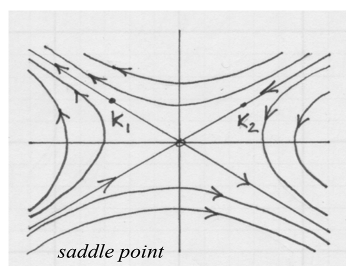
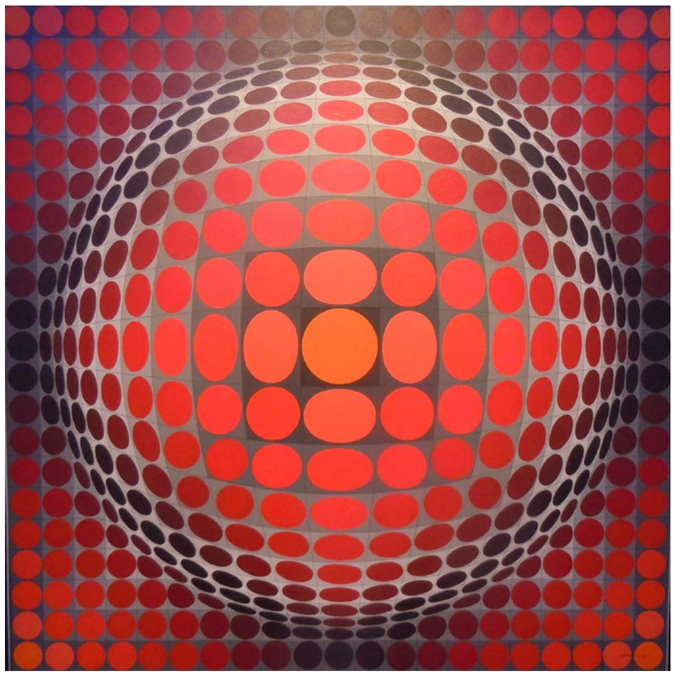
Chapter 7

Systems of 1st Order Linear Differential Equations



$\lambda_1 > 0, \quad \lambda_2 < 0$



7.1 LINEAR SYSTEMS OF THE 1st ORDER ODE's

Linear system $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$ $\mathbf{x}(t_0) = \mathbf{x}^0$ initial condition (14)

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{1n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t)
\end{aligned}
$$

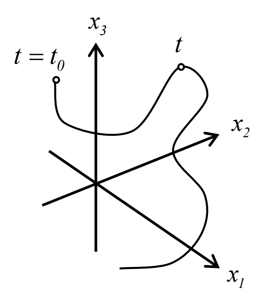
Homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$

Reduction of nth order linear ODE $y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = g$

to a system of *n* 1st order ODEs:

$$
\begin{aligned}
x_1 &= y & x_1' &= y' = x_2 \\
x_2 &= y' & x_2' &= y'' = x_3 \\
x_3 &= y''' & x_3' &= y''' = x_4 \\
&\ldots \\
x_{n-1} &= y^{(n-2)} & x_{n-1}' &= y^{(n-1)} = x_n \\
x_n &= y^{(n-1)} & x_n' &= y^{(n)} = -a_1 x_n - \ldots - a_n x_1 + g
\end{aligned}
$$

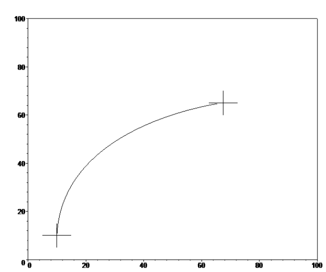
Solution, parametric graph $x_1(t)$, $x_2(t)$, $x_3(t)$



$$
\begin{cases}
x_1 = x_1(t) \\
x_2 = x_2(t) \\
x_3 = x_3(t)
\end{cases}
$$

Modeling of interconnecting tanks (7.1 #22)

$$
\begin{aligned}
x &= x_1(t) \\
y &= x_2(t)
\end{aligned}, \quad t \ge 0
$$



Existence Theorems (7.1.1 and 7.1.2)

7.2 Review of Matrices

Matrix $m \times n$ $\quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left(a_{ij}\right)_{n\times m}$

$n \times n$ $\quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \left(a_{ij}\right)_{n\times n}$ square matrix

Vector $n \times 1$ $\quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left(x_i\right)$ column vector

Transpose $\quad \mathbf{A}^T = \left(a_{ji}\right) \qquad \mathbf{x}^T = \left(x_1, \quad x_2, \quad \ldots, \quad x_n\right)$

Conjugate $\quad \overline{\mathbf{A}} = \left(\overline{a}_{ij}\right) \qquad \overline{\mathbf{x}} = \left(\overline{x}_i\right)$

Adjoint $\quad \mathbf{A}^* = \overline{\mathbf{A}}^T$

Self-adjoint (Hermitian) if $\quad \mathbf{A}^* = \mathbf{A}$ (for real matrices, $\mathbf{A}^T = \mathbf{A}$ symmetric)

Matrix Algebra: $\quad \mathbf{A} = \mathbf{B} \qquad a_{ij} = b_{ij}$ *for all i and j*

$\mathbf{I}_{n\times n} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

$\mathbf{A}+\mathbf{B} = \left(a_{ij}+b_{ij}\right) \qquad \mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$

$\mathbf{0} \qquad \mathbf{A}+\mathbf{0} = \mathbf{0}+\mathbf{A} = \mathbf{A}$

$k\mathbf{A} = \left(ka_{ij}\right)$

$\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \left(c_{ij}\right)_{m\times p} \qquad c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ (in general, $\mathbf{AB} \neq \mathbf{BA}$)

$\mathbf{IA} = \mathbf{AI}$ for square matrices

$\mathbf{0A} = \mathbf{A0}$ for square matrices

Matrix inverse $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ (if $det\,\mathbf{A} \neq 0$, then inverse $\mathbf{A}^{-1}$ exists)

$(2\times 2\ matrix)$ $\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\left[\ \mathbf{A} \mid \mathbf{I}\ \right] \xrightarrow{\substack{Gaussian \\ elimination}} \left[\ \mathbf{I} \mid \mathbf{A}^{-1}\right]$ row reduction

Products of vectors: $\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i$

$(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} x_i \overline{y}_i = \mathbf{x}^T\overline{\mathbf{y}}$ inner (scalar) product

Properties:

$(\mathbf{x},\mathbf{y}) = \overline{(\mathbf{y},\mathbf{x})}$

$(\alpha\mathbf{x},\mathbf{y}) = \alpha(\mathbf{x},\mathbf{y})$

$(\mathbf{x},\alpha\mathbf{y}) = \overline{\alpha}(\mathbf{x},\mathbf{y})$

$(\mathbf{x},\mathbf{y}+\mathbf{z}) = (\mathbf{x},\mathbf{y}) + (\mathbf{x},\mathbf{z})$

Norm $\|\mathbf{x}\| = \sqrt{(\mathbf{x},\mathbf{x})}$

Orthogonality $\mathbf{x} \perp \mathbf{y}\quad if \quad (\mathbf{x},\mathbf{y}) = 0$

3-D coordinate vectors $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Matrix Functions $\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{bmatrix} = \left(a_{ij}(t)\right), \qquad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

$\mathbf{A}'(t) = \left(a'_{ij}(t)\right)$

$\int \mathbf{A}(t)dt = \left(\int a'_{ij}(t)\,dt\right)$

7.3 Systems of Linear Algebraic Equations

System of algebraic equations $\mathbf{A}\mathbf{x} = \mathbf{b}$

Augmented matrix $\left[\mathbf{A} \mid \mathbf{b}\right]$

RREF

Solution

Linearly independence vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ are linearly independent if

$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + ... + c_n\mathbf{x}_n = \mathbf{0}$ only if all $c_n = 0$

n vectors of length n: $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ $\quad \mathbf{x}_m = \begin{bmatrix} x_{1m} \\ x_{2m} \\ \vdots \\ x_{nm} \end{bmatrix}$

Fact: $det\left[\mathbf{x}_1 \ \mathbf{x}_2 \ ... \ \mathbf{x}_n\right] \neq 0 \iff \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ are linearly independent

Eigenvalue problem: $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ $\quad \mathbf{x} \neq \mathbf{0}$

Solve characteristic equation: $\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \Rightarrow$ λ_n are called ***eigenvalues***

Find eigenvectors by solving $\left(\mathbf{A} - \lambda_n\mathbf{I}\right)\mathbf{k}_n = \mathbf{0} \Rightarrow$ $\mathbf{k}_n$ is called an ***eigenvector*** corresponding to eigenvalue λ_n

1) Real distinct eigenvalues $\left(\lambda_1 - \lambda\right)\left(\lambda_2 - \lambda\right)\cdots\left(\lambda_n - \lambda\right) = 0$ There exist n linearly independent eigenvectors $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n$ corresponding to $\lambda_1, \lambda_2, ..., \lambda_n$

2) Root of multiplicity s $\left(\lambda_1 - \lambda\right)^s = 0$ There can be more than one lin.indep. $\mathbf{k}_1, \cdots, \mathbf{k}_m$ corresponding to λ_1
(m is called ***geometric*** multiplicity)
(s is called ***algebraic*** multiplicity)

3) Complex roots $\lambda_1 = \alpha + \beta i$ $\quad \mathbf{k}_1 = \mathbf{b}_1 + i\mathbf{b}_2$ appear in conjugate pairs
$\lambda_2 = \alpha - \beta i$ $\quad \mathbf{k}_2 = \mathbf{b}_1 - i\mathbf{b}_2$

7.4 Basic Theory of Systems of 1st Order Linear Differential Equations

Matrix-vector notations:

$$\mathbf{k}=\begin{bmatrix}k_1\\k_2\\\vdots\\k_n\end{bmatrix},\ \mathbf{x}_m=\begin{bmatrix}x_{1m}\\x_{2m}\\\vdots\\x_{nm}\end{bmatrix},\ \mathbf{P}(t)=\begin{bmatrix}p_{11}(t)&p_{12}(t)&\cdots&p_{1n}(t)\\p_{21}(t)&p_{22}(t)&\cdots&p_{2n}(t)\\\vdots&\vdots&\ddots&\vdots\\p_{n1}(t)&p_{n2}(t)&\cdots&p_{nn}(t)\end{bmatrix},\ \mathbf{c}=\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

Homogeneous System $\mathbf{x}'(t)=\mathbf{P}(t)\mathbf{x}(t) \qquad t\in(a,b)$ (3)

Initial conditions $\mathbf{x}(t_0)=\mathbf{x}_0$

Superposition principle: If $\mathbf{x}_1,\mathbf{x}_2$ are solutions of (3), then $c_1\mathbf{x}_1+c_2\mathbf{x}_2$ is also a solution (Th 7.4.1)

Linear dependence

It is said that $\mathbf{x}_1(t),\mathbf{x}_2(t),...,\mathbf{x}_n(t)$ are ***linearly dependent*** on (a,b) if there exists a set of constants $c_1,c_2,...,c_n$ not all equal to zero, such that $c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t)+...+c_n\mathbf{x}_n(t)=\mathbf{0}$ for all $t\in(a,b)$.

Otherwise, $\mathbf{x}_1(t),\mathbf{x}_2(t),...,\mathbf{x}_n(t)$ are ***linearly independent*** on (a,b).

.

Wronskian

$W(t)=det\left[\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ ...\ \mathbf{x}_n(t)\right]$

Solutions $\mathbf{x}_1(t),\mathbf{x}_2(t),...,\mathbf{x}_n(t)$ are ***linearly independent*** at *t*, if $W(t)\neq 0$

Theorem 7.4.2 If $\mathbf{x}_1(t),\mathbf{x}_2(t),...,\mathbf{x}_n(t)$ are linearly independent solutions of $\mathbf{x}'(t)=\mathbf{P}(t)\mathbf{x}(t)$, then any solution of (3) can be written as $\boldsymbol{\varphi}(t)=c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t)+...+c_n\mathbf{x}_n(t)$

Theorem 7.4.3 If $\mathbf{x}_1(t),\mathbf{x}_2(t),...,\mathbf{x}_n(t)$ are solutions of (3) in (a,b), then

$W(t)=det\left[\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ ...\ \mathbf{x}_n(t)\right]\equiv 0$ in (a,b) or $W(t)\neq 0$ in (a,b).

$W(t)=ce^{\int\left[p_{11}(t)+...+p_{nn}(t)\right]dt}$

Theorem 7.4.4 *Existence of at least one fundamental solution*

Fundamental matrix $\boldsymbol{\Psi}=\left[\mathbf{x}_1\ \mathbf{x}_2\ ...\ \mathbf{x}_n\right] \qquad W=det\,\boldsymbol{\Psi}\neq 0,\ t\in(a,b)$

General solution $\mathbf{x}=\boldsymbol{\Psi}\mathbf{c}$ $\qquad \mathbf{x}(t)=c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t)+...+c_n\mathbf{x}_n(t)$

Solution of IVP $\mathbf{x}'=\mathbf{P}\mathbf{x} \qquad \mathbf{x}(t_0)=\mathbf{x}_0$

$\mathbf{x}(t)=\boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}_0$

Fundamental sets for homogeneous linear systems with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x}\ \text{, where}\quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}\qquad \text{Trial form: } \mathbf{x}(t) = \mathbf{k}e^{\lambda t},\quad \mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

7.5 I Real distinct eigenvalues

Characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad \Rightarrow \quad (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) = 0$

Find eigenvectors by solving $(\mathbf{A} - \lambda_k \mathbf{I})\mathbf{k}_k = \mathbf{0}$

There exist n linearly independent eigenvectors $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n$ corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$

The fundamental set: $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{k}_1,\ \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{k}_2,\ \cdots,\ \mathbf{x}_s = e^{\lambda_n t}\mathbf{k}_s$

7.8 II Repeated eigenvalues

Characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad \Rightarrow \quad (\lambda_1 - \lambda)^s = 0$ root of multiplicity s

Find eigenvectors by solving $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{k} = \mathbf{0}$ (***algebraic*** multiplicity)

Case 1 If there exist linearly independent eigenvectors $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_s$ corresponding to λ_1 (***geom***.)

The fund. set : $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{k}_1,\ \mathbf{x}_2 = e^{\lambda_1 t}\mathbf{k}_2,\ \cdots,\ \mathbf{x}_s = e^{\lambda_1 t}\mathbf{k}_s$

Case 2 If there exists only one independent eigenvector $\mathbf{k}$ corresponding to λ_1

Then solve
$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{p} = \mathbf{k}$$
$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{q} = \mathbf{p}$$
$$\vdots$$

To find vectors $\mathbf{p}, \mathbf{q}, \ldots$

The fund. Set: $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{k},\ \mathbf{x}_2 = e^{\lambda_1 t}(t\mathbf{k} + \mathbf{p}),\ \mathbf{x}_3 = e^{\lambda_1 t}\left(\frac{t^2}{2}\mathbf{k} + \mathbf{p}t + \mathbf{q}\right), \ldots$

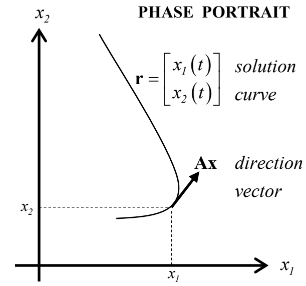
7.6 III Complex eigenvalues

Conjugate pair of complex roots $|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad \Rightarrow \quad \lambda_1 = \alpha + \beta i \qquad \lambda_2 = \alpha - \beta i$

Find eigenvectors by solving $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{k}_1 = \mathbf{0} \qquad \mathbf{k}_1 = \mathbf{a} + i\mathbf{b} \qquad \mathbf{k}_2 = \mathbf{a} - i\mathbf{b}$

The fundamental set:
$$\mathbf{x}_1 = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t)$$
$$\mathbf{x}_2 = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)$$

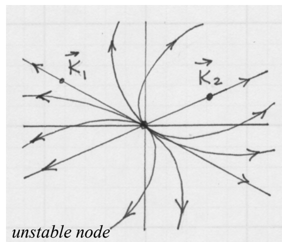
Plane System	$x' = ax + by$ $y' = cx + dy$ $\quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



Characteristic Equation: $\left|\mathbf{A} - \lambda \mathbf{I}\right| = \lambda^2 - (a+d)\lambda + ad - bc = 0$

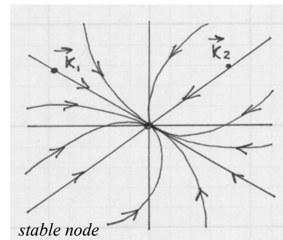
Eigenvalues: $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \dfrac{Tr\mathbf{A} \pm \sqrt{\Delta}}{2}$

I $\quad \Delta > 0, \quad \lambda_1 \neq \lambda_2 \in R, \quad \mathbf{x}(t) = c_1 \mathbf{k}_1 e^{\lambda_1 t} + c_2 \mathbf{k}_2 e^{\lambda_2 t}$



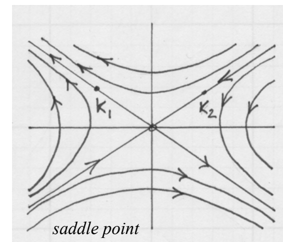
unstable node

$\lambda_1 > 0, \ \lambda_2 > 0$



stable node

$\lambda_1 < 0, \ \lambda_2 < 0$



saddle point

$\lambda_1 > 0, \ \lambda_2 < 0$

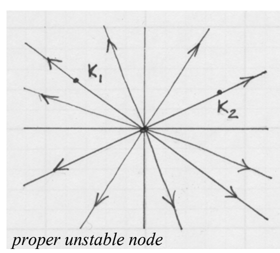
II $\quad \Delta = 0, \quad \lambda_1 = \lambda_2 = \lambda \in R$

a) Two independent $\mathbf{k}_1, \ \mathbf{k}_2$

$$\mathbf{x}(t) = c_1 \mathbf{k}_1 e^{\lambda t} + c_2 \mathbf{k}_2 e^{\lambda t}$$

b) One independent $\mathbf{k}$ (find $\mathbf{p}$)

$$\mathbf{x}(t) = c_1 \mathbf{k} e^{\lambda t} + c_2 (\mathbf{k}t + \mathbf{p}) e^{\lambda t}$$

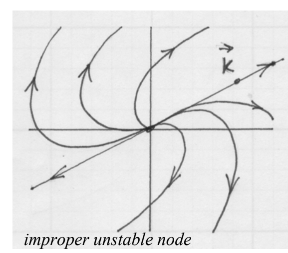


proper unstable node

$\lambda > 0$

proper stable node

$\lambda < 0$

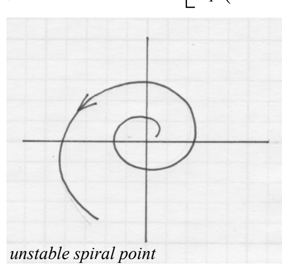


improper unstable node

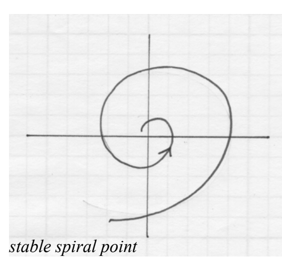
$\lambda > 0$

III $\quad \Delta < 0 \quad \lambda_{1,2} = \alpha \pm \beta i, \ \mathbf{k}_{1,2} = \mathbf{a} \pm i\mathbf{b}$

a) $\alpha \neq 0, \quad \mathbf{x} = \left[c_1 (\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t) + c_2 (\mathbf{a} \cos \beta t + \mathbf{b} \sin \beta t) \right] e^{\alpha t}$

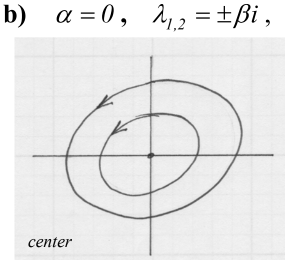

unstable spiral point

$\alpha > 0$


stable spiral point

$\alpha < 0$

b) $\alpha = 0, \quad \lambda_{1,2} = \pm \beta i, \quad x = c_1 (\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t) + c_2 (\mathbf{a} \cos \beta t + \mathbf{b} \sin \beta t)$


center

7.7 Fundamental Matrix

System of ODE's $\mathbf{x}' = P\mathbf{x}$, $\mathbf{x}(t_0) = \mathbf{x}^0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$ General solution: $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$

Ψ

$\mathbf{x}_k' = P\mathbf{x}_k$ $\quad \mathbf{\Psi} = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n]$ $\quad \mathbf{\Psi}' = P\mathbf{\Psi}$

$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}$ General solution

$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$ Solution of IVP

Φ

$\mathbf{x}_k' = P\mathbf{x}_k$

$\mathbf{x}_k(0) = \mathbf{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th}$ $\quad \mathbf{\Phi} = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n]$ $\quad \mathbf{\Phi}' = P\mathbf{\Phi}$

$\mathbf{\Phi}(0) = \mathbf{I}$

$\mathbf{\Phi}^{-1}(0) = \mathbf{I}$

$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}$

$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^0$ $\quad \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)$ $\quad \mathbf{\Phi} = e^{\mathbf{A}t}$

The ***matrix exponential function*** ($\mathbf{A}$ is a constant matrix):

$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \mathbf{x}(0) = \mathbf{x}^0$

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{(t\mathbf{A})^n}{n!} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \ldots + \frac{t^n}{n!}\mathbf{A}^n + \ldots$$

$e^{\mathbf{A}t} = \mathbf{\Phi}(t)$

$e^{\mathbf{A}t} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)$

$\left(e^{\mathbf{A}t}\right)' = \mathbf{A}e^{\mathbf{A}t} \qquad e^{\mathbf{A}\cdot 0} = \mathbf{I}$

$(\mathbf{\Phi})' = \mathbf{A}\mathbf{\Phi} \qquad \mathbf{\Phi}(0) = \mathbf{I}$

$\mathbf{\Phi}$ and $e^{\mathbf{A}t}$ are solutions of the same IVP

$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^0$

7.9 Solution of the non-homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \qquad \mathbf{x}(t_0) = \mathbf{x}^0$$

I Diagonalization

1) Solve Eigenvalue Problem: $|\mathbf{A} - \lambda\mathbf{I}| = 0 \;\Rightarrow\; \lambda_1, \lambda_2, \ldots, \lambda_n, \; \mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n$

2) Construct a transformation matrix $\mathbf{T} = [\mathbf{k}_1 \; \mathbf{k}_2 \; \cdots \; \mathbf{k}_n]$ (if eigenvalues are lin.ind.)

3) Find inverse $\mathbf{T}^{-1}$ (Transformation matrix diagonalizes **A**):

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}, \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

4) Calculate entries h_i $\quad \mathbf{T}^{-1}\mathbf{g} = (h_i)$

5) Define the new variable $\quad \mathbf{x} = \mathbf{T}\mathbf{y}$

Solve equations for $y_1, \ldots, y_n$: $\quad \mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$ (equations are uncoupled)

$$y_1(t) = c_1 e^{\lambda_1 t} + e^{\lambda_1 t}\int e^{-\lambda_1 t} h_1 dt$$
$$\vdots$$
$$y_n(t) = c_n e^{\lambda_n t} + e^{\lambda_n t}\int e^{-\lambda_n t} h_n dt$$

6) Obtain the general solution by $\quad \mathbf{x} = \mathbf{T}\mathbf{y}$

II Variation of parameter

Fundamental matrix $\quad \mathbf{\Psi} = [\mathbf{x}_1 \; \mathbf{x}_2 \; \ldots \; \mathbf{x}_n]$

Particular solution: $\quad \mathbf{x}_p(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)dt$

General solution: $\quad \mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)dt$

Solution of IVP with the help of $\mathbf{\Psi}$: $\quad \mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0 + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds$

Solution of IVP with the help of $\mathbf{\Phi}$: $\quad \mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds$

III Undetermined coefficients