5.1 Power Series

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{about } x = 0$$

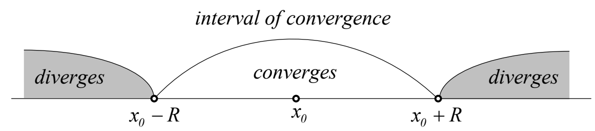
$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{about } x = x_0$$

convergence: series ***converges*** at x if the sequence of partial sums $\sum_{n=0}^{N} a_n (x - x_0)^n \to S$ converges.

absolute convergence: if the series of absolute values $\sum_{n=0}^{\infty} \left| a_n (x - x_0)^n \right|$ converges.

radius of convergence:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$



$\sum_{n=0}^{\infty} a_n (x - x_0)^n$ absolutely converges for all $x_0 - R < x < x_0 + R$

Convergence at boundary points $x = x_0 \pm R$ has to be investigated separately.

Taylor Series

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!} \cdot (x - x_0)^k$$

MacLauren series: $$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} \cdot x^k$$

Analytic Function

Power series defines an ***analytic function*** in its interval of convergence:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x \in (x_0 - R, x_0 + R)$$

Function is called ***analytic*** at x_0 if it has a Taylor series expansion about $x = x_0$.

Operations

Let $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$ be convergent in $(x_0 - R, x_0 + R)$:

summation
$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n$$

multiplication
$$f(x) \cdot g(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n \right] \cdot \left[\sum_{n=0}^{\infty} b_n (x - x_0)^n \right] = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

differentiation
$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$f''(x) = \sum_{n=1}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

Shift of index

$$\sum_{n=n_0}^{\infty} a_n x^{n+k} = \sum_{m=n_0+k}^{\infty} a_{m-k} x^m \qquad \begin{array}{l} m = n + k \\ n = m - k \end{array}$$

"dummy index"

Identity Theorem

If $\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$ for all x , then coefficients $a_n = 0$

Geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1 - x}, \text{ converges for } -1 < x < 1$$

5.2 Power Series Solution of the 2nd order linear ODE (Method of Frobenius)

Consider the linear ODE with variable coefficients

$$P(x)y''+Q(x)y'+R(x)y=0, \quad \text{where } P(x),\ Q(x),\ R(x) \text{ are polynomials} \tag{1}$$

If $P(x_0)\neq 0$, then x_0 is called an ***ordinary*** point.

If $P(x_1)=0$, then x_1 is called a ***singular*** point. In general, $x_1=a+ib$ can be a complex root.

1) Assume the solution to be in the form of ***power series***, and differentiate it

$$y(x)=\sum_{n=0}^{\infty}c_n\left(x-x_0\right)^n$$

$$y'(x)=\sum_{n=1}^{\infty}nc_n\left(x-x_0\right)^{n-1}$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)c_n\left(x-x_0\right)^{n-2}$$

2) Substitute into ODE

3) Combine in a single sigma summation term (change index of summation and write explicitly extra terms)

4) Derive the ***recurrence equation*** for coefficients c_n, using the Identity Theorem.

5) Collect terms with coefficient c_0: that yields one solution $y_1=c_0\cdot[\ \dots\ ]$

Collect terms with coefficient c_1: that yields the second solution $y_2=c_1\cdot[\ \dots\ ]$

6) General Solution: $y(x)=c_0y_1(x)+c_1y_2(x)$

7) Solution of IVP $c_0=y(x_0),\ c_1=y_2'(x_0)$

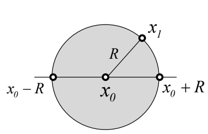
Theorem 5.3.1 If x_0 is an ***ordinary*** point, $P(x_0)\neq 0$, then the general solution of (1) is

$$y(x)=\sum_{n=0}^{\infty}c_n\left(x-x_0\right)^n \qquad =c_0y_1+c_1y_2 \qquad c_0,c_1\in\mathbb{R}$$

where y_1,y_2 are two power series solution of equation (1)

with the ***radius of convergence*** $R\geq\left|x_0-x_1\right|$, where

$\left|x_0-x_1\right|$ is the distance from the point of expansion x_0 to the closest singular point x_1, i.e. the root of $P(x_1)=0$, where x_1 can be a complex root.



5.3 Taylor series solution

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad y(x_0) = c_0 \qquad y'(x_0) = c_1$$

Look for solution in the form of the Taylor series about $x = x_0$:

$$y(x) = y(x_0) + y'(x_0)\cdot(x - x_0) + \frac{y''(x_0)}{2!}\cdot(x - x_0)^2 + \frac{y'''(x_0)}{3!}\cdot(x - x_0)^3 + \frac{y^{iv}(x_0)}{4!}\cdot(x - x_0)^4 + \dots$$

The first two coefficients are from the initial conditions:

$$y(x_0) = c_0$$

$$y'(x_0) = c_1$$

The third coefficient can be found from the given differential equation rewritten as

$$y''(x) = -p(x)\,y' - q(x)\,y\,, \qquad \text{then evaluate} \qquad y''(x_0) = -p(x_0)\overbrace{y'(x_0)}^{c_1} - q(x_0)\overbrace{y(x_0)}^{c_0}$$

To find the next coefficients, differentiate the equation and evaluate it at x_0:

$$y'''(x) = \left[-p(x)\,y' - q(x)\,y\right]' \qquad \Rightarrow \qquad y'''(x_0) = \dots$$

$$y^{iv}(x) = \left[y'''(x)\right]' \qquad \Rightarrow \qquad y^{iv}(x_0) = \dots$$

and so on …

Then with the found $y^{(n)}(x_0)$ construct the Taylor series solution

$$y(x) = y(x_0) + y'(x_0)\cdot(x - x_0) + \frac{y''(x_0)}{2!}\cdot(x - x_0)^2 + \frac{y'''(x_0)}{3!}\cdot(x - x_0)^3 + \frac{y^{iv}(x_0)}{4!}\cdot(x - x_0)^4 + \dots$$

5.4 Euler Equation is a linear ODE with specific dependence of coefficients on x

$$a_0 x^2 y'' + a_1 x y' + a_2 y = 0$$

I Change of variable: reduction to linear ODE with constant coefficients

$x = e^z$, $z = \ln|x|$

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$y'' = \frac{d}{dx}y' = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dz}{dx} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dz}{dx} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{1}{x^2}\frac{dz}{dx} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

Substitute into equation:

$$a_0 x^2\left(-\frac{1}{x^2}\frac{dz}{dx} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right) + a_1 x\left(\frac{1}{x}\frac{dy}{dz}\right) + a_2 y = 0$$

$$-a_0\frac{dy}{dz} + a_0\frac{d^2y}{dz^2} + a_1\frac{dy}{dz} + a_2 y = 0$$

$$a_0\frac{d^2y}{dz^2} + \left(a_1 - a_0\right)\frac{dy}{dz} + a_2 y = 0$$

Linear ODE with constant coefficients

II Find solution in the form:

$y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1)x^{m-2}$

Substitution into equation yields the linear ODE with constant coefficient:

$a_0 y'' + \left(a_1 - a_0\right)y' + a_2 y = 0$

$$a_0 m^2 + \left(a_1 - a_0\right)m + a_2 = 0$$

characteristic equation

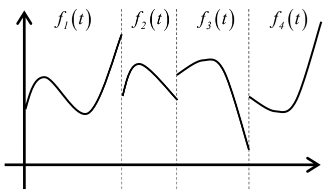
The fundamental solutions are (find solutions of ODE, first, for $x > 0$, then expand to $x < 0$):

a)	$m_1 \neq m_2 \in \mathbb{R}$	$\lvert x\rvert^{m_1}$, $\lvert x\rvert^{m_2}$
b)	$m_1 = m_2 = m \in \mathbb{R}$	$\lvert x\rvert^{m}$, $\lvert x\rvert^{m}\ln\lvert x\rvert$
c)	$m_{1,2} = a \pm ib$	$\lvert x\rvert^{a}\cos\left(b\ln\lvert x\rvert\right)$, $\lvert x\rvert^{a}\sin\left(b\ln\lvert x\rvert\right)$

Exercise: *5.4 #13*

6.1 The Laplace Transform

Piece-wise continuous function



$f(t)=\sum_k f_k(t)\cdot\left[u_{t_{k-1}}(t)-u_{t_k}(t)\right]$, where $f_k(t),\ t\in[t_{k-1},t_k]$ are continuous

$\int_a^\infty f(t)\,dt=\lim_{A\to\infty}\int_a^A f(t)\,dt$ if the limit exists, then **improper integral *converges***

Example: $\int_a^\infty \frac{1}{t^p}dt$ converges for $p>1$, and diverges for $p\le 1$

Let $f(t)\le g(t)$ and $\int_a^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ converges

Let $f(t)\ge g(t)$ and $\int_a^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ diverges

The Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Existence Theorem 6.1.2

If $f(t)$ is piece-wise continuous in any $[0,A]$, and if

$|f(t)|\le Ke^{at}$ for $t>M$, where $K>0, M>0, a\in R$, (of exponential order),

then the Laplace transform $F(s)=\mathcal{L}\{f(t)\}$ exists for $s>a$, and

$F(s)\le\frac{K}{s-a}$, $\lim_{s\to\infty}F(s)=0$, $\lim_{s\to\infty}sF(s)<\infty$

Functions for which the Laplace transform exists: $c,\ t,\ t^5,\ 2e^{9t},\ \cos t,\ t^6e^{10t},\ldots$

Functions for which the Laplace transform does not exist: e^{t^2}

Laplace transform is ***linear***

$$\mathcal{L}\{\alpha f(t)+\beta g(t)\}=\alpha F(s)+\beta G(s)$$

Transform of the derivatives

$$\mathcal{L}\{f(t)\}=F(s)$$

$$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$$

$$\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$$

Table of transforms:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$

The Inverse Laplace Transform: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

The Inverse Transform is linear: $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$

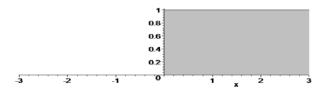
To find inverse Laplace transform invert the Table of Laplace transform.
Sometimes partial fractions or convolution theorem is used.

6.2 **Solution of the Initial Value Problems** with the help of Laplace Transform

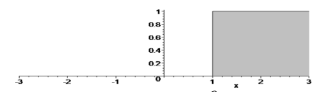
$a_0 y'' + a_1 y' + a_2 y = g(t)$

6.3 Unit Step Function $u_c(t)$ Heaviside function $H(t)$, typical name and notation in engineering literature

$$u_0(t)=\begin{cases}1 & t\ge 0\\ 0 & t<0\end{cases}\qquad H(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$

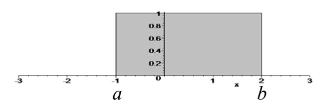


$$u_c(t)=\begin{cases}1 & t\ge c\\ 0 & t<c\end{cases}\qquad H(t-c)=\begin{cases}1 & t>c\\ 0 & t<c\end{cases}$$



Filter Function

$$u_a(t)-u_b(t)=\begin{cases}0 & t<a\\ 1 & a<t<b\\ 0 & t>b\end{cases}\qquad H(t-a)-H(t-b)=\begin{cases}0 & t<a\\ 1 & a<t<b\\ 0 & t>b\end{cases}$$



Express piece-wise continuous function in terms of the unit step function $u_c(t)$. Example:

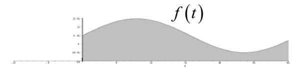
Express the function defined in terms of $u_c(t)$ as the piece-wise continuous function. Example:

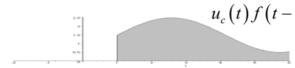
Laplace Transform:

$$\mathcal{L}\{u_c(t)\}=\int_0^\infty u_c(t)e^{-st}dt=\int_c^\infty e^{-st}dt=\frac{e^{-cs}}{s}$$

Translation of $f(t)$

$$u_c(t)f(t-c)$$





Theorem 6.3.1:

$$\mathcal{L}\{u_c(t)f(t-c)\}=e^{-cs}F(s)$$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\}=u_c(t)f(t-c)$$

$$\mathcal{L}^{-1}\left\{e^{-cs}\frac{1}{s}\right\}=u_c(t)$$

$$\mathcal{L}^{-1}\left\{e^{-cs}\frac{1}{s^2+1}\right\}=u_c(t)\sin(t-c)$$

$$\mathcal{L}^{-1}\left\{e^{-cs}\frac{s}{s^2+1}\right\}=u_c(t)\cos(t-c)$$

Theorem 6.3.2

$$\mathcal{L}\{e^{ct}f(t)\}=F(s-c)$$

$$\mathcal{L}^{-1}\{F(s-c)\}=e^{ct}f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}=e^{2t}\sin(t)$$

6.4 Differential Equations with Discontinuous Forcing Functions

Problem: 6.4 #1

$y'' + y = f(t)$ $\qquad\qquad$ $y(0) = y'(0) = 0$

$$y'' + y = u_0(t) - u_{3\pi}(t)$$

$$s^2 Y + Y = \frac{1}{s} - e^{-3\pi s}\frac{1}{s}$$

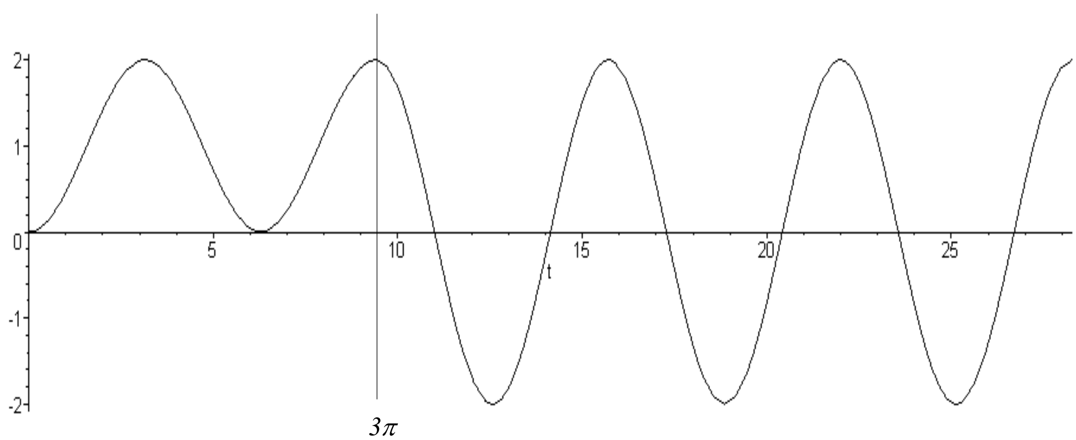
$$Y = \frac{1}{s(s^2+1)} - e^{-3\pi s}\frac{1}{s(s^2+1)}$$

$$Y = \frac{1}{s} - \frac{s}{(s^2+1)} - e^{-3\pi s}\frac{1}{s} + e^{-3\pi s}\frac{s}{s^2+1}$$ partial fractions

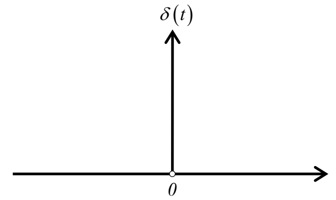
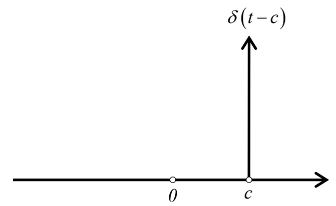
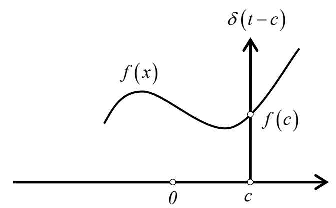
$$y = 1 - \cos t - H(t-3\pi)\cdot 1 + H(t-3\pi)\cos(t-3\pi)$$

$$y = 1 - \cos t - H(t-3\pi)\cdot\left[1+\cos(t)\right]$$

$y = 1 - \cos t\,,\qquad t < 3\pi$	$y = -2\cos t\,,\qquad t > 3\pi$



6.5 Impulse Function (or Dirac delta function) is defined by its properties:

$\delta(t)$	$\delta(t)=0$ *for all* $x\neq 0$ $\int_{-\infty}^{\infty}\delta(t)=1$ $\int_{-a}^{a}\delta(t)=1$ *for any* $a>0$	
$\delta(t-c)$	$\delta(t-c)=0$ *for all* $x\neq c$ $\int_{-\infty}^{\infty}\delta(t-c)=1$ $\int_{c-a}^{c+a}\delta(t-c)=1$ *for any* $a>0$	
Integration with $\delta(t-c)$	$\int_{-\infty}^{\infty}f(t)\delta(t-c)\,dt=f(c)$	
Laplace Transform	$\mathcal{L}\{\delta(t-c)\}=\int_{-\infty}^{\infty}e^{-st}\delta(t-c)dt=e^{-sc}$	

Solve: $y''+y=\delta(t-t_0)$ $\quad y(0)=y'(0)=0$

6.5 #16

(a) $y'' + y = f_k\,,\qquad y(0) = y'(0) = 0\,,\qquad f_k = \dfrac{u_{4-k}(t) - u_{4+k}(t)}{2k}\,,\qquad f_k \to \delta(t-4)$

$$s^2Y + Y = \frac{1}{2k}\frac{e^{-(4-k)s}}{s} - \frac{1}{2k}\frac{e^{-(4+k)s}}{s}$$

$$Y = \frac{1}{2k}\frac{e^{-(4-k)s}}{s(s^2+1)} - \frac{1}{2k}\frac{e^{-(4+k)s}}{s(s^2+1)}$$

$$Y = \frac{1}{2k}e^{-(4-k)s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] - \frac{1}{2k}e^{-(4+k)s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$$

$$y_k(t) = \frac{1}{2k}u_{4-k}(t)\big[1 - cos\big[t-(4-k)\big]\big] - \frac{1}{2k}u_{4+k}(t)\big[1 - cos\big[t-(4+k)\big]\big]$$

$$y_k(t) = 0 \qquad t < 4-k$$

$$y_k(t) = \frac{1}{2k}\big[1 - cos\big[t-(4-k)\big]\big] \qquad 4-k < t < 4+k$$

$$y_k(t) = \frac{1}{2k}\big[1 - cos\big[t-(4-k)\big]\big] - \frac{1}{2k}\big[1 - cos\big[t-(4+k)\big]\big] \qquad t > 4+k$$

$$\lim_{k\to 0} y_k(t) = \lim_{k\to 0}\frac{1}{2k}\big[1 - cos\big[t-(4-k)\big]\big] = \lim_{k\to 0}\frac{1}{2}\big[sin\big[t-(4-k)\big]\big] = 0 \qquad 4-k < t < 4+k \ \overset{k\to 0}{\to}\ t = 4$$

$$\lim_{k\to 0} y_k(t) = \lim_{k\to 0}\left\{\frac{1}{2k}\big[1 - cos\big[t-(4-k)\big]\big] - \frac{1}{2k}\big[1 - cos\big[t-(4+k)\big]\big]\right\}$$

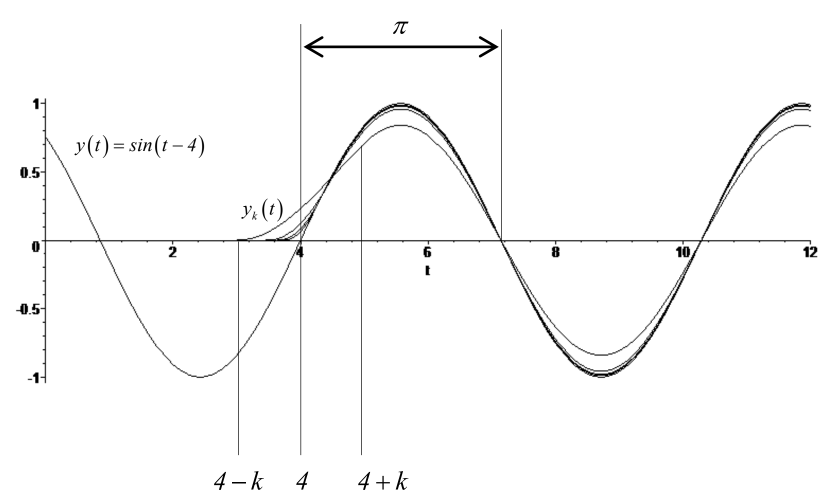
$$= \lim_{k\to 0}\left\{\frac{1}{2}\big[sin\big[t-(4-k)\big]\big] + \frac{1}{2}\big[sin\big[t-(4+k)\big]\big]\right\} = sin(t-4) \qquad t > 4+k \to t > 4$$

(b) $y'' + y = \delta(t-4)\,,\qquad y(0) = y'(0) = 0$

$$s^2Y + Y = e^{-4s}$$

$$Y = \frac{e^{-4s}}{s^2+1}$$

$$y(t) = u_4(t)\cdot sin(t-4)$$



6.6 The Convolution Theorem

Convolution

$$f*g = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t g(t-\tau)f(\tau)\,d\tau = g*f$$

Convolution Theorem

$$\mathcal{L}\{f*g\} = F(s)\cdot G(s)$$

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = f*g$$

Example 1: Let function $u(t)$ be defined by integral $u(t) = \int_0^t (t-\tau)e^{\tau}\,d\tau$. Find $\mathcal{L}\{u(t)\}$.

$u(t) = \int_0^t (t-\tau)e^{\tau}\,d\tau = f*g$ where $f(t) = t,\ g(t) = e^t$

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \mathcal{L}\{f*g\}\\ &= F(s)\cdot G(s)\\ &= \frac{1}{s^2}\cdot\frac{1}{s-1}\end{aligned}$$

Example 2: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s^2+1\right)}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{\left(s^2+1\right)}\right\}$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{t\}\cdot\mathcal{L}\{\sin t\}\}$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}\}$$

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\}$$

$$f*g = \int_0^t (t-\tau)\sin\tau\,d\tau = t-\sin t$$

Example 3: Find the solution of $y''+\omega^2 y = f(t)$, $y(0)=y'(0)=0$

in terms of the convolution integral.

Calculate solution for $\omega=3,\ f(t)=t$.

Find the inverse Laplace transform of