Why should an optimal sustainer rocket have an upward velocity the same as its terminal velocity?

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Required assumptions of the mathematical model:

1. The sustainer rocket motor has fixed total impulse: $I = Ft$, where F and t are the thrust force and thrust time, respectively. We are taking F to be constant in time until the point of burnout. While I is fixed, either F or t can be independently adjusted by other motor parameters such as the size of the exhaust aperture.

2. The sustainer has relatively constant total mass: $m = m_{\mathrm{body}} + m_{\mathrm{rxn}}$. Because the on-board reaction mass, m_{rxn}, must of necessity decrease during thrust (mass is expelled from the rocket and is the source of thrust), this assumption requires that m_{rxn} be much smaller than m_{body}, the mass of the rest of the rocket. One way to sidestep this requirement is to let m represent the *average* mass of the sustainer during thrust.

3. The sustainer spends most of its upward ascent in a steady-state thrusting situation, such that the additional height gained after the conclusion of thrusting (i.e. during coasting) is negligible when compared to the height gained during thrusting.

Required equations:

- Height of sustainer apogee relative to height at which booster separates:

$$h = vt\,, \tag{1}$$

  where v is the steady-state ascension velocity.

- Thrust force required to sustain a given value of v:

$$F = mg + \frac{1}{2}\rho A C_D v^2\,, \tag{2}$$

  where $g = 9.81\,\mathrm{m/s^2}$, ρ is air density, A is frontal cross-sectional area, and C_D is the coefficient of drag.

- Assumption (1) above, namely that thrust time is related to thrust force by

$$t = I/F\,. \tag{3}$$

Optimization solution:

We want to maximize h (Eq. 1) by varying the ascent velocity v, while satisfying Eqs. 2 and 3. We first insert Eqs. 2 and 3 into Eq. 1, to get

$$h = \frac{vI}{mg + \frac{1}{2}\rho A C_D v^2} \, . \tag{4}$$

To maximize h we need to do some calculus. First we take the derivative of Eq. 4 with respect to v. This gives us dh/dv as a function of v. We then say that the optimal v occurs when $dh/dv = 0$. The subsequent algebraic equation can be solved to give the optimum sustainer ascent velocity. It turns out to be $v = v_t$, where $v_t = \sqrt{(2mg)/(\rho A C_D)}$ is the "terminal velocity" or the speed of the sustainer in nose-down free-fall. The corresponding optimum thrust force has twice the magnitude of the force of gravity: $F = 2mg$. Also of note is that exactly half of the sustainer motor's thrust energy ($E_{\text{tot}} = Iv$) is expended in raising the altitude of the rocket and the other half is used to fight the air resistance.

Remember, these results are only as good as the assumptions made. In the real world assumptions 1 through 3 do not hold exactly and so the actual optimum thrust force or ascent velocity will differ from the predictions given with the simple model above. In fact, if one relaxes assumption 3 and accounts for the additional altitude achieved during the coasting phase, the optimal value of v will be in the range $v_t < v < 2v_t$, with the upper end of the range ($v > 1.5v_t$) for the case when the motor has a relatively smaller impulse, namely $I < 3.9mv_t$. In any case, the idea that the optimal ascent velocity is close to the terminal velocity can act as a starting point—one to be refined by more accurate computer simulations and by actual flight data.