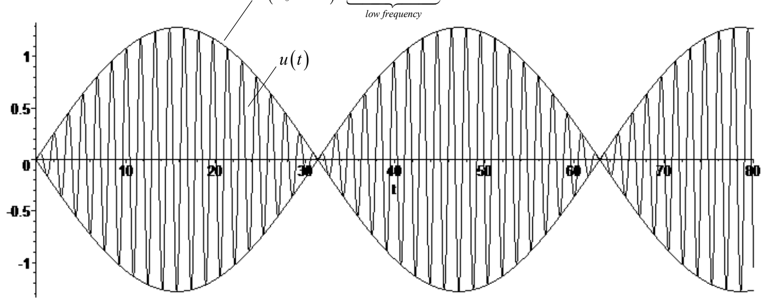
$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\cdot\sin\left(\frac{\omega_0 - \omega}{2}t\right)\cdot\sin\left(\frac{\omega_0 + \omega}{2}t\right)$$



If $|\omega - \omega_0| << |\omega + \omega_0|$ (***beat***)

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cdot\underbrace{\sin\left(\frac{\omega_0 - \omega}{2}t\right)}_{\text{low frequency}}\cdot\underbrace{\sin\left(\frac{\omega_0 + \omega}{2}t\right)}_{\text{high frequence}}$$

$\frac{F_0}{m(\omega_0^2 - \omega^2)}\cdot\underbrace{\sin\left(\frac{\omega_0 - \omega}{2}t\right)}_{\text{low frequency}}$ *amplitude modulation (**beat**)*



b) $\omega = \omega_0$ (***Resonance***) $\quad u'' + \omega_0^2 u = \frac{F_0}{m}\cos(\omega_0 t) \quad \omega_0^2 = \frac{k}{m} \quad u(0) = u_0, \quad u'(0) = v_0$

General solution:

$$u(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + U(t)$$

$U(t) = At\cos\omega_0 t + Bt\sin\omega_0 t$ (use undetermined coefficients to find a particular solution)

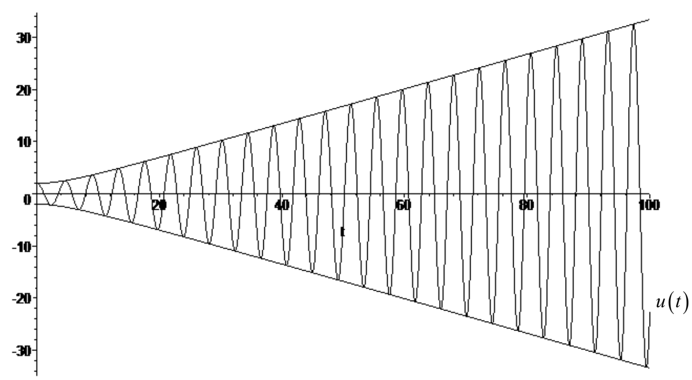
$A = 0$

$B = \frac{F_0}{2m\omega_0}$

$U(t) = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$

Solution of IVP:

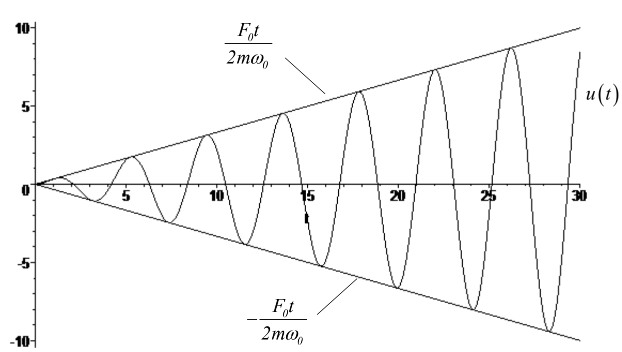
$$u(t) = u_0\cos\omega_0 t - \frac{v_0}{\omega_0}\sin\omega_0 t + \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$$



Example $\quad u(0) = u_0 = 0, \quad u'(0) = v_0 = 0$ (vibrations are driven only by external force)

$$u(t) = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$$

$$u'' + u = \frac{1}{2}\cos(t)$$

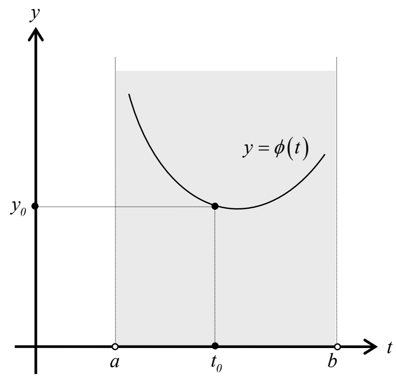


4.1 n^nt Order Linear Ordinary Differential Equations

$$Ly \equiv \quad y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{2}$$

$$y(0) = y_0, \; y'(0) = y_1, \ldots, \; y^{(n-1)}(t_0) = y_{n-1} \tag{3}$$

Theorem 3.2.1 and 4.1.1 (Existence and Uniqueness of the solution of the IVP for linear ODE)



Let $t_0 \in (a,b)$ and

let $p_k(t), g(t) \in C(a,b)$ (continuous).

Then the linear differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = g(t)$$

has a unique solution $y = \phi(t), \; t \in (a,b)$

such that $y(t_0) = y_0, y'(t_0) = y_1, \ldots, \; y^{(n-1)}(t_0) = y_{n-1}$

Homogeneous Equation

$$Ly \equiv \quad y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = 0 \tag{4}$$

Superposition Principle If the functions $y_1, y_2, \ldots, y_n$ are solutions of $Ly = 0$, then the linear combination

$$y(t) = c_1y_1 + c_2y_2 + \ldots + c_ny_n, \quad c_k \in \mathbb{R} \tag{5}$$

is also a solution of $Ly = 0$.

Wronskian

$$W(y_1, \ldots, y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \overset{\text{Abel's Formula}}{=} ce^{-\int p_1(t)dt} \tag{7}$$

Theorem 4.1.2 (General solution of homogeneous equation – a fundamental set of solutions)

Let $p_k(t), g(t) \in C(a,b)$ (continuous functions).

Let $y_1(t), y_2(t), \ldots, y_n(t)$ be solutions of homogeneous equation $Ly = 0$.

Then any solution of $Ly = 0$ can be expressed as a linear combination

$$y(t) = c_1y_1 + c_2y_2 + \ldots + c_ny_n, \quad c_k \in \mathbb{R}$$

if and only if $W(y_1, \ldots, y_n)(t) \neq 0$ at least at one point in (a,b).

Fundamental set The set $\{y_1(t), y_2(t), \ldots, y_n(t)\}$ is called a ***fundamental set*** of $Ly = 0$,

and $y(t) = c_1y_1 + c_2y_2 + \ldots + c_ny_n, \; c_k \in \mathbb{R}$ is called a ***general solution*** of $Ly = 0$.

Theorem 4.1.2' (Existence of a fundamental set of solutions)

Let $t_0 \in (a,b)$ and let $p_k(t), g(t) \in C(a,b)$ (continuous).

Let $y_1(t)$ be a solution of the IVP: $Ly = 0$, $y(t_0) = 1$, $y'(t_0) = 0$,…, $y^{(n-1)}(t_0) = 0$

Let $y_2(t)$ be a solution of the IVP: $Ly = 0$, $y(t_0) = 0$, $y'(t_0) = 1$,…, $y^{(n-1)}(t_0) = 0$

$\vdots$

Let $y_n(t)$ be a solution of the IVP: $Ly = 0$, $y(t_0) = 0$, $y'(t_0) = 0$,…, $y^{(n-1)}(t_0) = 1$

Then $\{y_1(t), y_2(t), ..., y_n(t)\}$ is a ***fundamental set*** of $Ly = 0$.

Linear Dependence ***The functions*** $y_1(t), y_2(t), ..., y_n(t)$ are said to be ***linearly dependent*** on (a,b) , if there exist a set of coefficients $c_1, c_2, ..., c_n \in \mathbb{R}$ not all zero, such that

$c_1 y_1(t) + c_2 y_2(t) + ... + c_n y_n(t) = 0$ for all $t \in (a,b)$.

Linear Independence ***The functions*** $y_1(t), y_2(t), ..., y_n(t)$ are said to be ***linearly independent*** on (a,b) , if they are not linearly dependent there.

The functions $y_1(t), y_2(t), ..., y_n(t)$ are said to be ***linearly independent*** on (a,b) , if

$c_1 y_1(t) + c_2 y_2(t) + ... + c_n y_n(t) = 0$ for all $t \in (a,b)$

only if all coefficients $c_1 = c_2 = ... = c_n = 0$.

Theorem 4.1.3 (Fundamental set of solutions)

The set $\{y_1(t), y_2(t), ..., y_n(t)\}$ is a ***fundamental set*** *of solutions* of $Ly = 0$ on (a,b) ,

if and only if $\{y_1(t), y_2(t), ..., y_n(t)\}$ is a ***linearly independent set*** *of solutions* on (a,b)

if $W(y_1, y_2, ..., y_n)(t_0) \neq 0$ *at some* t_0, *then* $y_1(t), y_2(t), ..., y_n(t)$ *are linearly independent*

if $W(y_1, y_2, ..., y_n)(t_0) = 0$ *at some* t_0, *then* $y_1(t), y_2(t), ..., y_n(t)$ *are linearly dependent*

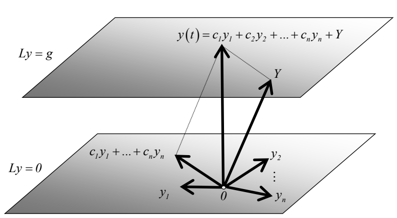
$f_1, ..., f_n$ *not solutions of Ly=0* *and* $W(f_1, f_2, ..., f_n)(t) = 0$ *at some* t_0, *however* $f_1(t), f_2(t), ..., f_n(t)$ *can be linearly independent*

NON-HOMOGENEOUS EQUATION

$$Ly \equiv \quad y^{(n)} + p_1(t) y^{(n-1)} + ... + p_{n-1}(t) y' + p_n(t) y \; = \; g(t) \qquad (2)$$

General Solution

Any solution (***general solution***) of non-homogeneous equation (2) can be written as a combination of the general solution of the corresponding homogeneous equation and some particular solution of non-homogeneous equation:



$$y(t) = c_1 y_1(t) + c_2 y_2(t) + ... + c_n y_n(t) + Y(t) \qquad (16)$$

4.2 **Fundamental sets** of a Linear ODE with constant coefficients

Homogeneous Linear ODE: $Ly = a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$

A polynoimial of degree n has n roots:

Characteristic equation: $a_0 r^n + a_1 r^{n-1} + \dots + a_n r = 0 \Rightarrow (r-r_1)(r-r_2)\cdots(r-r_n) = 0$

Fundamental Solutions		
	I $m_i \neq m_j \in \mathbb{R}$ $(r-m_1)(r-m_2)\cdots(r-m_n) = 0$	$e^{m_1 t},\ e^{m_2 t}, \dots,\ e^{m_n t}$
	II *repeated root of multiplicity s* $m_1 = \dots = m_s = m \in \mathbb{R}$ $(r-m)^s = 0$	$e^{mt},\ te^{mt}, \dots,\ t^{s-1}e^{mt}$
	III $z = a \pm ib\ ,\ a,b \in \mathbb{R}$ $(r-z) = 0$	$e^{at}\cos bt,\ e^{at}\sin bt$
	$(r-z)^s = 0$	$e^{at}\cos bt,\ e^{at}\sin bt$ $te^{at}\cos bt,\ te^{at}\sin bt$ $\vdots$ $t^{s-1}e^{at}\cos bt,\ t^{s-1}e^{at}\sin bt$

Complex Numbers

$$z = a + ib$$

$$z = re^{i\theta}$$

$$z = r\left[\cos\theta + i\sin\theta\right]$$

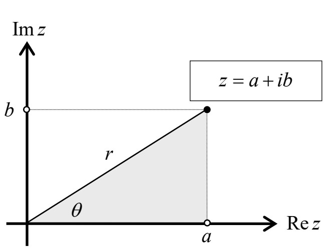
$$z = r\left[\cos(\theta + k\cdot 2\pi) + i\sin(\theta + k\cdot 2\pi)\right] = re^{i(\theta+2\pi k)}$$

$$a = r\cos\theta$$

$$b = r\sin\theta$$

$$\frac{b}{a} = \tan\theta$$

$$r^2 = a^2 + b^2$$

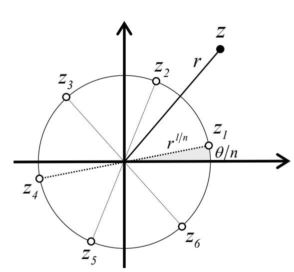


Roots of Complex Numbers

$$z_{k+1} = z^{\frac{1}{n}} \qquad k = 0,1,\dots,n-1$$

$$z_{k+1} = r^{\frac{1}{n}} e^{\frac{\theta + k\cdot 2\pi}{n}}$$

$$z_{k+1} = r^{1/n}\left[\cos\left(\frac{\theta + k\cdot 2\pi}{n}\right) + i\sin\left(\frac{\theta + k\cdot 2\pi}{n}\right)\right]$$



Non-Homogeneous Linear ODE $Ly \;=\; y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y \;=\; g(t)$

General Solution

$$y(t) \;=\; c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t) + Y(t) \qquad (16)$$

Fundamental Set $\{y_1, y_2, \ldots, y_n\}$ $\quad Ly_k(t) = 0 \quad W(y_1,\ldots,y_n)(t) \neq 0$

Particular solution of non-homogeneous eqn $Y(t)$ $\quad LY(t) = g(t)$

4.3 Undetermined Coefficients for $LY = g(t)$, where $g(t) \;=\; e^{at}\left[p_n(t)\cos bt + q_m(t)\sin bt\right]$, then the trial form is

$Y(t) \;=\; e^{at}\left[P_k(t)\cos bt + Q_k(t)\sin bt\right]$ if $a \pm ib$ **is not** a root

$Y(t) \;=\; t^s e^{at}\left[P_k(t)\cos bt + Q_k(t)\sin bt\right]$ if $a \pm ib$ **is** a root of multiplicity s

4.4 Variation of Parameters:

$$Y(t) \;=\; u_1 y_1 + u_2 y_2 + \ldots + u_n y_n$$

$$W(y_1,\ldots,y_n)(t) = \begin{vmatrix} y_1 & \cdots & y_k & \cdots & y_n \\ y_1' & \cdots & y_k' & \cdots & y_n' \\ \vdots & & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_k^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W_k(y_1,\ldots,y_n)(t) = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y_1' & \cdots & 0 & \cdots & y_n' \\ \vdots & & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & g(t) & \cdots & y_n^{(n-1)} \end{vmatrix}$$

k^{th} *column*

$$u_k' = \frac{W_k}{W} \quad \Rightarrow \quad u_k \;=\; \int \frac{W_k}{W}\, dt$$

Exercises: *2.9: 37, 44, 48*

Exercises *3.4: 28*

1) Find the general solution of the given differential equation $y'' - 4y' + 4y = \frac{e^{2t}}{t}$

2) Find the solution of the given initial value problem $y'' - y = e^{t}$ $\quad y(0)=1,\ y'(0)=0$

3) (a) $u''' - 5u'' + 3u' + u = 0$

   (b) $u''' + 5u'' + 6u' + 2u = 0$ $\qquad u(0)=0,\ u'(0)=1,\ u''(0)=-1$

4) $y^{(iv)} + 16y = 0$

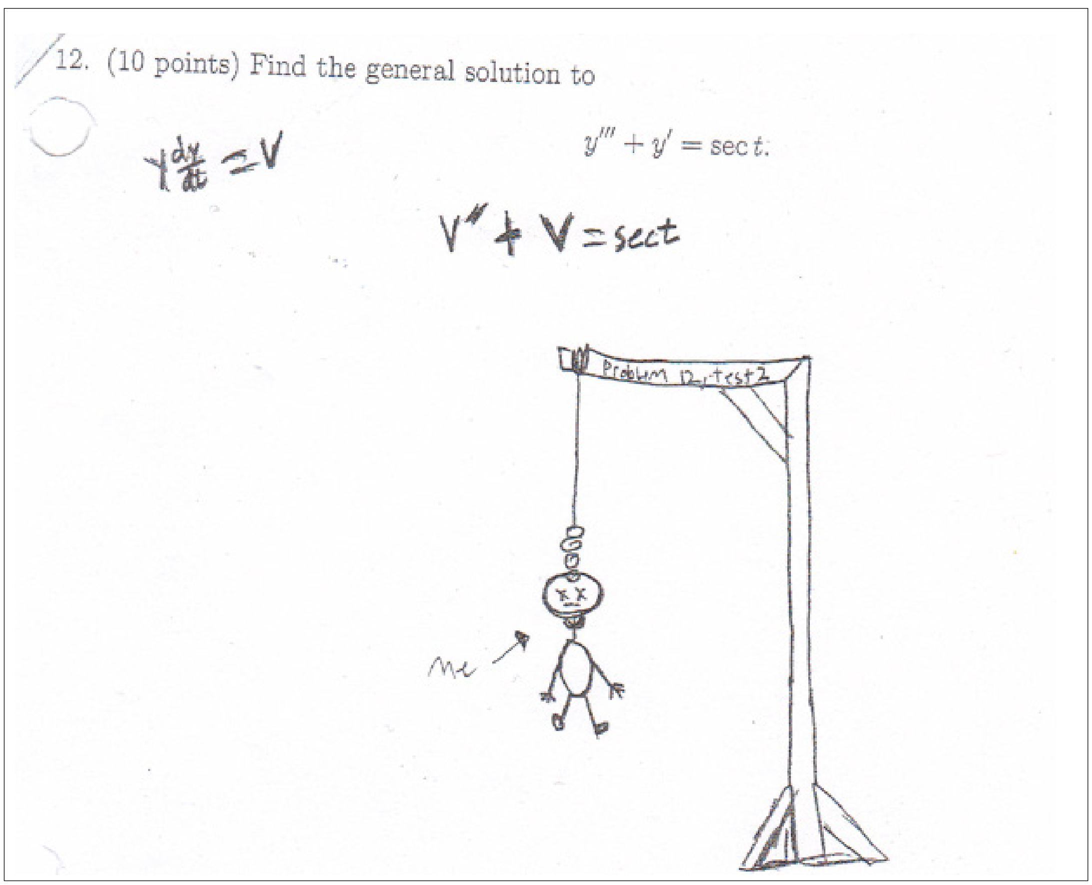
5) $u'' + 4u' + 4u = 0$ $\qquad u(0)=1,\ u'(0)=-5$ *Sketch the graph*

6) $u'' + 2u' + 2u = 4\cos(t) + 2\sin(t)$ $\qquad u(0)=0,\ u'(0)=0$

7) $u'' + u = 1/2\cos(t)$ $\qquad u(0)=0,\ u'(0)=0$

8) A mass 1 kg is attached to a spring with spring constant k=8 kg/s^2. The mass is in a medium that exerts a viscous resistance 4 N when the mass has a velocity 1 m/s. No external force is applied. Assume that mass is pulled down by 4 m below its equilibrium and released with initial downward velocity 2 m/s.
   Find the equation which describes the motion of the spring. *Sketch the graph.*

9) $y^{(vi)} - y''' = t$



Exercises

2.9 Reduction of order:

No y *Reduce* $y'' + t(y')^2 = 0$

No t *Reduce* $yy'' + (y')^2 = 0$

Linear 2nd order ODE, one solution y_1 is known, find the second solution:

$ty'' - 4ty' + 6y = 0$ $y_1 = t^2$

N^{th} Linear ODE

Normal form

Wronskian

Abel's Formula

Fundamental set

Fundamental set of the Linear ODE with constant coefficients:

General solution of homogeneous equation

General solution of non-homogeneous equation

Method of Undetermined coefficients

For what g(t) it can be used? *g(t)=*

Trial Form: *Y =*

Method of variation of parameter: *Y =*

Given W(1)=2 of the ODE $ty'' + 2y' + ty = 0$. What is W(5)=?

Find the general solution of $y''' + y' = \csc t$

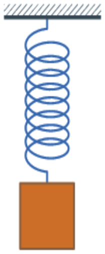
The roots of the complex number

$(z)^{\frac{1}{n}}$

Find the roots of (sketch):

$r^5 - 1 = 0$ $r^5 + 1 = 0$ $r^8 - 1 = 0$ $r^8 + 1 = 0$

Spring-mass system



Sketch the free-body diagram

Identify and describe the forces

Equilibrium relationship

Governing equation

Initial conditions

Unforced *undamped*

Damped *critical damping*

overdamped

Forced vibration *damped forced vibrations*

Undamped forced vibrations

Resonance

Undetermined Coefficients

If the function $g(t)$ is a combination of the exponential, trigonometric, and polynomial functions in the form:

$g(t) = e^{at}\left[p_n(t)\cos bt + q_m(t)\sin bt\right]$, where $p_n(t)$ *and* $q_m(t)$ are polynomials of order n *and* m, and

1) $a \pm ib$ **is not** a root of characteristic equation, then look for particular solution in the form:

$Y(t) = e^{at}\left[\left(A_k t^k + A_{k-1}t^{k-1} + \ldots + A_1 t + A_0\right)\cos bt + \left(B_k t^k + B_{k-1}t^{k-1} + \ldots + B_1 t + B_0\right)\sin bt\right]$, where $k = \max\{n,m\}$

the coefficients in which have to be found by substitution into $Ly = g$.

2) $a \pm ib$ **is** a root of characteristic equation of multiplicity *s*, then look for particular solution in the form:

$t^s \cdot Y(t)$

3.5 #17 $y'' - 2y' + y = te^t + 4$ $\qquad y(0) = 1,\ y'(0) = 1$

$r^2 - 2r + 1 = 0$ characteristic equation $(r-1)^2 = 0$, $r = 1$ is a root of multiplicity s=2

$y_c(t) = c_1 e^t + c_2 t e^t$ solution of homogeneous equation

I) $y'' - 2y' + y = te^t$ identify: $\alpha = 1, \beta = 0, n = 1, m = 0$ $\quad 1 \pm i \cdot 0$ is a root of multiplicity s=2

$Y = t^2(A + Bt)e^t = At^2e^t + Bt^3e^t$ trial form of solution

$$Y' = 2Ate^t + At^2e^t + 3Bt^2e^t + Bt^3e^t = 2Ate^t + (A+3B)t^2e^t + Bt^3e^t$$

$$Y'' = 2Ae^t + 2Ate^t + 2(A+3B)te^t + (A+3B)t^2e^t + 3Bt^2e^t + Bt^3e^t \qquad = 2Ae^t + (4A+6B)te^t + (A+6B)t^2e^t + Bt^3e^t$$

$$2Ae^t + (4A+6B)te^t + (A+6B)t^2e^t + Bt^3e^t - 2\left[2Ate^t + (A+3B)t^2e^t + Bt^3e^t\right] + At^2e^t + Bt^3e^t = te^t$$

$$2Ae^t + \underline{(4A+6B)te^t} + \underline{\underline{(A+6B)t^2e^t}} + Bt^3e^t - \underline{4Ate^t} + \underline{\underline{(-2A-6B)t^2e^t}} - 2Bt^3e^t + \underline{\underline{At^2e^t}} + Bt^3e^t = te^t$$

$$2Ae^t + (4A+6B-4A)te^t + (A+6B-2A-6B+A)t^2e^t + (B-2B+B)t^3e^t = te^t$$

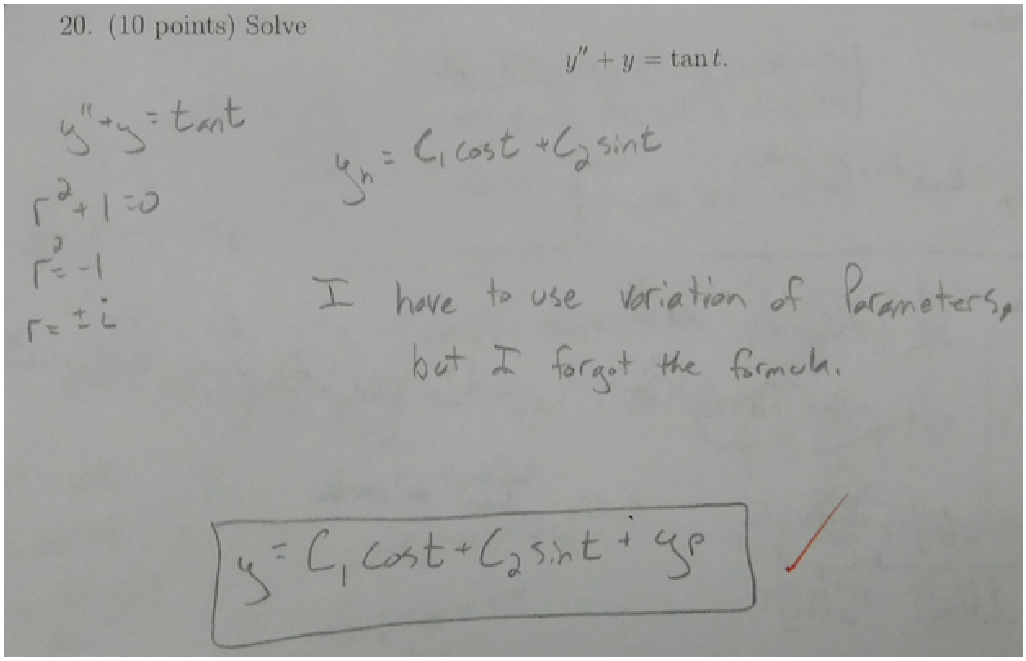
$$2Ae^t + 6Bte^t = te^t \qquad \Rightarrow \qquad A = 0,\ B = \frac{1}{6}$$

$$Y = \frac{1}{6}t^3e^t$$

II) $y'' - 2y' + y = 4$ $\qquad Y = A$ trial form of solution, substitution yields $A = 4$

$$y_c(t) = c_1e^t + c_2te^t + \underbrace{\frac{1}{6}t^3e^t + 4}_{Y}$$ General solution

Math303-3 Winter 2017 Quiz #5 Name:



20. (10 points) Solve

$$y'' + y = \tan t.$$

$y'' + y = \tan t$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

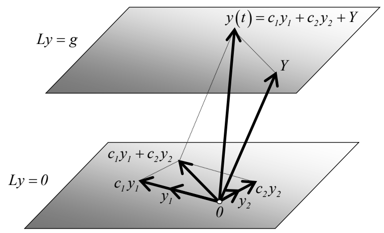
$y_h = C_1 \cos t + C_2 \sin t$

I have to use variation of Parameters, but I forgot the formula.

$y = C_1 \cos t + C_2 \sin t + y_p$

2nd order linear ODE:

$$Ly \equiv \quad y'' + p(t)y' + q(t)y \;=\; g(t)$$



Let $\{y_1(t), y_2(t)\}$ be a ***fundamental set***: $Ly_1 = 0$, $Ly_2 = 0$ and $W(y_1, y_2) \neq 0$.

Let $Y(t)$ be a particular solution of $Ly = g(t)$, then

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

general solution of $Ly = g(t)$.

2nd order linear ODE with constant coefficients $Ly = g(t)$

Homogeneous equation

$$Ly \equiv \quad a_0y'' + a_1y' + a\ y \;=\; 0$$

$a_0, a_1, a_2 = const \in \mathbb{R}$

Characteristic equation

$$a_0r^2 + a_1r + a \;=\; 0$$

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}$$

Method of Undetermined Coefficients for $Ly = g(t)$

$$g(t) \;=\; e^{at}\Big[\big(p_nt^n + \ldots + p_1t + p_0\big)\cos bt + \big(p_mt^m + \ldots + p_1t + p_0\big)\sin bt\Big]$$

1) $a \pm ib$ **is not** a root: $\quad Y(t) \;=\; e^{at}\Big[\big(A_kt^k + \ldots + A_1t + A_0\big)\cos bt + \big(B_kt^k + \ldots + B_1t + B_0\big)\sin bt\Big]$, $\quad k = max\{n, m\}$

2) $a \pm ib$ **is** a root of mult. s $\quad t^s \cdot Y(t)$

Variation of Parameter

$Y = u_1y_1 + u_2y_2$	$u_1 \;=\; -\int \frac{y_2}{W(y_1, y_2)} g(t)dt$	$u_2 \;=\; \int \frac{y_1}{W(y_1, y_2)} g(t)dt$

Example 1: $y'' - 2y' + y = te^t$

$r^2 - 2r + 1 = 0$ ch. eqn $(r-1)^2 = 0$ $r = 1$ is a root of multiplicity s=2

$g(t) = te^t$ identify: $a = 1, b = 0, n = 1, m = 0$

$a \pm bi = 1 \pm 0 \cdot i$ is a root of multiplicity s=2

$Y = (A + Bt)e^t$

$t^2 Y = t^2 (A + Bt)e^t = At^2 e^t + Bt^3 e^t$ Trial form of particular solution

Example 2: $y'' + y = t^2 \sin t$

Example 3: $y^{(iv)} + y''' = t^2 e^{-t}$